H^0 OF IGUSA VARIETIES VIA AUTOMORPHIC FORMS

ARNO KRET, SUG WOO SHIN

Abstract. Our main theorem describes the degree 0 cohomology of Igusa varieties in terms of one-dimensional automorphic representations in the setup of mod p Hodge-type Shimura varieties with hyperspecial level at p, mirroring the well known analogue for complex Shimura varieties. As an application, we obtain a completely new approach to two geometric questions. (See §1.5 for a comparison with independent results by van Hoften and Xiao via a different approach.) Firstly, we verify the discrete part of the Hecke orbit conjecture, which amounts to irreducibility of central leaves, generalizing preceding works by Chai, Oort, Yu, et al. Secondly, we deduce irreducibility of Igusa towers and its generalization to non-basic Igusa varieties in the same generality, extending previous results by Igusa, Ribet, Faltings–Chai, Hida, and others. Our proof is based on a Langlands–Kottwitz-type formula for Igusa varieties due to Mack-Crane, an asymptotic study of the trace formula, and an estimate for unitary representations and their Jacquet modules in representation theory of p-adic groups due to Howe–Moore and Casselman.

Contents

1. Introduction

Igusa varieties were studied by Igusa [Igu68] and Katz–Mazur [KM85] in the case of modular curves. Harris–Taylor and Mantovan [HT01, Man05] have generalized the construction to PEL-type Shimura varieties. Recently Caraiani–Scholze [CS17] gave a slightly different definition in the PEL case which gives the same cohomology. Hamacher, Zhang, and Hamacher–Kim went further to define Igusa varieties for Hodge-type Shimura varieties [Ham17, Zha, HK19]. In the (μ-)ordinary setting, Igusa varieties are also referred to as Igusa towers. (Often the definitions differ in a minor way.) There are versions of Igusa varieties as p-adic formal schemes or adic spaces over p-adic fields, but we concentrate on the characteristic p varieties in this paper. We also mention that function-field analogues of Igusa varieties are studied in a forthcoming paper by Sempliner.

The ℓ-adic cohomology of Igusa varieties (with $\ell \neq p$) has several arithmetic applications. In [HT01, Man05, HK19], the authors prove a formula computing the cohomology of Shimura varieties in terms of that of Igusa varieties and Rapoport–Zink spaces. This means that, if we understand the cohomology of Igusa varieties well enough, then our knowledge of cohomology can be propagated from Rapoport–Zink spaces to Shimura varieties or the other way around. This is the basic principle

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underlying [Shi11, Shi12] on the global Langlands correspondence and the Kottwitz conjecture. For another application, a description of ℓ-adic cohomology of Igusa varieties was one of the main ingredients in [CS17, CS] to prove vanishing of cohomology of certain Shimura varieties with ℓ-torsion coefficients, which in turn supplied a critical input for a recent breakthrough on the Ramanujan and Sato–Tate conjecture for cuspidal automorphic representations of GL_2 of "weight 2" over CM fields [ACC$^+$].

Thus an important long-term goal is to compute the ℓ-adic cohomology of Igusa varieties with a natural group action. A major first step would be a Langlands–Kottwitz-style trace formula for Igusa varieties, which has been obtained for Shimura varieties of Hodge type at hyperspecial level in [Shi09, Shi10, MC] building upon [HT01, Ch. 5] in analogy with [LR87, Kot92b, KSZ]. One wishes to turn that into an expression of the cohomology in terms of automorphic forms, but this requires a solution of various complicated problems; some should be tractable but others are out of reach in general, most notably an endoscopic classification and Arthur's multiplicity formula for the relevant groups.

The main objective of this paper is twofold. Firstly, we describe the H^0 of Igusa varieties via one-dimensional automorphic representations over non-basic Newton strata of Hodge-type Shimura varieties at hyperspecial level.[1] This mirrors the well-known fact that H^0 of complex Shimura varieties is governed by one-dimensional automorphic representations. Secondly, to achieve this, we develop a method and obtain various technical results with a view towards the entire cohomology of Igusa varieties (as an alternating sum over all degrees). Our method, partly inspired by Laumon [Lau05] and also by Flicker–Kazhdan [FK88], should prove useful for studying ℓ-adic cohomology of Shimura varieties as well.

Our result on H^0 not only sets a milestone in its own right, but also reveals deep geometric information. Namely, our theorem readily implies the discrete Hecke orbit conjecture for Shimura varieties and the irreducibility of Igusa varieties in the same generality as above. (The irreducibility means that Igusa varieties are no more reducible than the underlying Shimura varieties in some precise sense.) Our work provides a completely new approach and perspective to these two problems by means of automorphic forms and representation theory.

One of our main novelties consists in a careful asymptotic argument via the trace formula to single out H^0 (or compactly supported cohomology in the top degree) without reliance on any classification, a key to obtain an unconditional result. Since the "variable" for asymptotics is encoded in a test function at p, a good amount of local harmonic analysis naturally enters the picture. Another feature of our approach is to allow induction on the semisimple rank of the group; this would make little sense in a purely geometric argument (as endoscopy is hard to realize in the geometry of Shimura varieties).

Roughly speaking, cohomology of Igusa varieties is closely related to that of Shimura varieties via the Jacquet module operation at p, relative to a *proper* parabolic subgroup in the *non-basic* case. To show that only one-dimensional automorphic representations contribute to H^0 of Igusa varieties, the key representation-theoretic input is an estimate for the central action on Jacquet modules due to Casselman and Howe–Moore. Though there is no direct link, it would be interesting to remark that a similar situation occurs in the context of beyond endoscopy (e.g., [FLN10, §5]), where the leading term in asymptotics is accounted for by the "most non-tempered" (namely one-dimensional) representations.

1.1. **The main theorem.** Let (G, X) be a Shimura datum of Hodge type with reflex field $E \subset \mathbb{C}$. Assume that the reductive group G over $\mathbb{Q}$ admits a reductive model over $\mathbb{Z}_p$, and take $K_p := G(\mathbb{Z}_p)$. (Namely G is unramified at p, and K_p is a hyperspecial subgroup.) For simplicity, the adjoint group

[1] In the basic case, Igusa varieties are 0-dimensional, and their H^0 is expressed as the space of algebraic automorphic forms on an inner form of G, through a description of points with suitable group actions as in [MC]. This idea goes back to Serre [Ser96] for modular curves, and Fargues [Far04, Ch. 5] in the PEL case.

of G is assumed to be simple over $\mathbb{Q}$ throughout the introduction. (Otherwise the notion of basic elements needs to be modified. See Definition 5.3.2 below.) We do not assume $p > 2$ as the case $p = 2$ will be included in [KSZ, MC].

Fix field maps $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, $\overline{\mathbb{Q}}_p \simeq \mathbb{C}$, and $\overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$ (which will be mostly implicit). The resulting embedding $E \hookrightarrow \overline{\mathbb{Q}}_p$ induces a place $\mathfrak{p}$ of E above p. Let $k(\mathfrak{p})$ denote the residue field of E at $\mathfrak{p}$, which embeds into the residue field $\overline{\mathbb{F}}_p$ of $\overline{\mathbb{Q}}_p$. Thereby we identify $\overline{k(\mathfrak{p})} \simeq \overline{\mathbb{F}}_p$. Let $\mathscr{S}_{K_p}$ denote the integral canonical model over $\mathcal{O}_{E_\mathfrak{p}}$ with a $G(\mathbb{A}^{\infty,p})$-action. In the main text, we work with Shimura and Igusa varieties at finite level and then take limits over open compact subgroups $K^p \subset G(\mathbb{A}^{\infty,p})$. (For instance, the fixed-point formula should be applied at finite level.) However, we will ignore this point and pretend that we are always at infinite level away from p to simplify exposition.

A fixed symplectic embedding of (G, X) into a Siegel Shimura datum yields a $G(\mathbb{A}^{\infty,p})$-equivariant map from $\mathscr{S}_{K_p}$ to a suitable Siegel moduli scheme (over $\mathcal{O}_{E_\mathfrak{p}}$ after a base change from $\mathbb{Z}_p$). Via pullback, we obtain a universal abelian scheme $\mathscr{A}$ over $\mathscr{S}_{K_p}$, which can be equipped with a family of étale and crystalline tensors over geometric points $\overline{x} \to \mathscr{S}_{K_p,k(\mathfrak{p})}$. This assigns to $\overline{x}$ the p-divisible group $\mathscr{A}_{\overline{x}}[p^\infty]$ (with G-structure) up to isomorphism.

Let $\mu_p : \mathbb{G}_m \to G_{\overline{\mathbb{Q}}_p}$ denote the cocharacter arising from (G, X) (and via $\overline{\mathbb{Q}}_p \simeq \mathbb{C}$). This cuts out a finite subset $B(G_{\mathbb{Q}_p}, \mu_p^{-1})$ in the Kottwitz's set $B(G_{\mathbb{Q}_p})$ of G-isocrystals. Fix $b \in G(\breve{\mathbb{Q}}_p)$ whose image $[b]$ lies in $B(G_{\mathbb{Q}_p}, \mu_p^{-1})$. (Without the latter condition, C_b and N_b below are known to be empty.) Then b gives rise to an isomorphism class of p-divisible group Σ_b with G-structure over $\overline{\mathbb{F}}_p$ via the symplectic embedding above and the Dieudonné theory. We also obtain a Newton cocharacter ν_b from b. We may arrange that ν_b is dominant with respect to a suitable Borel subgroup of $G_{\mathbb{Q}_p}$ defined over $\mathbb{Q}_p$. For simplicity, assume that Σ_b is defined over $k(\mathfrak{p})$ and that $[k(\mathfrak{p}) : \mathbb{F}_p]\nu_b$ is a cocharacter, not just a fractional cocharacter. (In practice, these assumptions are unnecessary since it is sufficient to have a finite extension of $k(\mathfrak{p})$ in the last sentence.)

Write ρ for the half sum of all B-positive roots. Write J_b for the $\mathbb{Q}_p$-group of self-quasi-isogenies of Σ_b (preserving G-structure) over $\overline{\mathbb{F}}_p$, and J_b^{int} for the subgroup of $J_b(\mathbb{Q}_p)$ consisting of automorphisms. Then J_b^{int} is an open compact subgroup of J_b. As a general fact, J_b is an inner form of a $\mathbb{Q}_p$-rational Levi subgroup M_b of $G_{\mathbb{Q}_p}$. We say that b is *basic* if ν_b is a central in $G_{\mathbb{Q}_p}$, or equivalently if $M_b = G_{\mathbb{Q}_p}$ (namely if J_b is an inner form of $G_{\mathbb{Q}_p}$).

The central leaf C_b (resp. Newton stratum N_b) is the locus of $x \in \mathscr{S}_{K_p}$ on which the geometric fibers of $\mathscr{A}_{\overline{x}}[p^\infty]$ are isomorphic (resp. isogenous) to Σ_b. By construction, C_b and N_b are stable under the $G(\mathbb{A}^{\infty,p})$-action on $\mathscr{S}_{K_p}$. We also define the Igusa variety $\mathfrak{Ig}_b$ over $\mathscr{S}_{K_p,k(\mathfrak{p})}$ to be the parameter space of isomorphisms between Σ_b and $\mathscr{A}[p^\infty]$. The obvious action of J_b^{int} on $\mathfrak{Ig}_b$ naturally extends to a $J_b(\mathbb{Q}_p)$-action. Below are some basic facts (§5.3, §6.1, and §6.2). Put $q := \#k(\mathfrak{p})$.

Fact 1. C_b is (formally) smooth over $k(\mathfrak{p})$ and closed in N_b,
Fact 2. C_b is equidimensional of dimension $\langle 2\rho, \nu_b \rangle$,
Fact 3. $\mathfrak{Ig}_b$ is a J_b^{int}-torsor over the perfection of C_b,
Fact 4. the q-th power Frobenius on $\mathfrak{Ig}_b$ coincides with the action of $\nu_b(q) \in Z_{J_b}(\mathbb{Q}_p)$.

In particular $\dim \mathfrak{Ig}_b = \langle 2\rho, \nu_b \rangle$, and every connected component of $\mathfrak{Ig}_{b,\overline{\mathbb{F}}_p}$ (resp. $C_{b,\overline{\mathbb{F}}_p}$) is irreducible. (Fact 4 is actually considered only in the completely slope divisible case, cf. part (2) of Lemma 6.2.1.)

Our main theorem describes the connected components (= irreducible components) of Igusa varieties over $\overline{\mathbb{F}}_p$ with a natural group action. In the statement, writing $J_b(\mathbb{Q}_p)^{\mathrm{ab}}$ and $G(\mathbb{Q}_p)^{\mathrm{ab}}$ for the abelianizations as topological groups, the one-dimensional representation π_p of $G(\mathbb{Q}_p)$ is naturally viewed as a one-dimensional representation of $J_b(\mathbb{Q}_p)$ via the canonical map $J_b(\mathbb{Q}_p) \twoheadrightarrow J_b(\mathbb{Q}_p)^{\mathrm{ab}} \to G(\mathbb{Q}_p)^{\mathrm{ab}}$, cf. §6.1 below.

Theorem A (Theorem 6.1.3)**.** *Assume that b is non-basic with $[b] \in B(G_{\mathbb{Q}_p}, \mu_p^{-1})$. Then there is a $G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$-module isomorphism*

$$H^0(\mathfrak{Ig}_b, \overline{\mathbb{Q}}_\ell) \simeq \bigoplus_{\pi} \pi^{\infty,p} \otimes \pi_p,$$

where the sum runs over one-dimensional automorphic representations π of $G(\mathbb{A})$ whose real component is trivial on the preimage of the neutral component $G^{\mathrm{ad}}(\mathbb{R})^0$ in $G(\mathbb{R})$.

Before we sketch the idea of proof, let us discuss two geometric applications in some detail.

1.2. **Application to the discrete Hecke orbit conjecture.** Oort introduced a central leaf in the special fiber of PEL-type Shimura varieties modulo a prime p as a locus on which the universal p-divisible group is fiberwise constant. In 1995 [Oor95, §15] (also see [EMO01, Problem 18]), he proposed the Hecke Orbit (HO) conjecture that the prime-to-p Hecke orbit of a point in a central leaf should be Zariski dense in the leaf, if the point is outside the basic Newton stratum. He drew analogy with the André–Oort conjecture for a Shimura variety in characteristic zero, which asserts that the irreducible components of the Zariski closure of a set of special points are special subvarieties. (See [Tsi18] and references therein for recent results on the André–Oort conjecture.) A common feature is that a set of points with an extraordinary structure (being a prime-to-p Hecke orbit or special points) is Zariski dense in a distinguished class of subvarieties. We can also compare the HO conjecture with the Hecke equidistribution theorems for locally symmetric spaces in characteristic zero [COU01, EO06], stating roughly that the Hecke orbit of an arbitrary point is equidistributed in the locally symmetric space a suitable sense. (In particular the Hecke orbit is dense in the entire space even for the analytic topology, to be contrasted with the phenomenon in characteristic p.) It is also worth noting works to investigate Hecke orbits for the p-adic topology [GK, HMRL].

Chai and Oort verified the HO conjecture for Siegel modular varieties, in particular the irreducibility of leaves [Cha05, CO11]. The conjecture is also known for Hilbert modular varieties and due to Chai and Yu, cf. [YCO20]. The conjecture has seen several new results in very recent years. Shankar proved the conjecture for Deligne's "strange models" (in the sense of [Del71, §6]) in an unpublished preprint. Zhou [Zho] settled the HO conjecture in the ordinary locus of some quaternionic Shimura varieties along the way to realize a geometric level raising between Hilbert modular forms. Maulik–Shankar–Tang [MST] proved the HO conjecture in the ordinary locus of GSpin Shimura varieties. Xiao [Xia] proved partial results on the HO conjecture in the case of PEL type A and C.

Chai [Cha05, Cha06] proposed the strategy to divide the HO conjecture into two parts, that is, the discrete HO conjecture ($\mathrm{HO}_{\mathrm{disc}}$) and the continuous HO conjecture ($\mathrm{HO}_{\mathrm{cont}}$), corresponding to global and local geometry, respectively. In a nutshell ($\mathrm{HO}_{\mathrm{disc}}$) asserts that the prime-to-p Hecke orbit of a point x in a non-basic stratum meets every irreducible component of the central leaf through x. Then ($\mathrm{HO}_{\mathrm{cont}}$) is designed to tell us that the closure of the prime-to-p Hecke orbit has the same dimension as the central leaf in each irreducible component, so that ($\mathrm{HO}_{\mathrm{disc}}$) and ($\mathrm{HO}_{\mathrm{cont}}$) together imply the HO conjecture. To our knowledge, apart from some special cases mentioned above, no theorems of general type have been obtained on either ($\mathrm{HO}_{\mathrm{disc}}$) or ($\mathrm{HO}_{\mathrm{cont}}$) until the results by us and independently by van Hoften and Xiao (§1.5 below).

We deduce the following from Theorem A. (See §8.1 for details.)

Theorem B. *Conjecture* ($\mathrm{HO}_{\mathrm{disc}}$) *holds for Hodge-type Shimura varieties with hyperspecial level at p.*

Let $C_b \subset \mathrm{Sh}_{K_p}$ be a non-basic stratum. It is not difficult to observe that the theorem boils down to showing that the $G(\mathbb{A}^{\infty,p})$-equivariant immersion $C_b \to \mathrm{Sh}_{K_p}$ induces a bijection on the sets of geometric connected components. (Informally speaking, C_b is no more irreducible than Sh_{K_p}.) Since $G(\mathbb{A}^{\infty,p})$ acts transitively on $\pi_0(\mathrm{Sh}_{K_p,\overline{\mathbb{F}}_p})$, it is enough to show that $H^0(C_{b,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell) \simeq H^0(\mathrm{Sh}_{K_p,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell)$

as $G(\mathbb{A}^{\infty,p})$-modules (abstractly, not necessarily via the map induced by $C_b \subset \mathrm{Sh}_{K_p}$). It is standard to compute $H^0(\mathrm{Sh}_{K_p,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell)$ from $H^0(\mathrm{Sh}_{K_p,\mathbb{C}}, \overline{\mathbb{Q}}_\ell)$ as an application of compactification for integral models [MP19], so the problem is to compute $H^0(C_{b,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell)$. In light of Fact 3, this is done by taking the J_b^{int}-invariants of the formula in Theorem A.

1.3. **Application to irreducibility of Igusa towers and a generalization.** In Hida theory of p-adic automorphic forms, an important role is played by Igusa varieties over the *ordinary* Newton stratum, namely when the underlying p-divisible group is ordinary. In this case, Igusa varieties (and their natural extension to p-adic formal schemes) are usually referred to as Igusa towers. Recently Eischen and Mantovan [EM] developed Hida theory in the more general μ-ordinary PEL-type setup, where Howe [How20] (and its sequel) also shed new light on the role of Igusa varieties (à la Caraiani–Scholze). Igusa towers are also featured in Andreatta–Iovita–Pilloni's work [AIP18,AIP16] on overconvergent automorphic forms.

A key property of Igusa towers is irreducibility, roughly meaning that they are as irreducible as possible under the given constraints. The irreducibility has an application to the q-expansion principle for p-adic automorphic forms, which is a basic ingredient for the construction of p-adic L-functions. See p.96 in [Hid04] for the relevant remark, and also refer to Thm. 3.3 (Igusa), 4.21 (Ribet), 6.4.3 (Faltings–Chai), and Cor. 8.17 (Hida) therein for the known cases (elliptic modular, Hilbert, Siegel, and PEL type A/C cases, respectively, all over the ordinary stratum) and further references. Irreducibility in the μ-ordinary case of PEL type A was proven in [EM]. Such a result was obtained for Igusa varieties of a specific PEL type A by Boyer [Boy] without assuming μ-ordinariness.

There are various methods to show the irreducibility as explained in [Cha08] and the introduction of [Hid11], for instance by using the automorphism group of the function fields of Shimura varieties in characteristic 0 or by showing that the monodromy of the family of abelian varieties is large. As an application of Theorem A, we obtain an entirely different representation-theoretic proof and also a natural generalization from the μ-ordinary case to the general non-basic case (and from the PEL case to the Hodge-type case). In the non-μ-ordinary case, Igusa varieties do not live over the entire Newton stratum (that is, the Newton stratum is not a central leaf) but our method is insensitive to such a distinction.

Write $J_b(\mathbb{Q}_p)'$ for the kernel of the map $J_b(\mathbb{Q}_p) \to G(\mathbb{Q}_p)^{\mathrm{ab}}$ in §1.1. Our result is as follows.

Theorem C. *Assume that b is non-basic. The stabilizer subgroup in $J_b(\mathbb{Q}_p)$ of each connected component of $\mathfrak{Ig}_b$ is equal to $J_b(\mathbb{Q}_p)'$. The preimage of each connected component of C_b along $\mathfrak{Ig}_b \to C_b$ is the disjoint union of $J_b^{\mathrm{int}} \cap J_b(\mathbb{Q}_p)'$-torsors.*

Roughly speaking, the stabilizer subgroup cannot be larger than $J_b(\mathbb{Q}_p)'$, and this should be thought of as saying that Igusa varieties should be at least as reducible as Shimura varieties. The point of the theorem is that, conversely, the stabilizer is as large as possible under the given constraint. The proof is almost immediate from the $J_b(\mathbb{Q}_p)$-action on H^0 described in Theorem A.

The above theorem is easily translated into an irreducibility statement about Igusa varieties; identifying the stabilizer is the key here. The details are worked out in §8.2 below.

1.4. **Some details on the proof of Theorem A.** Changing Σ_b by a quasi-isogeny, as this does not affect $\mathfrak{Ig}_b$ up to isomorphism, we may assume that Σ_b is completely slope divisible and defined over a finite field. Then the main advantage is that $\mathfrak{Ig}_b$ can be written, up to taking perfection, as the projective limit of smooth varieties of finite type defined over $\mathbb{F}_{p^r}$ for some sufficiently divisible $r \in \mathbb{Z}_{>0}$. (The latter is denoted by Ig_b in the main text, but we do not distinguish $\mathfrak{Ig}_b$ from Ig_b in the introduction.) This allows us to apply a trace formula technique to compute the cohomology of $\mathfrak{Ig}_b$, noting that the trace of a Hecke algebra element can be computed at a finite level. Via

Poincaré duality, Theorem A may be rephrased as a description of the top degree compact-support cohomology $H_c^{\langle 4\rho,\nu_b\rangle}(\mathfrak{Ig}_b,\overline{\mathbb{Q}}_\ell)$, which we may hope to access by the Lang–Weil estimate.

Adapting the Langlands–Kottwitz method to Igusa varieties, as worked out in [Shi09] and [MC], one obtains a formula of the form

$$\mathrm{Tr}\,(\phi^{\infty,p}\phi_p \times \mathrm{Frob}_{p^r}^j | H_c(\mathfrak{Ig}_b,\overline{\mathbb{Q}}_\ell)) = (\text{geometric expansion}), \qquad j \in \mathbb{Z}_{\gg 1},$$

where $\phi^{\infty,p}\phi_p \in \mathcal{H}(G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p))$ and $j \in \mathbb{Z}_{\gg 1}$. In fact, one can show that the Frob_{p^r}-action on $\mathfrak{Ig}_b$ is represented by the action of a central element of $J_b(\mathbb{Q}_p)$. Thereby $\phi_p \times \mathrm{Frob}_{p^r}^j$ in (1.4.1) may be replaced with a translate $\phi_p^{(j)} \in \mathcal{H}(J_b(\mathbb{Q}_p))$ of ϕ_p by a central element. The right hand side of (1.4.1) is a linear combination of orbital integrals of $\phi^{\infty,p}\phi_p^{(j)}$ on $G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$ over a certain set of conjugacy classes. After stabilization (adapting [Shi10] to the Hodge-type case), we have a formula of the form

$$\mathrm{Tr}\,(\phi^{\infty,p}\phi_p^{(j)} | H_c(\mathfrak{Ig}_b,\overline{\mathbb{Q}}_\ell)) = \sum_{\mathfrak{e}} (\text{constant}) \cdot ST_{\mathrm{ell}}^{\mathfrak{e}}(f^{\mathfrak{e},p} f_p^{\mathfrak{e},(j)}), \qquad j \in \mathbb{Z}_{\gg 1}, \tag{1.4.1}$$

where the sum runs over endoscopic data $\mathfrak{e}$ for G (§2.6), and $f^{\mathfrak{e},p} f_p^{\mathfrak{e},(j)}$ is a suitable function on the corresponding endoscopic group $G^{\mathfrak{e}}$. By $ST_{\mathrm{ell}}^{\mathfrak{e}}$ we mean the stable elliptic terms in the trace formula for $G^{\mathfrak{e}}$. The most nontrivial point in the stabilization is the "transfer" at p. Indeed, as $G^{\mathfrak{e}}$ is not an endoscopic group of J_b, this requires a special construction as detailed in §3.

Ideally we would turn the right hand side of (1.4.1) into a spectral expansion and determine not only $H_c^{\langle 4\rho,\nu_b\rangle}(\mathfrak{Ig}_b,\overline{\mathbb{Q}}_\ell)$ but $H_c(\mathfrak{Ig}_b,\overline{\mathbb{Q}}_\ell)$ in the Grothendieck group of $G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$-representations. This is the long-term goal stated earlier. On the analogous problem for Shimura varieties, a road map has been laid out in [Kot90], which can be mimicked for Igusa varieties to some extent. However there are serious obstacles: (1) An endoscopic classification for most reductive groups is out of reach; exactly the same issue occurs for Shimura varieties as well. (2) The geometric side (stable elliptic terms) is very difficult to compare with the spectral side. One could imagine making the geometric side more tractable by passing from H_c to intersection cohomology, following the strategy for Shimura varieties to "fill in" the stable non-elliptic terms, but there is no available theory of compactification for Igusa varieties to allow it. (Franke's formula for H_c of locally symmetric spaces [Fra98] suggests that one should expect a similarly complicated answer for H_c of Igusa varieties.)

Our goal is to extract spectral information on $H_c^{\langle 4\rho,\nu_b\rangle}(\mathfrak{Ig}_b,\overline{\mathbb{Q}}_\ell)$ from the leading terms in (1.4.1) in the variable j via the Lang–Weil estimate. Thus we can get away with less by proving equalities up to error terms of lower order. To bypass (1) and (2), a key is to show that (stable) non-elliptic terms as well as endoscopic (a.k.a. unstable) terms have slower growth in j than the (stable) elliptic terms. This is the technical heart of our paper taking up §4. Let us provide more details.

The basic strategy is an induction on the semisimple rank, based on our observation that some key property of the function $f_p^{\mathfrak{e},(j)}$ is replicated after taking an endoscopic transfer or a constant term. (For instance, we need to pass along the Newton cocharacter through the inductive steps.) So we want to prove a bound on the trace formula for a quasi-split group over $\mathbb{Q}$, with a test function $f^p f_p^{(j)}$ satisfying such a property. The desired bound partly comes from a root-theoretic computation, involving a curious interaction between p and ∞ such as "evaluating" the Newton cocharacter (coming from p) at the infinite place (Lemma 4.1.1). The most interesting part in this part of the argument is

a spectral expansion of T_{ell}, the elliptic part of the trace formula.

(In (1.4.1) we can replace $ST_{\mathrm{ell}}^{\mathfrak{e}}$ with T_{ell} once the difference is known to be of lower order.) In our setup, where the archimedean test function is stable cuspidal, we have Arthur's simple trace formula [Art89] of the following shape:

$$T_{\mathrm{disc}}(f^p f_p^{(j)}) = T_{\mathrm{ell}}(f^p f_p^{(j)}) + (\text{geometric terms on proper Levi subgroups}). \tag{1.4.2}$$

The proper Levi terms at finite places look similar to the elliptic part of the trace formula for proper Levi subgroups, but a complicated behavior is seen at the infinite place as this comes from stable discrete series characters along non-elliptic maximal tori of the ambient group. On each open Weyl Chamber, we have a formula as a finite linear combination of finite dimensional characters of the Levi subgroup, but this quickly spirals out of control in the induction. Adapting an idea of Laumon [Lau97] from the non-invariant setup to the invariant setup, we overcome the difficulty by imposing a regularity condition on the test function at an auxiliary prime q ($\neq p$) and show that the $\mathbb{Q}$-conjugacy classes with nonzero contributions land in a single Weyl Chamber. Then a finite dimensional character of a Levi subgroup is itself a stable discrete series character of the same Levi subgroup along elliptic maximal tori, and the inductive argument can continue. (The auxiliary hypothesis at q is harmless in that no information is lost, as shown in §7.6.) This technique should prove useful for further understanding compactly supported cohomology of Igusa varieties and Shimura varieties alike.

Returning to our problem, the above argument turns (1.4.1) into

$$\mathrm{Tr}\,(\phi^{\infty,p}\phi_p^{(j)}|H_c(\mathfrak{Ig}_b, \overline{\mathbb{Q}}_\ell)) = \sum_{\pi^*} m(\pi^*)\mathrm{Tr}\,(f^{*,p}f_p^{*,(j)}) + (\text{error terms}),$$

where $f^{*,p}f_p^{*,(j)}$ is the test function on the quasi-split inner form G^* of G, and the sum runs over discrete automorphic representations of $G^*(\mathbb{A})$. At this point, we apply a trace identity. Let $\phi_p^{*,(j)}$ denote a transfer of $\phi_p^{(j)}$ from J_b to its quasi-split inner form M_b. For each irreducible smooth representation π^* of $G^*(\mathbb{Q}_p)$, we have (Lemma 3.1.3)

$$\mathrm{Tr}\,\pi_p^*(f_p^{*,(j)}) = \mathrm{Tr}\,J(\pi_p^*)(\phi_p^{*,(j)}),$$

where J is the normalized Jacquet module relative to the parabolic subgroup determined by ν_b whose Levi component is M_b. Since b is non-basic, M_b is a proper Levi subgroup. Moreover the translation (j) is given by a central element satisfying a positivity condition with respect to ν_b. In these circumstances, we make a crucial use of an estimate available by Casselman and Howe–Moore (§2.1), showing that $J(\pi_p^*)(\phi_p^{*,(j)})$ has the highest growth if and only if $\dim \pi_p^* = 1$. A strong approximation argument (§2.5) promotes this to the condition that $\dim \pi^* = 1$, under a group-theoretic condition guaranteed in our setting. Moreover, it is not hard to transfer one-dimensional representations from $M_b(\mathbb{Q}_p)$ to $J_b(\mathbb{Q}_p)$ compatibly with the transfer of functions (§2.3). We complete the proof of Theorem A by putting this final piece of the puzzle.

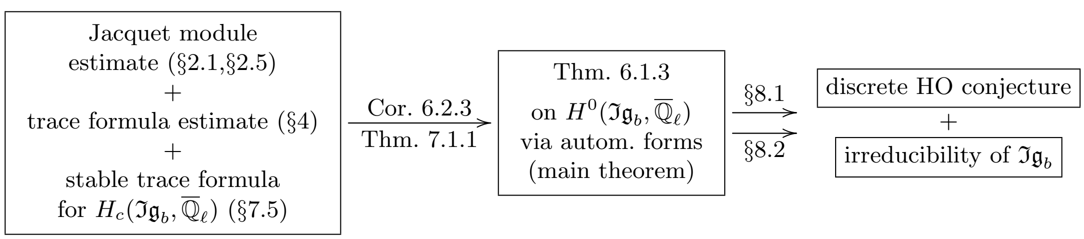
1.5. **Work of van Hoften and Xiao.** While this paper was being completed, Pol van Hoften announced [vH] as well as [vHX] with Luciena Xiao Xiao, which prove the discrete HO conjecture and the irreducibility of Igusa varieties. (Either this work or our work, combined with Xiao's earlier paper [Xia], implies the full HO conjecture in certain cases, cf. Corollary 5.4.6 below.) Their method is more geometric in nature and totally different from ours in that no use is made of automorphic forms. Further goals in their work and ours are disparate. For instance [vH] proves new results on stratification of Shimura varieties and the Langlands–Rapoport conjecture in the parahoric case, whereas our work is a stepping stone for computing the cohomology of Igusa varieties in all degrees (up to alternating sums). The two threads could have a future intersection though, as the Langlands–Rapoport conjecture in the parahoric case would be an important ingredient for deriving the analogue for Igusa varieties, extending Mack-Crane's thesis [MC] on the latter from the hyperspecial case to the parahoric case. This would in turn allow the current paper's method to extend to parahoric levels.

1.6. **A guide for the reader.** On a first reading, we suggest that all complexities with central characters and z-extensions should be skipped, assuming that all central character data are trivial and that z-extensions are unnecessary everywhere. In fact this should be the case in most examples. The central character datum is always trivial on the level of G appearing in the Hodge-type datum, but we allow it to be nontrivial mainly because we do not know whether $\mathcal{H}$ in the endoscopic datum (§2.6) can always be chosen to be an L-group. Another good idea is to start reading in §5, especially if one's main interests lie in geometry, referring to the earlier sections only as needed and taking the results there for granted.

Sections 2 and 3 consist of mostly background materials in local harmonic analysis and representation theory. Though we claim little originality, there may be some novelty in the way we organize and present them. Some statements would be of independent interest. Section 4 is perhaps the most technical as this is where the main trace formula estimates are obtained. As such, most readers may want to take the results in §4.2 on faith and proceed, returning to them as needed.

Sections 5 and 6 introduce the main geometric players, namely Shimura varieties, central leaves, and Igusa varieties. Except for §5.1, we are always in the Hodge-type case with hyperspecial level at p. Our main theorem on Igusa varieties are stated in §6.1. After reduction steps in §6.2–§7.1 and some recollection of the trace formula setup up to §7.5, the proof of the theorem is completed in §7.6. Lastly Section 8 is devoted to the main geometric applications on the discrete Hecke orbit conjecture and irreducibility of Igusa varieties.

The bare-bones structure of our argument is as follows.



1.7. **Notation.**

- The trivial character (of the group that is clear from the context) is denoted by $\mathbf{1}$.
- If T is a torus over a field k with algebraic closure $\overline{k}$, $X_*(T) := \mathrm{Hom}_{\overline{k}}(T, \mathbb{G}_m)$ and $X^*(T) := \mathrm{Hom}_{\overline{k}}(\mathbb{G}_m, T)$. When R is a $\mathbb{Z}$-algebra, we write $X_*(T)_R := X_*(T) \otimes_{\mathbb{Z}} R$ and $X^*(T)_R := X^*(T) \otimes_{\mathbb{Z}} R$.
- $\mathbb{D} := \varprojlim \mathbb{G}_m$ is the protorus (over an arbitrary base), where the transition maps are the n-th power maps.
- $\breve{\mathbb{Z}}_p := W(\overline{\mathbb{F}}_p)$, $\breve{\mathbb{Q}}_p := \mathrm{Frac}\, \breve{\mathbb{Z}}_p$, and $\sigma \in \mathrm{Aut}(\breve{\mathbb{Q}}_p)$ is the arithmetic Frobenius.
- $\mathscr{P}(S)$ is the power set of a set S.
- If H is an algebraic group over a field k, we write $H^0 \subset H$ for its neutral component.

Let G be a connected reductive group over a field k of characteristic 0.

- If k is a finite extension of k_0, then $\mathrm{Res}_{k/k_0} G$ denotes the restriction of scalars group.
- G_{der} is the derived subgroup, $\varrho : G_{\mathrm{sc}} \to G_{\mathrm{der}} \subset G$ the simply connected cover, Z_G the center (we also write $Z(G)$), $G^{\mathrm{ad}} := G/Z_G$ the adjoint group, and $G^{\mathrm{ab}} := G/G_{\mathrm{der}}$ the maximal commutative quotient. Write $A_G \subset Z_G$ for the maximal split subtorus over k.
- $G(k)_?$ is the set of semisimple (resp. regular semisimple, resp. strongly regular) elements in $G(k)$ for $? = \mathrm{ss}$ (resp. reg, resp. sr). We put $T(k)_? := T(k) \cap G(k)_?$ for $? \in \{\mathrm{reg}, \mathrm{sr}\}$.
- If k is a local field and G a reductive group over k, we then write $\mathcal{I}(G(k))$ and $\mathcal{S}(G(k))$ for the space of invariant and stable distributions on $G(k)$. (For more details, see § 2.2). By $\mathrm{Irr}(G(k))$ we mean the set of isomorphism classes of irreducible admissible representations of $G(k)$.

- When $k = \mathbb{Q}_p$, two elements $\delta, \delta' \in G(\breve{\mathbb{Q}}_p)$ are $(G(\breve{\mathbb{Q}}_p), \sigma)$-*conjugate* (resp. $(G(\breve{\mathbb{Z}}_p), \sigma)$-*conjugate*) if there exists a $g \in G(\breve{\mathbb{Q}}_p)$ (resp. $g \in G(\breve{\mathbb{Z}}_p)$) such that $\delta' = \sigma(g)\delta g^{-1}$.

Let T (resp. S) be a maximal torus (resp. maximal split torus) of G over k with $T \supset S$. Let M_0 be a minimal k-rational Levi subgroup containing T.

- $\Phi(T, G)$ is the set of absolute roots, $\Phi(S, G) = \Phi_k(S, G)$ the set of k-rational roots.
- $\overline{\Omega}^G = \Omega(T, G)$ for the Weyl group over $\overline{k}$, and $\Omega_k^G = \Omega(S, G)$ for the k-rational Weyl group. We often omit k from $\Phi_k(S, G)$ and Ω_k^G when it is clear from the context.
- $\mathcal{L}(G)$ or $\mathcal{L}_k(G)$ is the set of all k-rational Levi subgroups of G containing M_0. Write $\mathcal{L}^<(G) := \mathcal{L}(G)\backslash\{G\}$.

Lemma 1.7.1. *If G_{der} is simply connected then every k-rational Levi subgroup of G has simply connected derived subgroup.*

Proof. This can be checked after base change to $\overline{k}$, so assume $k = \overline{k}$. For every maximal torus $T \subset G$, the cocharacter lattice $X_*(T)$ modulo the coroot lattice is torsion free by hypothesis. Thus $X_*(T)$ modulo the lattice generated by an arbitrary subset of simple coroots is torsion free, implying that every Levi subgroup of G has simply connected derived subgroup. □

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2. Preliminaries in representation theory and endoscopy

2.1. **Estimates for Jacquet modules of unitary representations.** Here we recall some facts from work of Howe–Moore [HM79] and Casselman [Cas95] in order to bound the absolute value of central characters in the Jacquet modules of unitary representations of p-adic reductive groups.

We consider the following setup and notation.

- Let F be a non-archimedean local field of characteristic 0. We write val_F, $\mathcal{O}_F$, k, q, ϖ_F respectively for the normalized valuation of F, the ring of integers of F, the residue field of F, the cardinality of k and an uniformizer of F so that $\mathrm{val}_F(\varpi_F) = 1$,
- G is a connected reductive group over F with center $Z = Z_G$,
- $\mathrm{Rep}(G)$ is the category of smooth representations of $G(F)$,
- $P = MN$ is a Levi decomposition of an F-rational proper parabolic subgroup of G,
- A_M is the maximal F-split torus in the center of M,
- Δ is the set of roots of A_M in N,
- $A_P^- := \{x \in A_M(F) : |\alpha(x)| \le 1,\ \forall \alpha \in \Delta\}$,
- $A_P^{--} := \{x \in A_M(F) : |\alpha(x)| < 1,\ \forall \alpha \in \Delta\}$,
- $\delta_P : M(F) \to \mathbb{R}_{>0}^\times$ is the modulus character given by $\delta_P(m) := |\det(\mathrm{Ad}(m)|\mathrm{Lie}\, N(F))|$.
- $J_P : \mathrm{Rep}(G) \to \mathrm{Rep}(M)$ is the normalized Jacquet module functor, so $J_P(\pi) = \pi_N \otimes \delta_P^{-1/2}$ with π_N denoting the $N(F)$-coinvariants of π,
- $I_P^G : \mathrm{Rep}(M) \to \mathrm{Rep}(G)$ is the normalized parabolic induction functor, sending π_M to the smooth induction of $\pi_M \otimes \delta_P^{1/2}$ from $P(F)$ to $G(F)$.
- When $R \in \mathrm{Rep}(M)$ has finite length, write $\mathrm{Exp}(R)$ for the set of $A_M(F)$-characters appearing as central characters of irreducible subquotients of R.

Lemma 2.1.1. *If G is simply connected, F-simple, and F-isotropic, then every normal subgroup of $G(F)$ is either $G(F)$ itself or contained in $Z(F)$.*

Proof. A normal subgroup N of $G(F)$ not contained in $Z(F)$ is open of finite index in $G(F)$ by [PR94, Prop. 3.17] since G is F-simple. Since $G(F)$ is F-isotropic and simply connected, $G(F)$ is generated by the F-points of the unipotent radicals of F-rational parabolic subgroups. By Tit's theorem proven in [Pra82], every open proper subgroup of $G(F)$ is compact. On the other hand, N is easily seen to be non-compact by considering the adjoint action of a maximal F-split torus on a root subgroup.[2] Therefore $N = G(F)$. □

Proposition 2.1.2 (Howe–Moore)**.** *Assume that $G_{\rm sc}$ is F-simple. Let π be an infinite dimensional irreducible unitary representation of $G(F)$. Then there exists an integer $2 \le k < \infty$ such that every matrix coefficient of π belongs to $L^k(G(F)/Z(F))$.*

Proof. This follows from the explanation on pp.74–75 of [HM79] below Theorem 6.1, once we verify the following claim: if $\pi(g)$ is a scalar operator for $g \in G(F)$ then $g \in Z(F)$. Taking a z-extension of G, we reduce to the case when $G_{\rm der}$ is simply connected. Pulling back π via the multiplication map $Z(F) \times G_{\rm der}(F) \to G(F)$ and passing to one of the finitely many constituents (cf. [Xu16, Lem. 6.2]) which is infinite-dimensional, we may assume that G is itself F-simple and simply connected. Now Z' be the group of $g \in G(F)$ such that $\pi(g)$ is a scalar. Then Z' is a normal subgroup of $G(F)$. If $Z' = G(F)$ then $\dim \pi = 1$, contradicting the initial assumption. Therefore $Z' \subset Z(F)$ by Lemma 2.1.1, proving the claim. □

Proposition 2.1.3 (Casselman)**.** *Let π be an irreducible unitary representation of $G(F)$. For every $\omega \in \mathrm{Exp}(\pi_N)$ and every $a \in A_P^-$, we have the inequality*

$$|\omega(a)| \le 1. \tag{2.1.1}$$

Now suppose that $G_{\rm sc}$ is F-simple and that $a \in A_P^{--}$. Then the equality holds if and only if π is finite dimensional.

Proof. This follows from [Cas95, §4.4] (where $p < \infty$ is assumed), with the obvious extension to cover the case $p = \infty$. Let us explain more details, freely using the notation and definition of that paper.

The analogue of [Cas95, Lem. 4.4.1] for $p = \infty$ is the following. Let $\mathcal{F} : \mathbb{Z}^n \to \mathbb{C}$ be a $\mathbb{Z}^n$-finite function.[3] Then $\sup_{x \in \mathbb{Z}^n} \mathcal{F}(x) < \infty$ if and only if $|\chi(x)| \le 1$ for all nonzero $x \in \mathbb{N}^n$ and χ associated with $\mathcal{F}$. The proof is elementary. Similarly [Cas95, Lem. 4.4.3, Prop. 4.4.4] have the analogues for $p = \infty$, with "bounded" in place of "summable" and "$|\chi(x)| \le 1$" in place of "$|\chi(x)| < 1$". When $p = \infty$, the conclusion of [Cas95, Cor. 4.4.5] should read "$|\mathcal{F}|$ is bounded if and only if for every character χ associated to Φ and $a \in A_\Theta^-$, $|\chi(a)| \le 1$." (It is also true if "$a \in A_\Theta^-$" is replaced with "$a \in A_\Theta^- \backslash A_\emptyset(\mathcal{O}) A_\Delta$".) Let us refer to the $p = \infty$ analogue of [Cas95, Cor. 4.4.5] as Cor. 4.4.5$^\infty$.

Now we prove (2.5.2) by imitating the proof for the "only if" direction of [Cas95, Thm. 4.4.6]. Consider K, Γ as in that proof as well as the decomposition $G = \coprod_{i,j \in I} K \gamma_i A_\emptyset^- \gamma_j K$, where $\{\gamma_i\}_{i \in I}$ is a set of representatives for $\Gamma / K A_\Delta$. Let $v \in V$, $\tilde{v} \in \tilde{V}$ be given, and consider the matrix coefficient $c_{v,\tilde{v}}(g) = \langle \pi(g) v, \tilde{v} \rangle$ of π. We apply Cor. 4.4.5$^\infty$ to $\mathcal{F} = c_{v,\tilde{v}}$. Then every $\omega \in \mathrm{Exp}(\pi_N)$ is associated to the restriction Φ of $\mathcal{F}$ to ${}_\Theta A_\emptyset^-(\epsilon)$ with $0 < \epsilon \le 1$, and Cor. 4.4.5$^\infty$ tells us that $|\omega(a)| \le 1$ for every $a \in A_\Theta^-$. (His A_Θ^- corresponds to our A_P^-.)

It remains to check the last assertion of the theorem. Suppose that $\dim \pi = \infty$. Let $K, \Gamma, v, \tilde{v}, c_{v,\tilde{v}}$ be as above. By Proposition 2.1.2, $|c_{v,\tilde{v}}|^k$ is integrable modulo center for some $2 \le k < \infty$. Applying [Cas95, Cor. 4.4.5] in the same way as above to $p = k$ and $a \in A_P^{--}$, we obtain that $|\omega(a) \delta_P^{-1/k}(a)| < 1$. Therefore $|\omega(a)| < 1$. Now suppose that $\dim \pi < \infty$. Then $\ker \pi$ is an open subgroup of $G(F)$. As the open subgroup $N(F) \cap \ker \pi$ of the unipotent subgroup $N(F)$ acts trivially on π, we see that $N(F)$ itself acts trivially on π. (Use conjugation by $A_M(F)$.) Therefore $\mathrm{Exp}(\pi_N)$ consists of the central character ω of π (restricted to $M(F)$) only, which is unitary. In particular $|\omega(a)| = 1$ for all $a \in A_P^{--}$. □

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[2]For instance, see the proof of Proposition 3.9 in `http://virtualmath1.stanford.edu/~conrad/JLseminar/Notes/L2.pdf` for details.

[3]The roman F is used in [Cas95]. We change it to $\mathcal{F}$ to avoid conflict with the field F.

Remark 2.1.4. Proposition 2.1.3 is sharp in general (though some strengthening is available under a hypothesis, e.g., see [Oh02]). For example, consider $G = \mathrm{GL}_2(F)$ with P (resp. N) consisting of upper triangular (resp. upper triangular unipotent) matrices. The complementary series representations $\pi_\epsilon = I_P^G(|\cdot|^\epsilon, |\cdot|^{-\epsilon})$ with $\epsilon \in \mathbb{R}$ with $0 < \epsilon < 1/2$ are irreducible and unitary. We have

$$\begin{aligned}(\pi_\epsilon)_N = J_N(\pi_\epsilon) \otimes \delta_P^{1/2} &= \delta_P^{1/2} \otimes \big((|\cdot|^\epsilon, |\cdot|^{-\epsilon}) \oplus (|\cdot|^{-\epsilon}, |\cdot|^\epsilon)\big)\\ &= (|\cdot|^{\epsilon+\frac{1}{2}}, |\cdot|^{-\epsilon-\frac{1}{2}}) \oplus (|\cdot|^{-\epsilon+\frac{1}{2}}, |\cdot|^{\epsilon-\frac{1}{2}}).\end{aligned}$$

So in this case, $\mathrm{Exp}((\pi_\epsilon)_N)$ contains the character $\omega = (|\cdot|^{-\epsilon+\frac{1}{2}}, |\cdot|^{\epsilon-\frac{1}{2}})$ of $\mathbb{Q}_p^\times \times \mathbb{Q}_p^\times$. Then $a = \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \in A_P^{--}$. We get $\omega(a) = p^{\epsilon-\frac{1}{2}}$ which gets arbitrarily close to 1 as ϵ tends to 1/2.

Lemma 2.1.5. *Assume that G^{ad} has no F-anisotropic factor. Then every irreducible smooth representation of $G(F)$ is either one-dimensional or infinite-dimensional.*

Proof. We may assume that G_{der} is simply connected via z-extensions. Suppose that π is a finite-dimensional irreducible smooth representation of $G(F)$. Then the normal subgroup $\ker \pi \cap G_{\mathrm{der}}(F)$ of $G_{\mathrm{der}}(F)$ is open. Lemma 2.1.1 implies that $\ker \pi \cap G_{\mathrm{der}}(F) = G_{\mathrm{der}}(F)$, thus π factors through the abelian quotient $G(F)/G_{\mathrm{der}}(F)$. Therefore $\dim \pi = 1$, completing the proof. ☐

2.2. **Local Hecke algebras and their variants.** We retain the notation from the preceding section but allow the local field F to be either nonarchimedean or archimedean. Some basic setup of local Hecke algebras will be introduced, partly following [Art96, §1].

Fix a Haar measure on $G(F)$ and a maximal compact subgroup $K \subset G(F)$. Let $G(F)_{\mathrm{sr}}$ denote the subset of strongly regular elements $g \in G(F)$, namely the semisimple elements whose centralizers in G are (maximal) tori. By [Ste65, 2.15], $G(F)_{\mathrm{sr}}$ is open and dense in $G(F)$ (for both the Zariski and nonarchimedean topologies). Write $R(G)$ for the space of finite $\mathbb{C}$-linear combinations of irreducible characters of $G(F)$, which is a subspace in the space of functions on $G(F)_{\mathrm{sr}}$. We also identify $R(G)$ with the Grothendieck group of smooth finite-length representations of $G(F)$ with $\mathbb{C}$-coefficients. Let $\mathcal{H}(G) = \mathcal{H}(G(F))$ denote the space of smooth compactly supported bi-K-finite functions on $G(F)$. Let $\mathcal{I}(G)$ denote the invariant space of functions on $G(F)$, namely the quotient of $\mathcal{H}(G)$ by the ideal generated by functions of the form $g \mapsto f(g) - f(hgh^{-1})$ with $h \in G(F)$ and $f \in \mathcal{H}(G)$. According to [Kaz86, Thm. 0], $f \in \mathcal{H}(G)$ has trivial image in $\mathcal{I}(G)$ if and only if its orbital integral vanishes on $G(F)_{\mathrm{sr}}$ if and only if $\mathrm{Tr}\,\pi(f) = 0$ for all irreducible tempered representations of $G(F)$; moreover, the same is true if $G(F)_{\mathrm{sr}}$ is replaced with $G(F)$ and if the temperedness condition is dropped. By abuse of notation, we frequently write $f \in \mathcal{I}(G)$ to mean a representative $f \in \mathcal{H}(G)$ of an element in $\mathcal{I}(G)$. The trace Paley–Wiener theorem [BDK86] describes $\mathcal{I}(G)$ as a subspace of $\mathbb{C}$-linear functionals on $R(G)$ via

$$f \mapsto \left(\Theta \mapsto \int_{G(F)_{\mathrm{reg}}} f(g)\Theta(g)dg\right). \tag{2.2.1}$$

If $R(G)$ is thought of as a Grothendieck group, the above map is simply $f \mapsto (\pi \mapsto \mathrm{Tr}\,\pi(f))$.

Denote by $\mathcal{S}(G)$ the quotient of $\mathcal{H}(G)$ by the ideal generated by functions each of which has vanishing stable orbital integrals on $G(F)_{\mathrm{sr}}$. Thus we have natural surjections $\mathcal{H}(G) \twoheadrightarrow \mathcal{I}(G) \twoheadrightarrow \mathcal{S}(G)$. By $R(G)^{\mathrm{st}}$ we mean the subspace of $R(G)$ consisting of stable linear combinations (i.e., constant on each stable conjugacy class in $G(F)_{\mathrm{sr}}$). Then $\mathcal{S}(G)$ is identified with a subspace of functions on $R(G)^{\mathrm{st}}$ via (2.2.1) (since $\Theta \in R(G)^{\mathrm{st}}$ now, the image depends only on the image of f in $\mathcal{S}(G)$); the subspace is characterized by [Art96, Thm. 6.1, 6.2] in the p-adic case, cf. last paragraph on p.491 of [Xu17]. Via the obvious quotient map $\mathcal{I}(G) \to \mathcal{S}(G)$ and the restriction map from $R(G)$

to $R(G)^{\mathrm{st}}$, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{I}(G) & \longrightarrow & \mathcal{S}(G) \\ \downarrow{\scriptstyle \mathrm{Tr}} & & \downarrow{\scriptstyle \mathrm{Tr}} \\ \mathrm{Hom}_{\mathbb{C}\text{-linear}}(R(G),\mathbb{C}) & \longrightarrow & \mathrm{Hom}_{\mathbb{C}\text{-linear}}(R(G)^{\mathrm{st}},\mathbb{C}). \end{array}$$

Let us extend the setup so far to allow a fixed central character. By a **local central character datum** for G, we mean a pair $(\mathfrak{X},\chi)$, where

- $\mathfrak{X}$ is a closed subgroup of $Z(F)$ with a Haar measure $\mu_{\mathfrak{X}}$ on $\mathfrak{X}$,
- $\chi:\mathfrak{X}\to\mathbb{C}^\times$ is a smooth character.

Let $\mathcal{H}(G,\chi^{-1})=\mathcal{H}(G(F),\chi^{-1})$ denote the local Hecke algebra consisting of smooth bi-K-finite functions f on $G(F)$ which have compact support modulo $\mathfrak{X}$ and satisfy $f(xg)=\chi^{-1}(x)f(g)$ for $x\in\mathfrak{X}$ and $g\in G(F)$. The χ-averaging map

$$\mathcal{H}(G)\to\mathcal{H}(G,\chi^{-1}),\qquad f\mapsto\left(g\mapsto\int_{\mathfrak{X}}f(gz)\chi(z)d\mu_{\mathfrak{X}}\right),$$

is a surjection. We have the obvious definitions of $\mathcal{I}(G,\chi^{-1})$ and $\mathcal{S}(G,\chi^{-1})$, the χ-averaging maps $\mathcal{I}(G)\to\mathcal{I}(G,\chi^{-1})$ and $\mathcal{S}(G)\to\mathcal{S}(G,\chi^{-1})$, as well as the quotient maps

$$\mathcal{H}(G,\chi^{-1})\twoheadrightarrow\mathcal{I}(G,\chi^{-1})\twoheadrightarrow\mathcal{S}(G,\chi^{-1}).$$

We can think of $\mathcal{I}(G,\chi^{-1})$ as a subspace of functions on $R(G,\chi)$, the subspace of $R(G)$ generated by irreducible characters with central character χ. Analogously $\mathcal{S}(G,\chi^{-1})$ is a subspace of functions on $R(G,\chi)^{\mathrm{st}}$ defined similarly.

2.3. **Transfer of one-dimensional representations.** Let G and G^* be connected reductive groups over a nonarchimedean local field F of characteristic zero, related by an $\overline{F}$-isomorphism $\xi:G_{\overline{F}}\overset{\sim}{\to}G^*_{\overline{F}}$. We assume that G^* is quasi-split over F and that ξ is an inner twisting, namely that $\xi^{-1}\sigma(\xi)$ is an inner automorphism of $G_{\overline{F}}$ for every $\sigma\in\mathrm{Gal}(\overline{F}/F)$. As in §1.7, we have canonical F-morphisms $\varrho:G_{\mathrm{sc}}\to G$ and $\varrho^*:G^*_{\mathrm{sc}}\to G^*$. Define an F-torus and two groups

$$G^\flat:=G/G_{\mathrm{der}},\qquad G(F)^\flat:=\mathrm{cok}(G_{\mathrm{sc}}(F)\overset{\varrho}{\to}G(F)),\qquad G(F)^{\mathrm{ab}}:=G(F)/G(F)_{\mathrm{der}},$$

where $G(F)_{\mathrm{der}}$ is the commutator subgroup (as an abstract group). Then $G(F)_{\mathrm{der}}$ is contained in $\mathrm{im}(G(F)_{\mathrm{sc}}\to G(F))$ [Del79, 2.0.2], so there are natural morphisms

$$G(F)\twoheadrightarrow G(F)^{\mathrm{ab}}\twoheadrightarrow G(F)^\flat\twoheadrightarrow G(F)/G_{\mathrm{der}}(F)\hookrightarrow G^\flat(F).\tag{2.3.1}$$

In particular, $G(F)^\flat$ is an abelian group. We view $G(F)^\flat$ and $G(F)^{\mathrm{ab}}$ as topological groups using the quotient topology as $G_{\mathrm{sc}}(F)$ and $G(F)_{\mathrm{der}}$ have closed images in $G(F)$. Only the latter case requires explanation: if G is a torus then $G_{\mathrm{der}}=\{1\}$, and if not, $G(F)_{\mathrm{der}}$ is not contained in $Z_{G_{\mathrm{der}}}(F)$ so an open subgroup of finite index in $G_{\mathrm{der}}(F)$ by [PR94, Prop. 3.17] (reduce to the simply connected and F-simple case via z-extensions).

The last two maps in (2.3.1) are isomorphisms if $G_{\mathrm{der}}=G_{\mathrm{sc}}$ by Kneser's vanishing theorem for H^1 of simply connected groups (applicable since F is nonarchimedean). The definition and discussion above applies to G^* in the same way.

Let $1\to Z_1\to G_1\overset{\alpha}{\to}G\to 1$ be a z-extension of G over F. Since $G_1\to G$ induces $G_1^{\mathrm{ad}}\overset{\sim}{\to}G^{\mathrm{ad}}$, the classifying data for inner twists of G_1 and those of G are identified (up to isomorphism). Thus we may assume that there is a z-extension $1\to Z_1\to G_1^*\overset{\alpha^*}{\to}G^*\to 1$ with an inner twisting $\xi_1:G_{1,\overline{F}}\overset{\sim}{\to}G^*_{1,\overline{F}}$ such that ξ_1 and ξ form a commutative square together with the maps α and α^*. The map $G_{1,\mathrm{der}}\to G_{\mathrm{der}}$ induced by α is a simply connected cover, allowing an identification $G_{1,\mathrm{der}}=G_{\mathrm{sc}}$. Likewise we have $G^*_{1,\mathrm{der}}=G^*_{\mathrm{sc}}$.

Lemma 2.3.1. *There is a commutative diagram in which rows are exact and vertical maps are isomorphisms:*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & Z_1(F)/Z_1(F)\cap G_{1,\mathrm{der}}(F) & \longrightarrow & G_1(F)^\flat = G_1^\flat(F) & \longrightarrow & G(F)^\flat & \longrightarrow & 1 \\ & & \downarrow\wr & & \downarrow\wr & & \downarrow\wr & & \\ 1 & \longrightarrow & Z_1(F)/Z_1(F)\cap G^*_{1,\mathrm{der}}(F) & \longrightarrow & G'_1(F)^\flat = G_1^{*,\flat}(F) & \longrightarrow & G^*(F)^\flat & \longrightarrow & 1. \end{array}$$

Here the second vertical map is given by the isomorphism $G_1^\flat \stackrel{\sim}{\to} G_1^{*,\flat}$ induced by ξ, and the first and third vertical maps are induced by the second. Moreover the isomorphism $G(F)^\flat \stackrel{\sim}{\to} G^*(F)^\flat$ is canonical, i.e., independent of the choice of z-extensions.

We will write $\xi^\flat : G(F)^\flat \stackrel{\sim}{\to} G^*(F)^\flat$ for the canonical isomorphism.

Proof. Let us verify that the first row is exact in the lemma; the exactness of the second row follows in the same way. Consider the following commutative diagram where all maps are the natural ones:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G_{1,\mathrm{sc}}(F) & \longrightarrow & G_1(F) & \longrightarrow & G_1^\flat(F) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathrm{im}(G_{1,\mathrm{sc}}(F)\to G_{1,\mathrm{der}}(F)) & \longrightarrow & G(F) & \longrightarrow & G(F)^\flat & \longrightarrow & 1. \end{array}$$

Applying the snake lemma, we obtain the exact sequence in the first row of the lemma.

Since inner automorphisms are trivial on the derived subgroups of G_1 and G_1^*, the map ξ induces an F-isomorphism $G_1^\flat \stackrel{\sim}{\to} G_1^{*,\flat}$ and restricts to an F-isomorphism from Z_1 onto Z_1. Thus the first two vertical maps are isomorphisms, which implies that the last vertical map is also.

As for the last assertion, if $1 \to Z_1 \to G_1 \stackrel{\alpha_1}{\to} G \to 1$ and $1 \to Z_2 \to G_2 \stackrel{\alpha_2}{\to} G \to 1$ are two z-extensions then there is a third z-extension $1 \to Z_1 \times Z_2 \to G_3 \to G \to 1$ by $G_3 = \{(g_1, g_2) \in G_1 \times G_2 : \alpha_1(g_1) = \alpha_2(g_2)\}$, equipped with projections $G_3 \twoheadrightarrow G_1$ and $G_3 \twoheadrightarrow G_1$. Thus we are reduced to showing that G_3 and G_1 (and likewise G_3 and G_2) induce the same isomorphism $G(F)^\flat \stackrel{\sim}{\to} G^*(F)^\flat$. This is an elementary compatibility check. □

Lemma 2.3.2. *If G_{sc} has no F-anisotropic factor, then $G(F)^\flat = G(F)^{\mathrm{ab}}$.*

Proof. We may assume that G is not a torus. Via a z-extension, reduce to the case when $G_{\mathrm{sc}} = G_{\mathrm{der}}$. Then $G(F)_{\mathrm{der}}$ is a noncentral normal subgroup of $G_{\mathrm{der}}(F)$. Applying Lemma 2.1.1 to each F-simple factor of G_{der}, we deduce that $G(F)_{\mathrm{der}} = G_{\mathrm{der}}(F)$, hence $G(F)^\flat = G(F)^{\mathrm{ab}}$. □

Corollary 2.3.3. *If G_{sc} has no F-anisotropic factor, then the following four groups (under multiplication) are in canonical isomorphisms with each other:*

(1) the group of smooth characters $G(F) \to \mathbb{C}^\times$,
(2) the group of smooth characters $G(F)^\flat \to \mathbb{C}^\times$,
(3) the group of smooth characters $G^*(F)^\flat \to \mathbb{C}^\times$,
(4) the group of smooth characters $G^*(F) \to \mathbb{C}^\times$,

where the maps from (2) to (1) and from (3) to (4) are given by pullbacks, and the map between (2) and (3) is via the isomorphism of the preceding lemma. With no assumption on G_{sc}, we still have canonical isomorphisms between (2), (3), and (4), and a canonical embedding from (2) to (1).

Proof. We have seen that $G(F)^{\mathrm{ab}}$ is the maximal abelian topological quotient of $G(F)$, so $G(F)$ may be replaced with $G(F)^{\mathrm{ab}}$ in (1). The analogue holds for (4). From (2.3.1) and Lemma 2.1.1, we have

$$G(F)^{\mathrm{ab}} \twoheadrightarrow G(F)^\flat \simeq G^*(F)^\flat \twoheadleftarrow G^*(F)^{\mathrm{ab}}.$$

Lemma 2.3.2 tells us that the last map is always an isomorphism (since G^* has no F-anisotropic factor); so is the first map if G has no F-anisotropic factor. The corollary follows. □

Remark 2.3.4. The only nontrivial F-anisotropic simply connected simple group over F is of type A, more precisely of the form $\mathrm{Res}_{F'/F}\mathrm{SL}_1(D)$ for a central division algebra D over a finite extension F' of F with $[D:F]=n^2$ and $n\geq 2$. So the condition in the corollary is that G_{sc} has no such factor. Two exemplary cases are (i) $G=\mathrm{GL}_1(D)$, $G^*=\mathrm{GL}_n(F)$ and (ii) $G=\mathrm{SL}_1(D)$, $G^*=\mathrm{SL}_n(F)$. In (i), it is standard (e.g., [Rie70, Intro.]) that $G(F)_{\mathrm{der}}=G_{\mathrm{der}}(F)$, and the four sets are still isomorphic. However, in (ii), $G(F)_{\mathrm{der}}$ is the group of 1-units in the maximal order of D by [Rie70, §5, Cor.]. In particular (1) is a nontrivial group, whereas (2) and (3) are evidently trivial, thus (4) is trivial by the corollary.

Remark 2.3.5. One can also construct a natural map from (4) to (1) through the continuous cohomology $H^1(W_F,Z(\widehat{G}))=H^1(W_F,Z(\widehat{G}^*))$ following Langlands. (This works for archimedean local fields F as well.) Indeed, [Xu16, App. A] explains the isomorphism between $H^1(W_F,Z(\widehat{G}^*))$ and (4), and a map from $H^1(W_F,Z(\widehat{G}^*))$ to (1).[4]

Let us define the notion that semisimple elements $g\in G(F)$ and $g^*\in G^*(F)$ are *matching*, depending only on the $G(\overline{F})$-conjugacy class of the inner twisting $\xi: G_{\overline{F}}\stackrel{\sim}{\to} G^*_{\overline{F}}$. (In practice, only the $G(\overline{F})$-conjugacy class of ξ is well-defined.) When $G_{\mathrm{der}}=G_{\mathrm{sc}}$, stable conjugacy classes are the same as $\overline{F}$-conjugacy classes, and g and g^* are considered matching if their stable conjugacy classes are identified via ξ. In general, matching can be defined by lifting ξ to an inner twisting between z-extensions of G and G^* as in [Kot82, pp.799–800] (specializing to the case $E=F$), which also shows that the definition is independent of the choice of z-extensions and the choice of ξ in its $G(\overline{F})$-conjugacy class. Since G^* is quasi-split, every g admits a matching element in $G^*(F)$.

When $g\in G(F)$ and $g^*\in G^*(F)$ are matching, we have an inner twisting between the connected centralizers I_g, I_{g^*} in G, G^* by [Kot82, Lem. 5.8]. Fix Haar measures on the pairs of inner forms $(G(F),G^*(F))$ and $(I_g(F),I_{g^*}(F))$ compatibly in the sense of [Kot88, p.631] to define orbital integrals at g and g^*. Write $e(G)\in\{\pm1\}$ for the Kottwitz sign. Now $f\in\mathcal{H}(G(F))$ and $f^*\in\mathcal{H}(G^*(F))$ are said to be *matching* if for every semisimple $g\in G(F)$,

$$SO_{g^*}(f^*)=e(G)SO_g(f) \tag{2.3.2}$$

if there exists a matching $g^*\in G^*(F)$, and $SO_{g^*}(f^*)=0$ if there is no such g^*. A standard fact (cf. §2.6 below) proved in [Wal97] is that given f, there always exists a matching f^* as above, called a *transfer* of f. If the Harish-Chandra character Θ_{π^*} of $\pi^*\in\mathrm{Irr}(G^*(F))$ is stable, i.e., $\Theta_{\pi^*}(g_1^*)=\Theta_{\pi^*}(g_2^*)$ whenever two regular semisimple elements $g_1^*,g_2^*\in G^*(F)$ are stably conjugate, then the value $\mathrm{Tr}\,\pi^*(f^*)=\int_{G^*(F)_{\mathrm{sr}}}f^*(g^*)\Theta_{\pi^*}(g^*)dg^*$ is determined by stable orbital integrals of f^* on (strongly) regular semisimple elements. This follows from the stable version of the Weyl integration formula, cf. (2.3.3) below. This discussion applies to π^* with $\dim\pi^*=1$ for example, since the characters of such π^* are clearly stable. The analogue holds true with G and f in place of G^* and f^*.

Lemma 2.3.6. *Let $f\in\mathcal{H}(G(F))$ and $f^*\in\mathcal{H}(G^*(F))$ be matching functions. Let π (resp. π^*) be a one-dimensional smooth representation of $G(F)$ (resp. $G^*(F)$). If π^* corresponds to π via the preceding corollary then*

$$\mathrm{Tr}\,\pi(f)=e(G)\mathrm{Tr}\,\pi^*(f^*).$$

[4] The latter map is asserted to be also an isomorphism in [Xu16, App. A], but this is false for $G=\mathrm{SL}_1(D)$ (in which case $Z(\widehat{G})=\{1\}$) as explained in Remark 2.3.4. In *loc. cit.*, for a simply connected group G' over F, it is said that all continuous characters $G'(F)\to\mathbb{C}^\times$ are trivial, but this is not guaranteed unless G_{sc} has no F-anisotropic factor (e.g., this is okay for G^*).

Proof. As $\dim \pi = \dim \pi^* = 1$, the characters Θ_π and Θ_{π^*} are stable, and $\Theta_\pi(g) = \pi(g)$, $\Theta_{\pi^*}(g^*) = \pi^*(g^*)$ for $g \in G(F)$, $g^* \in G^*(F)$. For a maximal torus T of G over F, write $W(T)$ for the Weyl group, and $T(F)_{\mathrm{sr}}$ for the strongly regular subset of $T(F)$. By the stable Weyl integration formula,

$$\operatorname{Tr} \pi(f) = \int_{G(F)_{\mathrm{sr}}} f(g)\Theta_\pi(g)dg = \sum_T \frac{1}{|W(T)|} \int_{T(F)_{\mathrm{sr}}} SO_t(f)\Theta_\pi(t)dt, \tag{2.3.3}$$

where the first sum runs over the set of stable conjugacy classes of maximal tori of G over F. The analogous formula holds for $G^*(F)$, with the sum over the set of stable conjugacy classes of maximal tori T^* of G^* over F. Since f and f^* are matching, if T^* does not transfer to G over F (in the sense of [Kot84b, 9.5]) then $SO_{t^*}(f^*) = 0$ on $T^*(F)_{\mathrm{sr}}$. (Since G^* is quasi-split, T is always the transfer of a maximal torus of G^*.) We observe that, if T is a transfer of T^* to G, and if $t \in T(F)_{\mathrm{sr}}$ and $t^* \in T^*(F)_{\mathrm{sr}}$ have matching conjugacy classes, then the image of t in $G(F)^{\mathrm{ab}}$ and that of t^* in $G^*(F)^{\mathrm{ab}}$ correspond via the surjection $G(F)^{\mathrm{ab}} \twoheadrightarrow G^*(F)^{\mathrm{ab}}$ in the proof of Corollary 2.3.3. Hence $\Theta_\pi(t) = \Theta_{\pi^*}(t^*)$ in view of Corollary 2.3.3. It follows from (2.3.2) and (2.3.3) that $\operatorname{Tr} \pi(f) = e(G)\operatorname{Tr} \pi^*(f^*)$. □

Remark 2.3.7. In Lemma 2.3.6, π and π^* need not be related by the Jacquet–Langlands correspondence when $G^* = \mathrm{GL}_n$, $n > 1$. For instance, consider the case $G = \mathrm{GL}_1(D)$ for a central division algebra D over a p-adic field F. The trivial representation of $D^\times$ corresponds to the Steinberg representation of $\mathrm{GL}_n(F)$ under Jacquet–Langlands, but to the trivial representation of $\mathrm{GL}_n(F)$ in the lemma.

2.4. **Lefschetz functions on real reductive groups.** Let G be a connected reductive group over $\mathbb{R}$ containing an elliptic maximal torus. Fix a maximal compact subgroup $K_\infty \subset G(\mathbb{R})$. Denote by $G(\mathbb{R})_+$ the preimage of the neutral component $G^{\mathrm{ad}}(\mathbb{R})^0$ (for the real topology) under the natural map $G(\mathbb{R}) \to G^{\mathrm{ad}}(\mathbb{R})$.

Lemma 2.4.1. *We have $G(\mathbb{R})_+ = Z(\mathbb{R}) \cdot \varrho(G_{\mathrm{sc}}(\mathbb{R}))$.*

Proof. Since $G_{\mathrm{sc}}(\mathbb{R})$ is connected, clearly $\varrho(G_{\mathrm{sc}}(\mathbb{R}))$ maps into $G^{\mathrm{ad}}(\mathbb{R})^0$. Therefore $G(\mathbb{R})_+ \supset Z(\mathbb{R}) \cdot \varrho(G_{\mathrm{sc}}(\mathbb{R}))$. We have surjections $G_{\mathrm{sc}}(\mathbb{R})^0 \times Z(\mathbb{R})^0 \twoheadrightarrow G(\mathbb{R})^0 \twoheadrightarrow G^{\mathrm{ad}}(\mathbb{R})^0$ by [Mil05, Prop. 5.1]. This implies that $G(\mathbb{R})_+ \subset Z(\mathbb{R})G(\mathbb{R})^0 = Z(\mathbb{R}) \cdot \varrho(G_{\mathrm{sc}}(\mathbb{R}))$. □

Let ξ be an irreducible algebraic representation of $G_{\mathbb{C}}$, and $\zeta : G(\mathbb{R}) \to \mathbb{C}^\times$ be a continuous character. By restriction ξ yields a representation of $G(\mathbb{R})$ on a complex vector space, which we still denote by ξ. Write $\omega_\xi : Z(\mathbb{R}) \to \mathbb{C}^\times$ for the central character of ξ. By $\Pi_\infty(\xi, \zeta)$ we mean the set of isomorphism classes of irreducible discrete series representations whose central and infinitesimal characters are equal to those of the contragredient of $\xi \otimes \zeta$. This is a discrete series L-packet by the construction of [Lan89], which assigns to $\Pi_\infty(\xi, \zeta)$ an L-parameter

$$\varphi_{\xi,\zeta} \colon W_{\mathbb{R}} \to {}^L G.$$

Thus we also write $\Pi_\infty(\varphi_{\xi,\zeta})$ for $\Pi_\infty(\xi, \zeta)$. We have $\xi \otimes \zeta \simeq \xi' \otimes \zeta'$ as representations of $G(\mathbb{R})$ if and only if there exists an algebraic character χ of $G_{\mathbb{C}}$ such that $\xi' = \xi \otimes \chi$ and $\zeta' = \zeta \otimes \chi^{-1}$. In this case $\Pi_\infty(\xi, \zeta) = \Pi_\infty(\xi', \zeta')$, and $\varphi_{\xi,\zeta} \simeq \varphi_{\xi',\zeta'}$.

Write A_G for the maximal split torus in the center of G. Let $\chi : A_G(\mathbb{R})^0 \to \mathbb{C}^\times$ be a continuous character. Let $\mathrm{Irr}_{\mathrm{temp}}(G(\mathbb{R}), \chi)$ be the set of (isomorphism classes of) irreducible tempered representations of $G(\mathbb{R})$ whose central character equals χ on $A_G(\mathbb{R})^0$. Following [Art89, §4], a function $f \in \mathcal{H}(G(\mathbb{R}), \chi^{-1})$ is said to be *stable cuspidal* if $\operatorname{Tr} \pi(f) = 0$ for every $\pi \in \mathrm{Irr}_{\mathrm{temp}}(G(\mathbb{R}), \chi)$ that is not a discrete series representation, and if $\operatorname{Tr} \pi(f)$ is constant as π varies over each discrete series L-packet.

Fix a Haar measure on $G(\mathbb{R})$ and the Lebesgue measure on $A_G(\mathbb{R})^0$, so as to determine a Haar measure on $G(\mathbb{R})/A_G(\mathbb{R})^0$. Choose a pseudo-coefficient f_π for each $\pi \in \Pi_\infty(\xi, \zeta)$ à la [CD85], so that $f_\pi \in \mathcal{H}(G(\mathbb{R}), \omega_\xi \zeta)$. The characterizing property of f_π is that $\operatorname{Tr} \pi(f_\pi) = 1$ and $\operatorname{Tr} \pi'(f_\pi) = 0$

for $\pi' \in \mathrm{Irr}_{\mathrm{temp}}(G(\mathbb{R}), (\omega_\xi\zeta)^{-1})$. Although f_π is not unique, its orbital integrals are uniquely determined. It depends on the Haar measure on $G(\mathbb{R})$: if the measure is multiplied by $c \in \mathbb{C}^\times$ then f_π is to be replaced with $c^{-1}f_\pi$. A *Lefschetz function* associated with (ξ,ζ), to be denoted by either $f_{\xi,\zeta}$ or $f_{\varphi_{\xi,\zeta}}$, is defined as

$$f_{\xi,\zeta} := |\Pi_\infty(\xi,\zeta)|^{-1} \sum_{\pi\in\Pi_\infty(\xi,\zeta)} f_\pi \quad \in \mathcal{H}(G(\mathbb{R}), \omega_\xi\zeta).$$

By construction, $f_{\xi,\zeta}$ is stable cuspidal in the above sense.

In fact $|\Pi_\infty(\xi,\zeta)|$ is a constant depending only on G, namely the ratio of the Weyl groups for G and a maximal compact subgroup. Write $d(G) \in \mathbb{Z}_{\geq 1}$ for this constant. When $\xi = \mathbf{1}$ (the trivial representation), we also write $\Pi_\infty(\zeta)$ and f_ζ for $\Pi_\infty(\xi,\zeta)$ and $f_{\xi,\zeta}$.

For elliptic $\gamma \in G(\mathbb{R})$, let I_γ denote its connected centralizer in $G(\mathbb{R})$, with Kottwitz sign $e(I_\gamma) \in \{\pm 1\}$. Let I_γ^{cpt} denote an inner form of I_γ over $\mathbb{R}$ that is anisotropic modulo $Z_G(\mathbb{R})$. From [Kot92a, p.659] (as our $O_\gamma(f_{\xi,\zeta})$ equals $d(G)^{-1}\mathrm{SO}_{\gamma_\infty}(f_\infty)$ there), we see that

$$O_\gamma(f_{\xi,\zeta}) = \begin{cases} d(G)^{-1}\mathrm{vol}(A_G(\mathbb{R})^0\backslash I_\gamma^{\mathrm{cpt}}(\mathbb{R}))^{-1}\zeta(\gamma)e(I_\gamma)\mathrm{Tr}\,\xi(\gamma), & \gamma : \text{elliptic},\\ 0, & \gamma : \text{non-elliptic}.\end{cases} \tag{2.4.1}$$

In (2.4.1), the Haar measure on $I_\gamma^{\mathrm{cpt}}(\mathbb{R})$ is chosen to be compatible (in the sense of [Kot88, p.631]) with the measure on $I_\gamma(\mathbb{R})$ used in the orbital integral, to compute $\mathrm{vol}(A_G(\mathbb{R})^0\backslash I_\gamma^{\mathrm{cpt}}(\mathbb{R}))$ with respect to the Lebesgue measure on $A_G(\mathbb{R})^0$. Again by *loc. cit.* we have

$$SO_\gamma(f_{\xi,\zeta}) = \begin{cases} \mathrm{vol}(A_G(\mathbb{R})^0\backslash I_\gamma^{\mathrm{cpt}}(\mathbb{R}))^{-1}\zeta(\gamma)\mathrm{Tr}\,\xi(\gamma), & \gamma : \text{elliptic},\\ 0, & \gamma : \text{non-elliptic}.\end{cases} \tag{2.4.2}$$

Let G^* be a quasi-split group over $\mathbb{R}$ with inner twisting $G_{\mathbb{C}} \xrightarrow{\sim} G^*_{\mathbb{C}}$, through which ξ,ζ above are transported to G^*. Thereby we obtain an averaged Lefschetz function $f^*_{\xi,\zeta}$ on $G^*(\mathbb{R})$.

Lemma 2.4.2. *The function $f^*_{\xi,\zeta}$ is a transfer of $e(G)f_{\xi,\zeta}$.*

Proof. This is immediate from (2.4.2). □

Lemma 2.4.3. *Assume that $\xi = \mathbf{1}$. Let $\pi : G(\mathbb{R}) \to \mathbb{C}^\times$ be a continuous character whose central character equals ζ^{-1} when restricted to $A_G(\mathbb{R})^0$. Then $\pi|_{G(\mathbb{R})_+} = \zeta^{-1}|_{G(\mathbb{R})_+}$ if and only if $\pi|_{Z(\mathbb{R})} = \zeta^{-1}|_{Z(\mathbb{R})}$. If the equivalent conditions hold then $\mathrm{Tr}\,(f_\zeta|\pi) = 1$ if $\pi = \zeta^{-1}$; otherwise $\mathrm{Tr}\,(f_\zeta|\pi) = 0$.*

Proof. The first assertion is clear from Lemma 2.4.1. For the second assertion, it follows from (2.4.1) via the Weyl integration formula that $\mathrm{Tr}\,(f_\zeta|\pi) = \mathrm{vol}(K)^{-1}\int_K \zeta(k)\pi(k)dk$, where K is a maximal compact-modulo-$A_G(\mathbb{R})^0$ subgroup of $G(\mathbb{R})$. The integral vanishes unless $\pi = \zeta^{-1}$ on K, in which case $\pi = \zeta^{-1}$ on the entire $G(\mathbb{R})$ (since K meets every component of $G(\mathbb{R})$) and $\mathrm{Tr}\,(f_\zeta|\pi) = 1$. □

2.5. **One-dimensional automorphic representations.** Now let G be a connected reductive group over a *number field* F. Let v be a place of F and set $G_v := G \otimes_F F_v$. We have a finite decomposition of G_{sc} into F-simple factors

$$G_{\mathrm{sc}} = \prod_{i\in I} G_i, \qquad \text{with} \quad G_i = \mathrm{Res}_{F_i/F} H_i, \tag{2.5.1}$$

for a finite extension F_i/F and an absolutely F-simple simply connected group H_i over each F_i. Accordingly $G^{\mathrm{ad}} = \prod_{i\in I} G_i^{\mathrm{ad}}$. Note that we have a natural composite map $G \to G^{\mathrm{ad}} \to G_i^{\mathrm{ad}}$ for each $i \in I$, where the last map is the projection onto the i-component. Let $P_v = M_vN_v$ be a Levi decomposition of a parabolic subgroup of G_v. We consider the following assumption, where "nb" stands for non-basic (cf. Definition 5.3.2).

($\mathbb{Q}$-nb(P_v)) The image of P_v in $(G_i^{\mathrm{ad}})_v$ is a *proper* parabolic subgroup for every $i \in I$.

The assumption implies that G^{ad} has no (nontrivial) F-simple factor that is anisotropic over F_v. This leads to a useful fact that the embedding $G_{\mathrm{sc}}(F) \hookrightarrow G_{\mathrm{sc}}(\mathbb{A}_F^v)$ has dense image by strong approximation. When G^{ad} is itself F-simple, ($\mathbb{Q}$-nb(P_v)) is equivalent to the simple condition that P_v is a *proper* parabolic subgroup of G_v.

Lemma 2.5.1. *Assume that $G_{\mathrm{der}} = G_{\mathrm{sc}}$ and that G_i is isotropic over F_v for every $i \in I$. Let π be a discrete automorphic representation of $G(\mathbb{A}_F)$, and π' an irreducible $G_{\mathrm{der}}(\mathbb{A}_F)$-subrepresentation of π. Decompose $\pi' = \otimes_i \pi'_i$ according to $G_{\mathrm{der}}(\mathbb{A}_F) = \prod_{i\in I} G_i(\mathbb{A}_F)$. Write*

$$G_i(F_v) = H_i(F_i \otimes_F F_v) = \prod_{w|v} H_i(F_{i,w}),$$

where w runs over the set of places of F_i above v, and decompose $\pi'_{i,v} = \otimes_{w|v}\pi'_{i,w}$ accordingly. If for every $i \in I$, there exists $w|v$ such that $\pi'_{i,w}$ is trivial, then $\dim \pi = 1$.

Proof. Thanks to the isotropicity assumption, we apply the strong approximation to see that the embedding $H_i(F_i) \hookrightarrow H_i(\mathbb{A}_{F_i}^w)$ has dense image. Since the underlying space of π'_i consists of automorphic forms which are left-invariant under $H_i(F_i)$, and since $\pi'_{i,w}$ is trivial, we argue as in the proof of [KST, Lem. 6.2] to see that π'_i is trivial on the entire $H_i(\mathbb{A}_{F_i})$. The same argument applies to every $i \in I$ to imply that π' is trivial. Since $G(\mathbb{A}_F)/G_{\mathrm{der}}(\mathbb{A}_F)$ is abelian, we deduce $\dim\pi = 1$ noting that π is generated by π' as a $G(\mathbb{A}_F)$-module. □

Corollary 2.5.2. *Let π be an irreducible $G(\mathbb{A}_F)$-subrepresentation of $L^2_{\mathrm{disc}}(G(F)\backslash G(\mathbb{A}_F)/A_{G,\infty})$ and let $\omega_v \in \mathrm{Exp}(J_{P_v}(\pi_v))$. Then*

$$\delta_{P_v}^{1/2}(a) \le |\omega_v(a)| \le \delta_{P_v}^{-1/2}(a), \qquad a \in A_{P_v}^{--}. \tag{2.5.2}$$

Now assume ($\mathbb{Q}$-nb(P_v)). Then the left (resp. right) equality holds for some $a \in A_{P_v}^{--}$ if and only if the left (resp. right) equality holds for all $a \in A_{P_v}^{--}$ if and only if π is one-dimensional.

Proof. The assertion about the (in)equality on the left follows from the right counterpart by considering the contragredient of π. From now on, we concentrate on the right (in)equality.

The right inequality in (2.5.2) is immediate from Proposition 2.1.3 and the normalization $J_{P_v}(\pi_v) = (\pi_v)_{N_v} \otimes \delta_{P_v}^{-1/2}$. It remains to check the three conditions for the equality are equivalent. The only nontriviality is to show that $\dim\pi = 1$, assuming that $|\omega_v(a)| = \delta_{P_v}^{-1/2}(a)$ for some $a \in A_{P_v}^{--}$.

We may assume $G_{\mathrm{der}} = G_{\mathrm{sc}}$ via z-extensions. Then the assumptions of Lemma 2.5.1 are satisfied, so we adopt the setup and notation from there. By the lemma, it suffices to show that for every i, there exists a place w of F_i above v such that $\pi'_{i,w} = \mathbf{1}$. In fact we only need to find w such that $\dim \pi'_{i,w} < \infty$ by Lemma 2.1.5 and Corollary 2.3.3. (The latter tells us that all 1-dimensional representations are trivial for simply connected groups.)

We decompose

$$P'_v := P_v \cap G_{\mathrm{der}} = \prod_{\substack{i\in I,\\ w|v}} P_{v,i,w} \quad \text{according as} \quad (G_{\mathrm{der}})_v = \prod_{\substack{i\in I,\\ w|v}} (H_i)_w,$$

where w runs over places of F_i above v. Similarly $A_{P'_v} = \prod_{i,w} A_{P_{v,i,w}}$. Assumption ($\mathbb{Q}$-nb($P_v$)) tells us that for every i, there exists $w|v$ such that $P_{v,i,w}$ is a *proper* parabolic subgroup of $(H_i)_w$.

The central isogeny $Z \times G_{\mathrm{der}} \to G$ induces a map $A_{G_v} \times A_{P'_v} \to A_{P_v}$, which has finite kernel and cokernel on the level of F_v-points. Replacing a with a finite power, we may assume that a is the image of $(a_0, a') \in A_{G_v}(F_v) \times A_{P'_v}(F_v)$, so that $|\omega(a)| = |\omega(a')|$. (The central character of π'_v is unitary on $A_{G_v}(F_v)$, so $|\omega(a_0)| = 1$.) Write $a' = (a_{i,w})_{i,w}$ and $\omega_v|_{A_{P'_v}(F_v)} = (\omega_{v,i,w})_{i,w}$ according to the decomposition. We have $|\omega_{v,i,w}(a_{i,w})| \le \delta_{P_{v,i,w}}^{-1/2}(a_i)$ by Proposition 2.1.3, whereas

$\prod_{i,w} |\omega_{v,i,w}(a_{i,w})| = \prod_{i,w} \delta^{-1/2}_{P_{v,i,w}}(a_{i,w})$ from our running assumption. Therefore

$$|\omega_{v,i,w}(a_{i,w})| = \delta^{-1/2}_{P_{v,i,w}}(a_{i,w}), \quad \forall i \in I.$$

Via $J_{P'_v}(\pi'_v) \subset J_{P_v}(\pi_v)$, we see that $\omega_v|_{A_{P'_v}(F_v)} \in \mathrm{Exp}(J_{P'_v}(\pi'_v))$. Thus we have

$$\omega_{v,i,w} \in \mathrm{Exp}(J_{P_{v,i,w}}(\pi'_{v,i,w})).$$

Finally for each i, we apply the equality criterion of Proposition 2.1.3 at a place w where $P_{v,i,w}$ is proper in $(H_i)_w$. Thereby we deduce that $\dim \pi'_{i,w} < \infty$ as desired. □

Let $\xi : G_{\overline{F}} \xrightarrow{\sim} G^*_{\overline{F}}$ be an inner twisting, with G^* a connected reductive group over F.

Lemma 2.5.3. *There is a canonical bijection between one-dimensional automorphic representations of $G(\mathbb{A}_F)$ and those of $G^*(\mathbb{A}_F)$, compatible with the bijection of Corollary 2.3.3 at every place of F.*

Proof. Define $G(\mathbb{A}_F)^\flat := \mathrm{cok}(G_{\mathrm{sc}}(\mathbb{A}_F) \xrightarrow{\varrho} G(\mathbb{A}_F))$. Similarly we have $G^*(\mathbb{A}_F)^\flat$, $G(F)^\flat$, and $G^*(F)^\flat$. The arguments of §2.3 via z-extensions are easily adapted to show that $G(\mathbb{A}_F)^\flat$ is an abelian group and that there exists a canonical isomorphism $G(\mathbb{A}_F)^\flat \simeq G^*(\mathbb{A}_F)^\flat$ compatible with the isomorphism of Lemma 2.3.1 at every place of v and that the above isomorphism carries $G(F)^\flat$ onto $G^*(F)^\flat$.

With this input, the proof is analogous to that of Corollary 2.3.3. In fact we can assume that $G_{\mathrm{sc}} = G_{\mathrm{der}}$ again by taking a z-extension, and it suffices to show that the inclusion $G_{\mathrm{der}}(F)G(\mathbb{A}_F)_{\mathrm{der}} \subset G_{\mathrm{der}}(\mathbb{A}_F)$ is an equality so that every one-dimensional automorphic representations of $G(\mathbb{A}_F)$ factors through $G(\mathbb{A}_F)^\flat$ (and likewise for G^*). Since $G(\mathbb{A}_F)_{\mathrm{der}}$ contains $G(F_v)_{\mathrm{der}} = G_{\mathrm{der}}(F_v)$ whenever G is quasi-split over F_v (Lemma 2.3.2), the desired equality follows from the strong approximation for G_{der}. □

To state the next lemma, define a **(global) central character datum** to be a pair $(\mathfrak{X}, \chi)$ as follows, where $\prod'_v$ means the restricted product over all places of F.

- $\mathfrak{X} = \prod'_v \mathfrak{X}_v$ is a closed subgroup of $Z(\mathbb{A}_F)$ such that $Z(F)\mathfrak{X}$ is closed in $Z(\mathbb{A}_F)$, and
- $\chi = \prod_v \chi_v : \mathfrak{X} \cap Z(F)\backslash\mathfrak{X} \to \mathbb{C}^\times$, with $\chi_v : \mathfrak{X}_v \to \mathbb{C}^\times$ a continuous character. (It is implicit that for each $x = (x_v) \in \mathfrak{X}$, we have $\chi_v(x_v) = 1$ for almost all v, so that χ is well defined on $\mathfrak{X}$.) .)

Lemma 2.5.4. *Let $(\mathfrak{X}, \chi)$ be a central character datum for G. Let v be a finite place of F, and $g_v \in G(F_v)$ such that the image of g_v in $G(F_v)^{\mathrm{ab}}$ is not contained in the image of $\mathfrak{X}_v$. Then there exists a one-dimensional automorphic representation π of $G(\mathbb{A}_F)$ with $\pi|_{\mathfrak{X}} = \chi$ such that $\pi_v(g_v) \neq 1$.*

Proof. Replacing G with a z-extension and $(\mathfrak{X}, \chi)$ with its pullback to the z-extension, we may assume that $G_{\mathrm{der}} = G_{\mathrm{sc}}$. Then we may replace G with G^{ab} as $(\mathfrak{X}, \chi)$ factors through a central character datum for G^{ab}.

Thus we assume that $G = T$ is a torus. By assumption $g_v \in T(F_v)$ lies outside $\mathfrak{X}_v$ and thus $g_v \notin T(F)\mathfrak{X}$ in $T(\mathbb{A}_F)$, in which g_v is an element via the obvious embedding $T(F_v) \hookrightarrow T(\mathbb{A}_F)$. Thus the proof is complete by standard fourier analysis on locally compact Hausdorff abelian groups X, telling us that for every non-identity element $x \in X$, there exists a unitary character of X that is nontrivial at x. (Take $X = T(\mathbb{A}_F)/T(F)\mathfrak{X} \to \mathbb{C}^\times$ and $x = g_v$.) □

2.6. **Endoscopy with fixed central character.** Let F be a local or global field of characteristic 0. Let G be a connected reductive group over F with an inner twisting $G_{\overline{F}} \xrightarrow{\sim} G^*_{\overline{F}}$ with G^* quasi-split over F. Let $\mathcal{E}(G)$ (resp. $\mathcal{E}_{\mathrm{ell}}(G)$) denote a set of representatives for isomorphism classes of endoscopic (resp. elliptic endoscopic) data for G as defined in [LS87, KS99]. A member of $\mathcal{E}(G)$ is represented by a quadruple $\mathfrak{e} = (G^{\mathfrak{e}}, \mathcal{G}^{\mathfrak{e}}, s^{\mathfrak{e}}, \eta^{\mathfrak{e}})$ consisting of a quasi-split group $G^{\mathfrak{e}}$, a split extension

$\mathcal{G}^{\mathfrak{e}}$ of W_F by $\widehat{G}^{\mathfrak{e}}$, $s^{\mathfrak{e}} \in Z_{G^{\mathfrak{e}}}$, and $\eta^{\mathfrak{e}} : \mathcal{G}^{\mathfrak{e}} \hookrightarrow {}^L G$ satisfying the conditions detailed in *loc. cit.* In particular $\mathfrak{e}^* := (G^*, {}^L G^*, 1, \mathrm{id}) \in \mathcal{E}_{\mathrm{ell}}(G)$. Write $\mathcal{E}^{<}_{\mathrm{ell}}(G) := \mathcal{E}_{\mathrm{ell}}(G)\backslash\{\mathfrak{e}^*\}$.

From now on, let $\mathfrak{e} \in \mathcal{E}(G)$. Set

$$\iota(G, G^{\mathfrak{e}}) := \tau(G)\tau(G^{\mathfrak{e}})^{-1}\zeta(\mathfrak{e})^{-1} \in \mathbb{Q}.$$

Throughout §2.6, we make the following assumption, which will be removed via z-extensions in the next subsection. (The assumption is known to be true if $\mathfrak{e} = \mathfrak{e}^*$, when it is evident, or if G^{der} is simply connected, by [Lan79, Prop. 1].)

- (assumption) $\mathcal{G}^{\mathfrak{e}} = {}^L G^{\mathfrak{e}}$.

For now we restrict to the case when F is local. Let $\mathfrak{e}$ be as above. Consider a local central character datum $(\mathfrak{X}, \chi)$ for G as in §2.2. Let $\mathfrak{X}^{\mathfrak{e}} \subset Z_{G^{\mathfrak{e}}}(F)$ denote the image of $\mathfrak{X}$ under the canonical embedding $Z_G \hookrightarrow Z_{G^{\mathfrak{e}}}$. Thus we can identify $\mathfrak{X} = \mathfrak{X}^{\mathfrak{e}}$. We say a semisimple element $\gamma^{\mathfrak{e}} \in G^{\mathfrak{e}}(F)$ is strongly G-regular if $\gamma^{\mathfrak{e}}$ corresponds to (the $G(\overline{F})$-conjugacy class of) an element of $G(F)_{\mathrm{sr}}$ via the correspondence between the semisimple conjugacy classes in $G^{\mathfrak{e}}(\overline{F})$ and those in $G(\overline{F})$ [LS87, 1.3]. Write $G^{\mathfrak{e}}(F)_{G\text{-sr}} \subset G^{\mathfrak{e}}(F)$ for the subset of strongly G-regular elements.

Thanks to the proof of the transfer conjecture and the fundamental lemma [Wal06, CL10, Ngô10], we know that each $f \in \mathcal{H}(G(F))$ admits a *transfer* $f^{\mathfrak{e}} \in \mathcal{H}(G^{\mathfrak{e}}(F))$ whose stable orbital integrals on strongly G-regular semisimple elements are determined by the following formula, where the sum runs over strongly regular $G(F)$-conjugacy classes, and $\Delta(\cdot,\cdot)$ denotes the transfer factor as in [LS87] (see the remark below on normalization).

$$SO_{\gamma^{\mathfrak{e}}}(f^{\mathfrak{e}}) = \sum_{\gamma \in G(F)_{\mathrm{sr}}/\sim} \Delta(\gamma^{\mathfrak{e}}, \gamma) O_\gamma(f), \qquad \gamma^{\mathfrak{e}} \in G^{\mathfrak{e}}(F)_{G\text{-sr}}. \tag{2.6.1}$$

The assignment of $f^{\mathfrak{e}}$ to f is not unique on the level of Hecke algebras, but (2.6.1) determines a well-defined map $\mathrm{LS}^{\mathfrak{e}} : \mathcal{I}(G) \to \mathcal{S}(G^{\mathfrak{e}})$.

The transfer satisfies an equivariance property. For each $z \in Z_G(F) \subset Z_{G^{\mathfrak{e}}}(F)$, define the translates $f_z, f^{\mathfrak{e}}_z$ of $f, f^{\mathfrak{e}}$ by $f_z(g) = f(zg)$ and $f^{\mathfrak{e}}_z(h) = f^{\mathfrak{e}}(zh)$. The equivariance of transfer factors under translation by central elements (see [LS87, Lem. 4.4.A]) implies that $f^{\mathfrak{e}}_z$ is a transfer of $\lambda^{\mathfrak{e}}(z) f_z$ for a smooth character $\lambda^{\mathfrak{e}} : Z_G(F) \to \mathbb{C}^\times$. The character $\lambda^{\mathfrak{e}}$ is independent of $f^{\mathfrak{e}}$ and f, and its restriction $\lambda^{\mathfrak{e}}|_{Z^0_G(F)}$ can be described as follows. Consider the composite map

$$W_F \to {}^L G^{\mathfrak{e}} \stackrel{\eta^{\mathfrak{e}}}{\to} {}^L G \to {}^L Z^0_G, \tag{2.6.2}$$

where the last map is dual to the embedding $Z^0_G \hookrightarrow G$. Then $\lambda^{\mathfrak{e}}|_{Z^0_G(F)}$ is the character of $Z^0_G(F)$ corresponding to the composite map above. See [KSZ, §10.3] for details. Define a smooth character $\chi^{\mathfrak{e}} : \mathfrak{X}^{\mathfrak{e}} \to \mathbb{C}^\times$ by the relation

$$\chi^{\mathfrak{e}}(z) = \lambda^{\mathfrak{e}}(z)^{-1}\chi(z), \qquad z \in \mathfrak{X} = \mathfrak{X}^{\mathfrak{e}}. \tag{2.6.3}$$

In light of the equivariance property above, the transfer map $\mathrm{LS}^{\mathfrak{e}} : \mathcal{I}(G) \to \mathcal{S}(G^{\mathfrak{e}})$ descends to

$$\mathrm{LS}^{\mathfrak{e}} : \mathcal{I}(G, \chi^{-1}) \to \mathcal{S}(G^{\mathfrak{e}}, \chi^{\mathfrak{e},-1}) \tag{2.6.4}$$

via averaging, still denoted by $\mathrm{LS}^{\mathfrak{e}}$ for simplicity. The identity (2.6.1) still holds if $f^{\mathfrak{e}} = \mathrm{LS}^{\mathfrak{e}}(f)$ under (2.6.4). In the special case of $\mathfrak{e} = \mathfrak{e}^*$ (so that $\chi^{\mathfrak{e}} = \chi$), we write $f^* \in \mathcal{H}(G^*(F), \chi^{-1})$ for a transfer of $f \in \mathcal{H}(G(F), \chi^{-1})$. If $\mathfrak{X} = \{1\}$ then f^* here coincides with the one in §2.3, noting that $e(G)$ in (2.3.2) plays the role of transfer factor.

The fundamental lemma tells us the following. Assume that G and $\mathfrak{e}$ are unramified; the latter means that $G^{\mathfrak{e}}$ is an unramified group and that the L-morphism $\eta^{\mathfrak{e}}$ is inflated from a morphism of L-groups with respect to an unramified extension of F. We also assume that χ is unramified, i.e., χ is trivial on $\mathfrak{X} \cap K$ for some (thus every) hyperspecial subgroup K of $G(F)$. We normalize the transfer factors canonically as in [LS87] (which is possible as G is quasi-split). Then $\mathrm{LS}^{\mathfrak{e}}$ can be realized by an algebra morphism on the unramified Hecke algebras

$$\xi^{\mathfrak{e},*}\colon \mathcal{H}^{\mathrm{ur}}(G(F),\chi^{-1}) \to \mathcal{H}^{\mathrm{ur}}(G^{\mathfrak{e}}(F),\chi^{\mathfrak{e},-1}).$$

We turn to the case of global field F. Recall that Z is the center of G. Let $(\mathfrak{X},\chi)$ be a global central character datum (§2.5). As in the local case, we define $\mathfrak{X}^{\mathfrak{e}} = \prod_v \mathfrak{X}^{\mathfrak{e}}_v$ to be the image of $\mathfrak{X}$ under the canonical embedding $Z_G(\mathbb{A}_F) \hookrightarrow Z_{G^{\mathfrak{e}}}(\mathbb{A}_F)$. We have $\chi^{\mathfrak{e}} := \prod_v \chi^{\mathfrak{e}}_v : \mathfrak{X}^{\mathfrak{e}} \to \mathbb{C}^\times$, where $\chi^{\mathfrak{e}}_v$ was given by the local consideration above, so that functions in $\mathcal{H}(G(\mathbb{A}_F),\chi^{-1})$ transfer to those in $\mathcal{H}(G^{\mathfrak{e}}(\mathbb{A}_F),(\chi^{\mathfrak{e}})^{-1})$. Denote by $\lambda^{\mathfrak{e}} = \prod_v \lambda^{\mathfrak{e}}_v : Z_G(F)\backslash Z_G(\mathbb{A}_F) \to \mathbb{C}^\times$ the character with $\lambda^{\mathfrak{e}}_v$ as in the local context above. (The $Z_G(F)$-invariance of $\lambda^{\mathfrak{e}}$ follows from the equivariance of transfer factors [LS87, Lem. 4.4.A] and the product formula [LS87, Cor. 6.4.B].) The restriction of λ to $Z^0_G(\mathbb{A}_F)$ corresponds to the composite map (2.6.2) (with F now global). There is an equality $\chi^{\mathfrak{e}} = \lambda^{\mathfrak{e},-1}\chi$ as characters on $\mathfrak{X} = \mathfrak{X}^{\mathfrak{e}}$ as in (2.6.3) since this holds at every place of F. In particular $\chi^{\mathfrak{e}}$ is trivial on $Z_G(F) \cap \mathfrak{X}^{\mathfrak{e}}$, and $(\mathfrak{X}^{\mathfrak{e}},\chi^{\mathfrak{e}})$ is a central character datum for $G^{\mathfrak{e}}$.

Remark 2.6.1. The local transfer factors are well defined only up to a nonzero scalar (there is no canonical choice unless G is quasi-split or $G^{\mathfrak{e}} = G^*$, without further rigidification), so we always choose a normalization implicitly, for instance throughout §3. Observe that scaling the transfer factor results in scaling the transfer map (2.6.4). However the Langlands–Shelstad product formula [LS87, §6.4] tells us that it is possible to choose a normalization at every place such that the product of local transfer factors over all places *is* the canonical global transfer factor. We will always make such a consistent choice across all places without further comments. This will not introduce ambiguity in our main argument as it takes place in the global context.

It simplifies some later arguments if $\mathfrak{e}$ is chosen to enjoy a boundedness property. We say that a subgroup of ${}^LG = \widehat{G} \rtimes W_F$ is *bounded* if its projection to $\widehat{G} \rtimes \mathrm{Gal}(E/F)$ is contained in a compact subgroup for some (thus every) finite Galois extension E/F containing the splitting field of G.

Lemma 2.6.2. *In either local or global case, we can choose the representative $\mathfrak{e} = (G^{\mathfrak{e}},\mathcal{G}^{\mathfrak{e}},s^{\mathfrak{e}},\eta^{\mathfrak{e}})$ in its isomorphism class to satisfy the following condition: $\eta^{\mathfrak{e}}(W_F)$ is a bounded subgroup of ${}^LG$. (We restrict $\eta^{\mathfrak{e}}$ to W_F via the splitting $W_F \to \mathcal{G}^{\mathfrak{e}}$ built into the data.)*

Proof. Since $\eta^{\mathfrak{e}}|_{\widehat{G}^{\mathfrak{e}}}$ will be fixed throughout, we use it to identify $\widehat{G}^{\mathfrak{e}}$ with a subgroup of $\widehat{G}$. We take the convention that all cocycles/cohomology below are continuous cocycles/cohomology.

It suffices to show that there exists an L-morphism $\eta^{\mathfrak{e}}_0 : \mathcal{G}^{\mathfrak{e}} \to {}^LG$ extending $\eta^{\mathfrak{e}}|_{\widehat{G}^{\mathfrak{e}}}$ such that $\eta^{\mathfrak{e}}_0(W_F)$ is bounded. Indeed, given such an $\eta^{\mathfrak{e}}_0$, write $\eta^{\mathfrak{e}}(w) = a(w)\rtimes w$ and $\eta^{\mathfrak{e}}_0(w) = a_0(w)\rtimes w$ for each $w \in W_F$. Since $\eta^{\mathfrak{e}}(wgw^{-1}) = \eta^{\mathfrak{e}}_0(wgw^{-1})$ for every $g \in \widehat{G}^{\mathfrak{e}}$, one deduces that $c(w) := a(w)^{-1}a_0(w)$ centralizes $\eta^{\mathfrak{e}}(\widehat{G}^{\mathfrak{e}})$, thus $c(w)$ is a 1-cocycle representing a class in $H^1(W_F, Z(\widehat{G}^{\mathfrak{e}}))$. By defining $\eta^{\mathfrak{e}}_0$ by $g \rtimes w \in \mathcal{G}^{\mathfrak{e}} \mapsto c(w)\eta^{\mathfrak{e}}(g \rtimes w)$, we see that $\mathfrak{e}_0 = (G^{\mathfrak{e}},\mathcal{G}^{\mathfrak{e}},s^{\mathfrak{e}},\eta^{\mathfrak{e}}_0)$ belongs to the isomorphism class of $\mathfrak{e}$ and satisfies the boundedness condition by hypothesis.

To prove the existence of $\eta^{\mathfrak{e}}_0$ as above, we assume for the moment that $G_{\mathrm{der}} = G_{\mathrm{sc}}$ and that $\mathcal{G}^{\mathfrak{e}} = {}^LG^{\mathfrak{e}}$. There may be a more direct proof, but we will show this by slightly refining the proof of [Lan79, Prop. 1], where Langlands shows that $\eta^{\mathfrak{e}}|_{\widehat{G}^{\mathfrak{e}}}$ extends to an L-morphism $\eta^{\mathfrak{e}}_0 : {}^LG^{\mathfrak{e}} \to {}^LG$ under the hypothesis but without guaranteeing boundedness of image. To construct $\eta^{\mathfrak{e}}_0$ (denoted ξ therein), Langlands reduces to the elliptic endoscopic case, chooses a sufficiently large finite extension K/F, and then constructs $\xi' : W_{K/F} \to {}^LG$ such that $\eta^{\mathfrak{e}}_0(g \rtimes w) := \eta^{\mathfrak{e}}(g)\xi'(w)$ gives the desired L-morphism. (From here on, in the current proof, we follow Langlands to use the Weil group $W_{K/F}$ to form the L-group, i.e., ${}^LG = \widehat{G} \rtimes W_{K/F}$.)

It is enough to arrange that ξ' has bounded image in Langlands's construction. Write $\widehat{N}$ for the normalizer of $\widehat{T}$ (which is ${}^LT^0$ in *loc. cit.*) in $\widehat{G}$. Let $\widehat{N}_c$ (resp. $Z(\widehat{G}^{\mathfrak{e}})_c$) denote the maximal compact subgroup of $\widehat{N}$ (resp. $Z(\widehat{G}^{\mathfrak{e}})$). The starting point is a choice of $\xi' : W_{K/F} \to {}^LG$ as a set-theoretic map satisfying the second displayed formula on p.709 therein. Such a ξ' can be chosen using the Langlands–Shelstad representatives of each Weyl group element ω, denoted by $n(\omega) \in \widehat{N}$ in [LS87, §2.1]. (The point is that the σ-action $\omega_{T/G}(\sigma)$ on ${}^L T^0$ and the action $\omega^1(\sigma)$ differ by the Weyl action $\omega^2(\sigma)$ in his notation. See the seventh displayed formula on p.703.) In fact $n(\omega) \in \widehat{N}_c$ since it is a product of finite-order elements in $\widehat{N}$. Thereby ξ' has image in $\widehat{N}_c \rtimes W_{K/F}$ (thus bounded). It follows that the 2-cocycle of $W_{K/F}$ given by

$$a_{w_1,w_2} = \xi'(w_1)\xi'(w_2)\xi'(w_1w_2)^{-1}$$

has values in $Z(\widehat{G}^{\mathfrak{e}})_c$ (not just $Z(\widehat{G}^{\mathfrak{e}})$ as in [Lan79, p.709]). We need to verify the claim that the 2-cocycle is trivial in $H^2(W_{K/F}, Z(\widehat{G}^{\mathfrak{e}})_c)$; then ξ' can be made a homomorphism after multiplying a $Z(\widehat{G}^{\mathfrak{e}})_c$-valued 1-cocycle, without affecting boundedness of image. Moreover, thanks to Lemma 2 therein (stated for $Z(\widehat{G}^{\mathfrak{e}})$ but also applicable for $Z(\widehat{G}^{\mathfrak{e}})_c$ since both groups have the same group of connected components), we may assume that $a_{w_1,w_2} \in (Z(\widehat{G}^{\mathfrak{e}})_c)^0$. Then the claim follows from a variant of Lemma 4 therein, with S replaced by the maximal compact subtorus in the statement and proof. (In particular, the map (1) on p.719 is still surjective if S_1 and S_2 are replaced with their maximal compact subtori, by considering unitary characters.) This completes the proof that there exists a choice of ξ' under which the Weil group has bounded image, thus proving the lemma in the case of simply connected derived subgroup.

Going back to the general case, let us remove the preceding hypothesis by replacing G with a z-extension $1 \to Z_1 \to G_1 \to G \to 1$, which gives rise to a Γ_F-equivariant exact sequence

$$1 \to \widehat{G} \to \widehat{G}_1 \to \widehat{Z}_1 \to 1. \tag{2.6.5}$$

From $\mathfrak{e}$, we obtain $\mathfrak{e}_1 = (G_1^{\mathfrak{e}}, \mathcal{G}_1^{\mathfrak{e}}, s_1^{\mathfrak{e}}, \eta_1^{\mathfrak{e}})$ with $\eta^{\mathfrak{e}_1}(\mathcal{G}_1^{\mathfrak{e}}) = \eta^{\mathfrak{e}}(\mathcal{G}^{\mathfrak{e}})\widehat{G}_1$ and $\mathcal{G}_1^{\mathfrak{e}} \simeq {}^L G_1^{\mathfrak{e}}$ as split extensions of W_F by $\widehat{G}_1$. Our preceding proof tells us that there exists a 1-cocycle $c : W_F \to Z(\widehat{G}_1^{\mathfrak{e}})$ such that the image of W_F under $c \cdot \eta_1^{\mathfrak{e}}$ is bounded in ${}^L G_1$. It is enough to prove the claim that there exists a choice of c such that $(c \cdot \eta_1^{\mathfrak{e}})(W_F)$ is contained and bounded in ${}^L G$ (viewed as a subgroup of ${}^L G_1$). Indeed, we can then take $\eta_0^{\mathfrak{e}}$ to be the restriction of $\eta_1^{\mathfrak{e}}$ via $\mathcal{G}^{\mathfrak{e}} \subset \mathcal{G}_1^{\mathfrak{e}}$.

To prove the claim, we may work with respect to a sufficiently finite extension K/F as above. Let us designate maximal compact subtori by the subscript c as before. Fix a 1-cocycle $c : W_{K/F} \to Z(\widehat{G}_1)$ such that $(c \cdot \eta_1^{\mathfrak{e}})(W_{K/F})$ is a bounded subgroup of ${}^L G_1$. Composing $(c \cdot \eta_1^{\mathfrak{e}})|_{W_{K/F}}$ with the natural projection ${}^L G_1 \to {}^L Z_1$, we obtain an L-morphism $\epsilon : W_{K/F} \to {}^L Z_1$ with bounded image, which is also viewed as a 1-cocycle valued in $\widehat{Z}_{1,c}$. From the surjection $Z(\widehat{G}_1)_c \twoheadrightarrow \widehat{Z}_{1,c}$, we obtain a surjection $H^1(W_{K/F}, Z(\widehat{G}_1)_c) \to H^1(W_{K/F}, \widehat{Z}_{1,c})$ in the same way as on p.719 of [Lan79] (as we remarked above). Therefore, after multiplying a $Z(\widehat{G}_1)_c$-valued cocycle whose image in $H^1(W_{K/F}, \widehat{Z}_{1,c})$ is the same as ϵ^{-1}, we may assume that $(c \cdot \eta_1^{\mathfrak{e}})|_{W_{K/F}}$ maps to the trivial cocycle $W_{K/F} \to \widehat{Z}_{1,c})$ via the projection ${}^L G_1 \to {}^L Z_1$. This means that $(c \cdot \eta_1^{\mathfrak{e}})|_{W_{K/F}}$ has image contained in ${}^L G$ in light of (2.6.5), and the image remains bounded. This completes the proof the claim, and we are done. □

2.7. **Endoscopy and z-extensions.** Here we explain a general endoscopic transfer with fixed central character by removing the assumption that $\mathcal{G}^{\mathfrak{e}} = {}^L G^{\mathfrak{e}}$ in §2.8 via z-extensions. For the time being, let the base field F of G be either local or global. Fix a z-extension (defined in [Kot82, §1])

$$1 \to Z_1 \to G_1 \to G \to 1$$

over F. If F is local and G is unramified over F, then we may and will choose G_1 to be also unramified over F. If F is global, we choose G_1 to be unramified at every finite place where G is unramified. Such a choice of G_1 is possible by [KSZ, Lem. 10.1.4].

Let $\mathfrak{e} = (G^{\mathfrak{e}}, \mathcal{G}^{\mathfrak{e}}, s^{\mathfrak{e}}, \eta^{\mathfrak{e}}) \in \mathcal{E}^{<}_{\mathrm{ell}}(G)$. As explained in [LS87, §4.4], we have a z-extension

$$1 \to Z_1 \to G^{\mathfrak{e}}_1 \to G^{\mathfrak{e}} \to 1,$$

and $\mathfrak{e}$ can be promoted to an endoscopic datum $\mathfrak{e}_1 = (G^{\mathfrak{e}}_1, {}^LG^{\mathfrak{e}}_1, s^{\mathfrak{e}}_1, \eta^{\mathfrak{e}}_1)$ for G_1 such that $\eta^{\mathfrak{e}}_1 : {}^LG^{\mathfrak{e}}_1 \hookrightarrow {}^LG$ extends $\eta^{\mathfrak{e}} : \mathcal{G}^{\mathfrak{e}} \hookrightarrow {}^LG$. Moreover, changing $\mathfrak{e}_1$ and $\mathfrak{e}$ in their isomorphism classes if necessary, we may ensure that $\eta^{\mathfrak{e}}_1(W_F)$ and $\eta^{\mathfrak{e}}(W_F)$ are bounded subgroups in ${}^LG_1$ and ${}^LG$, respectively. Indeed, this is done in the course of proof of Lemma 2.6.2 in the general case. Write $\mathfrak{X}_1$ (resp. $\mathfrak{X}^{\mathfrak{e}}_1$) for the preimage of $\mathfrak{X}$ in G_1 (resp. $G^{\mathfrak{e}}_1$), and $\chi_1 : \mathfrak{X}_1 \to \mathbb{C}^\times$ for the character pulled back from χ.

To describe endoscopic transfers, it is enough to work locally, so let F be a local field. Applying §2.8 to G_1 and $\mathfrak{e}_1$ in place of G and $\mathfrak{e}$, we obtain an identification $\mathfrak{X}^{\mathfrak{e}}_1 = \mathfrak{X}_1$ under the canonical embedding $Z_{G_1} \hookrightarrow Z_{G^{\mathfrak{e}}_1}$ as well as characters $\lambda^{\mathfrak{e}}_1 : Z_{G_1}(F) \to \mathbb{C}^\times$ and $\chi^{\mathfrak{e}}_1 : \mathfrak{X}^{\mathfrak{e}}_1 = \mathfrak{X}_1 \to \mathbb{C}^\times$ such that $\chi^{\mathfrak{e}}_1 = \lambda^{\mathfrak{e},-1}_1 \chi_1$ as characters on $\mathfrak{X}^{\mathfrak{e}}_1 = \mathfrak{X}_1$. Again $\lambda^{\mathfrak{e}}_1|_{Z^0_{G_1}(F)}$ corresponds to the parameter (2.6.2) (with $G^{\mathfrak{e}}_1, G_1$ replacing $G^{\mathfrak{e}}, G$). We also have a transfer

$$\mathrm{LS}^{\mathfrak{e}} : \mathcal{I}(G, \chi^{-1}) = \mathcal{I}(G_1, \chi_1^{-1}) \stackrel{(2.6.4)}{\to} \mathcal{S}(G^{\mathfrak{e}}_1, \chi^{\mathfrak{e},-1}),$$

where the equality is induced by $G_1(F) \twoheadrightarrow G(F)$.

2.8. **The trace formula with fixed central character.** In this subsection, G is a connected reductive group over $\mathbb{Q}$. Let A_G denote the maximal $\mathbb{Q}$-split torus in Z_G, and $A_{G_{\mathbb{R}}}$ denote the maximal $\mathbb{R}$-split torus in $Z_{G_{\mathbb{R}}}$. Put

$$A_{G,\infty} := A_G(\mathbb{R})^0, \qquad A_{G_{\mathbb{R}},\infty} := A_{G_{\mathbb{R}}}(\mathbb{R})^0.$$

Let $\chi_0 : A_{G,\infty} \to \mathbb{C}^\times$ denote a continuous character. By $L^2_{\mathrm{disc},\chi_0}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ we mean the discrete spectrum in the space of square-integrable functions (modulo $A_{G,\infty}$) on $G(\mathbb{Q})\backslash G(\mathbb{A})$ which transforms under $A_{G,\infty}$ by χ_0.

Let $(\mathfrak{X} = \prod_v \mathfrak{X}_v, \chi = \prod_v \chi_v)$ be a central character datum as in §2.5. Henceforth we always assume that

$$A_{G,\infty} \subset \mathfrak{X}_\infty.$$

We can define $L^2_{\mathrm{disc},\chi}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ in the same way as $L^2_{\mathrm{disc},\chi_0}(G(\mathbb{Q})\backslash G(\mathbb{A}))$. Let $\mathcal{A}_{\mathrm{disc},\chi}(G)$ stand for the set of isomorphism classes of irreducible $G(\mathbb{A})$-subrepresentations in $L^2_{\mathrm{disc},\chi}(G(\mathbb{Q})\backslash G(\mathbb{A}))$. The multiplicity of $\pi \in \mathcal{A}_{\mathrm{disc},\chi}(G)$ in $L^2_{\mathrm{disc},\chi}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ is denoted $m(\pi)$.

Define $\mathcal{H}(G(\mathbb{A}), \chi^{-1}) := \otimes'_v \mathcal{H}(G(\mathbb{Q}_v), \chi_v^{-1})$ as a restricted tensor product. Each $f \in \mathcal{H}(G(\mathbb{A}), \chi^{-1})$ defines a trace class operator, yielding the discrete part of the trace formula:

$$T^G_{\mathrm{disc},\chi}(f) := \mathrm{Tr}\,\left(f \,|\, L^2_{\mathrm{disc},\chi}(G(\mathbb{Q})\backslash G(\mathbb{A}))\right) = \sum_{\pi \in \mathcal{A}_{\mathrm{disc},\chi}(G)} m(\pi) \mathrm{Tr}\,(f|\pi). \qquad (2.8.1)$$

Fix a minimal $\mathbb{Q}$-rational Levi subgroup $M_0 \subset G$. Write $\mathcal{L}$ for the set of $\mathbb{Q}$-rational Levi subgroups of G containing M_0. Define the subset $\mathcal{L}_{\mathrm{cusp}} \subset \mathcal{L}$ of *relatively cuspidal* Levi subgroups; by definition, $M \in \mathcal{L}$ belongs to $\mathcal{L}_{\mathrm{cusp}}$ exactly when the natural map $A_{M,\infty}/A_{G,\infty} \to A_{M_{\mathbb{R}},\infty}/A_{G_{\mathbb{R}},\infty}$ is an isomorphism. Let $M \in \mathcal{L}$ and $\gamma \in M(\mathbb{Q})$ be a semisimple element. Write M_γ for the centralizer of γ in M, and $I^M_\gamma := (M_\gamma)^0$ for the identity component. Write $\iota^M(\gamma) \in \mathbb{Z}_{\geq 1}$ for the number of connected components of M_γ containing $\mathbb{Q}$-points. Write $|\Omega^M|$ for the order of the Weyl group of M. For $\gamma \in M(\mathbb{Q})$, let $\mathrm{Stab}^M_{\mathfrak{X}}(\gamma)$ denote the set of $x \in \mathfrak{X}$ such that γ and $x\gamma$ are $M(\mathbb{Q})$-conjugate. Note that $\mathrm{Stab}^M_{\mathfrak{X}}(\gamma)$ is necessarily finite, cf. [KSZ, §9.1]. When $M = G$, we often omit M from the notation, e.g., $I_\gamma = I^G_\gamma$ and $\iota(\gamma) = \iota^G(\gamma)$.

Fix Tamagawa measures on $M(\mathbb{A})$ and $I^M_\gamma(\mathbb{A})$ for $M \in \mathcal{L}_{\mathrm{cusp}}$ and fix their decomposition into Haar measures on $M(\mathbb{A}^\infty)$ and $M(\mathbb{R})$ (resp. $I^M_\gamma(\mathbb{A}^\infty)$ and $I^M_\gamma(\mathbb{R})$). This determines a measure on the quotient $I^M_\gamma(\mathbb{A})\backslash M(\mathbb{A})$, which is used to define the adèlic orbital integral at γ in M, and similarly over finite-adèlic groups. We also fix Haar measures on $\mathfrak{X}$ and $\mathfrak{X}_\infty$. We equip $I^M_\gamma(\mathbb{Q})$ and

$\mathfrak{X}_{\mathbb{Q}} := \mathfrak{X}\cap Z(\mathbb{Q})$ with the counting measures and $A_G(\mathbb{R})^0$ with the multiplicative Lebesgue measure. Thereby we have quotient measures on $I^M_\gamma(\mathbb{Q})\backslash I^M_\gamma(\mathbb{A})/\mathfrak{X}$, $\mathfrak{X}_{\mathbb{Q}}\backslash\mathfrak{X}/A_G(\mathbb{R})^0$, and $\mathfrak{X}_\infty/A_G(\mathbb{R})^0$.

We define the elliptic part of the trace formula as

$$T^G_{\mathrm{ell},\chi}(f) := \sum_{\gamma\in\Gamma_{\mathrm{ell},\mathfrak{X}}(G)} |\mathrm{Stab}^G_{\mathfrak{X}}(\gamma)|^{-1}\iota(\gamma)^{-1}\mathrm{vol}(I_\gamma(\mathbb{Q})\backslash I_\gamma(\mathbb{A})/\mathfrak{X})O_\gamma(f), \quad f\in\mathcal{H}(G(\mathbb{A}),\chi^{-1}), \tag{2.8.2}$$

where $\Gamma_{\mathrm{ell},\mathfrak{X}}(G)$ is the set of $\mathfrak{X}$-orbits of elliptic conjugacy classes of G.

Now we assume that $G_{\mathbb{R}}$ contains an elliptic maximal torus. Let ξ be an irreducible algebraic representation of $G_{\mathbb{C}}$, and $\zeta : G(\mathbb{R}) \to \mathbb{C}^\times$ a continuous character. Let $M \in \mathcal{L}_{\mathrm{cusp}}$ and T_∞ an $\mathbb{R}$-elliptic torus in M. Arthur introduced the function $\Phi_M(\gamma,\xi)$ in $\gamma\in T_\infty(\mathbb{R})$ in [Art89, (4.4), Lem. 4.2]. (We will see a concrete description of $\Phi_M(\gamma,\xi)$ later.) Now let $\gamma\in M(\mathbb{Q})$ and suppose that γ is elliptic in $M(\mathbb{R})$. Let $I^{M,\mathrm{cpt}}_\gamma$ denote a compact-mod-center inner form of $(I^M_\gamma)_{\mathbb{R}}$. We choose a Haar measure on $I^{M,\mathrm{cpt}}_\gamma(\mathbb{R})$ compatibly with that on $I^M_\gamma(\mathbb{R})$. Write $q(I_\gamma)\in\mathbb{Z}_{\ge 0}$ for the real dimension of the symmetric space associated with the adjoint group of $(I^M_\gamma)_{\mathbb{R}}$. Following [Art89, (6.3)], define

$$\chi(I^M_\gamma) := (-1)^{q(I_\gamma)}\tau(I^M_\gamma)\mathrm{vol}(A_{I^M_\gamma,\infty}\backslash I^{M,\mathrm{cpt}}_\gamma(\mathbb{R}))^{-1}d(I^M_\gamma). \tag{2.8.3}$$

For $f^\infty\in\mathcal{H}(G(\mathbb{A}^\infty),(\chi^\infty)^{-1})$, let $f^\infty_M\in\mathcal{H}(M(\mathbb{A}^\infty),(\chi^\infty)^{-1})$ denote the constant term, cf. §3.2 and §3.5 below. Dalal extended Arthur's Lefschetz number formula [Art89, Thm. 6.1] to the fixed central character setup. The condition on $\mathfrak{X}$ is imposed below because it is also in [Dal19]. It is a harmless condition that is satisfied in our setup, but we expect it to be superfluous.

Proposition 2.8.1. *Assume that $\mathfrak{X} = Y(\mathbb{A})A_{G,\infty}$ for a central torus $Y\subset Z_G$ over $\mathbb{Q}$. Let ξ,ζ be as above. Then for each $f^\infty\in\mathcal{H}(G(\mathbb{A}^\infty),(\chi^\infty)^{-1})$,*

$$T^G_{\mathrm{disc},\chi}(f_{\xi,\zeta}f^\infty) = \frac{1}{\mathrm{vol}(\mathfrak{X}_{\mathbb{Q}}\backslash\mathfrak{X}/A_{G,\infty}}\sum_{M\in\mathcal{L}_{\mathrm{cusp}}}(-1)^{\dim(A_M/A_G)}\frac{|\Omega^M|}{|\Omega^G|}\sum_\gamma\frac{\chi(I^M_\gamma)\zeta(\gamma)\Phi_M(\gamma,\xi)O^M_\gamma(f^\infty_M)}{\iota^M(\gamma)\cdot|\mathrm{Stab}^M_{\mathfrak{X}}(\gamma)|},$$

where the second sum runs over $\mathfrak{X}$-orbits on the set of $\mathbb{R}$-elliptic conjugacy classes of $M(\mathbb{Q})$.

Proof. This is [Dal19, Cor. 6.5.1]. □

2.9. **The stable trace formula.** Let H be a quasi-split group over $\mathbb{Q}$. Let $(\mathfrak{X}_H,\chi_H)$ be a central character datum for H. Write $\Sigma_{\mathrm{ell},\chi_H}(H)$ denote the set of stable elliptic conjugacy classes in $H(\mathbb{Q})$ up to $\mathfrak{X}_H$-equivalence, where two stable conjugacy classes are considered $\mathfrak{X}_H$-equivalent if there exists representative γ_H and γ'_H such that $\gamma'_H = x\gamma_H$ for some $x\in\mathfrak{X}_H$. Following [KSZ, §11.3], define

$$ST^H_{\mathrm{ell},\chi_H}(h) := \tau_{\mathfrak{X}_H}(H)\sum_{\gamma_H\in\Sigma_{\mathrm{ell},\chi_H}(H)}|\mathrm{Stab}_{\mathfrak{X}_H}(\gamma_H)|^{-1}SO^{H(\mathbb{A})}_{\gamma_H}(h), \qquad h\in\mathcal{H}(H(\mathbb{A}),\chi_H^{-1}).$$

Consider a central character datum $(\mathfrak{X},\chi)$ for G as well as $f = \otimes'_v f_v\in\mathcal{H}(G(\mathbb{A}),\chi^{-1})$. For each $\mathfrak{e}\in\mathcal{E}^<_{\mathrm{ell}}(G)$, we have $\mathfrak{e}_1$ and a central character datum $(\mathfrak{X}^{\mathfrak{e}}_1,\chi^{\mathfrak{e}}_1)$ (whose v-components are given as in the preceding subsection). Write $f^{\mathfrak{e}}_{1,v}\in\mathcal{H}(G^{\mathfrak{e}}_1(\mathbb{A}),(\chi^{\mathfrak{e}}_1)^{-1})$ for a transfer of f_v at each v. Put $f^{\mathfrak{e}}_1 := \otimes'_v f^{\mathfrak{e}}_{1,v}$. For $\mathfrak{e} = \mathfrak{e}^*$, we transfer f to $f^*\in\mathcal{H}(G^*(\mathbb{A}),\chi^{-1})$ as in §2.6.

Proposition 2.9.1. *Let $f = \otimes'_v f_v\in\mathcal{H}(G(\mathbb{A}),\chi^{-1})$. Assume that there exists a finite place q such that $O_g(f_q) = 0$ for every non-regular semisimple $g\in G(\mathbb{Q}_q)$. If f^* and $f^{\mathfrak{e}}_1$ are associated with f as above, then*

$$T^G_{\mathrm{ell},\chi}(f) = ST^{G^*}_{\mathrm{ell},\chi}(f^*) + \sum_{\mathfrak{e}\in\mathcal{E}^<_{\mathrm{ell}}(G)}\iota(G,G^{\mathfrak{e}})ST^{G^{\mathfrak{e}}_1(\mathbb{A})}_{\mathrm{ell},\chi^{\mathfrak{e}}_1}(f^{\mathfrak{e}}_1).$$

Proof. By hypothesis, the stable orbital integral of $f^{\mathfrak{e}}_{1,q}$ vanishes outside G-regular semisimple conjugacy classes. Suppose that the central character datum is trivial. In this case, the stabilization of regular elliptic terms is due to Langlands [Lan83], cf. [Kot86, Thm. 9.6] (which assumes $G_{\mathrm{der}} = G_{\mathrm{sc}}$ but stabilizes all elliptic terms) and [KS99, §7.4] (which stabilizes strongly G-regular terms in the twisted trace formula). The latter can be specialized to the untwisted case, but extended to all G-regular terms (or all elliptic terms, in fact) following [Lab99].

For general central character data, the argument is essentially the same by using the Langlands–Shelstad transfer with fixed central character as in §2.6. Such a stabilization with fixed central character is worked out in [KSZ, §11] in the setting of Shimura varieties (without the regularity assumption). The method carries over to the current case (if we perform the usual transfer at p and ∞ rather than what is done in *loc. cit.* specific to Shimura varieties). ☐

Remark 2.9.2. If we do not assume the existence of q as above, then the same proposition should hold with $ST^{G^{\mathfrak{e}}_1(\mathbb{A})}_{\mathrm{ell},\chi^{\mathfrak{e}}_1}$ restricted to conjugacy classes of $G^{\mathfrak{e}}_1(\mathbb{Q})$ whose images in $G^{\mathfrak{e}}(\mathbb{Q})$ are $(G, G^{\mathfrak{e}})$-regular, cf. [KS99, §7.4]. The restriction is superfluous when f_∞ is stable cuspidal (§2.4), which implies that the stable orbital integral of $f^{\mathfrak{e}}_{1,\infty}$ vanishes outside the $(G, G^{\mathfrak{e}})$-regular locus.

The following finiteness result is going to be useful. In particular it tells us that the sum in Theorem 7.5.1 (and a similar sum in Theorem 7.1.1 below) is finite for each choice of $\phi^{\infty,p}$.

Lemma 2.9.3. *The following are true.*

1. *Let v be a rational prime such that $G_{\mathbb{Q}_v}$ and χ_v are unramified. Let $f_v \in \mathcal{H}^{\mathrm{ur}}(G(\mathbb{Q}_v), \chi_v^{-1})$. Then f_v transfers to the zero function on $G^{\mathfrak{e}}_1(\mathbb{Q}_v)$ for each $\mathfrak{e} = (G^{\mathfrak{e}}, \mathcal{G}^{\mathfrak{e}}, s^{\mathfrak{e}}, \eta^{\mathfrak{e}}) \in \mathcal{E}^{<}_{\mathrm{ell}}(G)$ if $G^{\mathfrak{e}}$ is ramified over $\mathbb{Q}_v$.*
2. *Let S be a finite set of rational primes. The set of $\mathfrak{e} \in \mathcal{E}^{<}_{\mathrm{ell}}(G)$ such that $G^{\mathfrak{e}}_{\mathbb{Q}_v}$ is unramified at every rational prime $v \notin S$ is finite.*

Proof. The first point follows from [Kot86, Prop. 7.5]. The second point is well known; see [Lan83, Lem. 8.12]. ☐

3. Jacquet modules, regular functions, and endoscopy

Throughout this section, let F be a finite extension of $\mathbb{Q}_p$ with a uniformizer ϖ and residue field cardinality q. The valuation on F is normalized so that $|\varpi| = q^{-1}$. Let G be a connected reductive group over F. We study how certain maps of invariant distributions between G and its Levi subgroups interact with Jacquet modules and endoscopy. The material here is largely based on [Shi10, Xu17].

3.1. **ν-ascent and Jacquet modules.** Let $\nu : \mathbb{G}_m \to G$ be a cocharacter defined over F. Let M_ν denote the centralizer of ν in G, which is an F-rational Levi subgroup. The maximal F-split torus in the center of M_ν is denoted by A_{M_ν}.

Write P_ν (resp. P^{op}_ν) for the F-rational parabolic subgroup of G which contains M_ν as a Levi component and such that $\langle \alpha, \nu \rangle < 0$ (resp. $\langle \alpha, \nu \rangle > 0$) for every root α of A_{M_ν} in P_ν (resp. P^{op}_ν). The set of α as such is denoted by $\Phi^+(P_\nu)$ (resp. $\Phi^+(P^{\mathrm{op}}_\nu)$). Let $N_\nu, N^{\mathrm{op}}_\nu$ denote the unipotent radical of $P_\nu, P^{\mathrm{op}}_\nu$. For every $\alpha \in \Phi^+(P^{\mathrm{op}}_\nu)$, we have $|\alpha(\nu(\varpi))| = q^{-\langle \alpha, \nu \rangle} < 1$. Therefore $\nu(\varpi) \in A^{--}_{P^{\mathrm{op}}_\nu}$. The following definition is a rephrase of [Shi10, Def. 3.1].

Definition 3.1.1. An element $\gamma \in M_\nu(\overline{F})$ is considered **acceptable** (with respect to ν) if the action of $\mathrm{Ad}(\gamma)$ on $(\mathrm{Lie}\, N^{\mathrm{op}}_\nu)_{\overline{F}}$ is contracting, i.e., all its eigenvalues $\lambda \in \overline{F}$ have the property that $|\lambda| < 1$.

Remark 3.1.2. By definition, $a \in A_{M_\nu}(F)$ is acceptable if and only if $a \in A^{--}_{P^{\mathrm{op}}_\nu}$.

Evidently the subset of acceptable elements is nonempty, open, and stable under $M_\nu(\overline{\mathbb{Q}}_p)$-conjugacy. Define $\mathcal{H}_{\mathrm{acc}}(M_\nu(F)) \subset \mathcal{H}(M_\nu(F))$ as the subspace of functions supported on acceptable elements. We also write $\mathcal{H}_{\nu\text{-acc}}(M_\nu(F))$ to emphasize the dependence on ν. As in §2.2 we often omit F for simplicity.

Lemma 3.1.3. *Let $\phi \in \mathcal{H}_{\mathrm{acc}}(M_\nu)$. There exists $f \in \mathcal{H}(G)$ with the following properties.*

(1) For every $g \in G(F)_{\mathrm{ss}}$,

$$O_g^{G(F)}(f) = \delta_{P_\nu}(m)^{-1/2} O_m^{M_\nu(F)}(\phi)$$

if there exists an acceptable $m \in M_\nu(F)$ which is conjugate to g in $G(F)$ (in which case m is unique up to $M_\nu(F)$-conjugacy, and the Haar measures are chosen compatibly on the connected centralizers of m and g), and $O_g^{G(F)}(f) = 0$ otherwise.

(2) $\mathrm{Tr}\,(f|\pi) = \mathrm{Tr}\,\left(\phi|J_{P_\nu^{\mathrm{op}}}(\pi)\right)$ for $\pi \in \mathrm{Irr}(G(F))$.

Proof. This is [Shi10, Lem. 3.9] except that we corrected typos in the statement. The same proof still works with two remarks. Firstly, we removed the assumption in *loc. cit.* that orbital integrals of ϕ vanish on semisimple elements with disconnected centralizers. This is possible by reducing to the case of G with simply connected derived subgroup (then $M_{\nu,\mathrm{der}}$ is also simply connected by Lemma 1.7.1) so that the centralizers of semisimple elements are connected in both M_ν and G. Secondly, the mistake in *loc. cit.* occurs in line 1, p.806, where it should read $\phi^0 := \phi \cdot \delta_{P_\nu}^{-1/2}$. □

Corollary 3.1.4. *Let ϕ and f be as in Lemma 3.1.3. For every $g \in G(F)_{\mathrm{ss}}$,*

$$SO_g^{G(F)}(f) = \delta_{P_\nu}(m)^{-1/2} SO_m^{M_\nu(F)}(\phi)$$

if there exists an acceptable $m \in M_\nu(F)$ which is conjugate to g in $G(F)$, and $SO_g^{G(F)}(f) = 0$ otherwise.

Proof. The proof is immediate from the preceding lemma, using the following fact [Shi10, Lem. 3.5]: if $m \in M_\nu(F)$ is acceptable then the natural map from the set of $M_\nu(F)$-conjugacy classes in the stable conjugacy class of m in M_ν to the set of $G(F)$-conjugacy classes in the stable conjugacy class of m in G is a bijection. □

Definition 3.1.5. In the setup of Lemma 3.1.3, we say that f is a **ν-ascent** of ϕ.

Recall the definition of $\mathcal{I}(\cdot)$ and the trace Paley–Wiener theorem from §2.2. According to [BDK86, Prop. 3.2], the Jacquet module induces the map

$$\mathscr{J}_\nu \colon \mathcal{I}(M_\nu) \to \mathcal{I}(G), \qquad \mathcal{F} \mapsto \left(\pi \mapsto \mathcal{F}(J_{P_\nu^{\mathrm{op}}}(\pi))\right). \tag{3.1.1}$$

Write $\mathcal{I}_{\mathrm{acc}}(M_\nu)$ for the image of $\mathcal{H}_{\mathrm{acc}}(M_\nu)$ in $\mathcal{I}(M_\nu)$. Then Lemma 3.1.3 means that, when $\phi \in \mathcal{I}_{\mathrm{acc}}(M_\nu)$, a ν-ascent of ϕ is well-defined as an element of $\mathcal{I}(G)$, which is nothing but $\mathscr{J}_\nu(\phi)$. The lemma yields extra information on orbital integrals. Xu [Xu17, Prop. C.4] showed that (3.1.1) induces a similar map for the stable analogues:

$$\mathscr{J}_\nu : \mathcal{S}(M_\nu) \to \mathcal{S}(G). \tag{3.1.2}$$

Write $X_F^*(G)$ for the group of F-rational cocharacters of G. We denote $X_F^*(G)_{\mathbb{Q}} := X_F^*(G) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathfrak{a}_G := \mathrm{Hom}(X_F^*(G)_{\mathbb{Q}}, \mathbb{R})$. We have the map

$$H^G \colon G(F) \to \mathfrak{a}_G, \qquad g \mapsto (\chi \mapsto \log|\chi(g)|).$$

It is easy to see that H^G is invariant under $G(\overline{F})$-conjugacy, i.e., if $g_1, g_2 \in G(F)$ are conjugate in $G(\overline{F})$ then $H^G(g_1) = H^G(g_2)$. Indeed, if g_1, g_2 become conjugate in $G(F')$ for a finite extension F'/F, then since the map H^G is functorial with respect to $G \hookrightarrow G' := \mathrm{Res}_{F'/F} G$, and $\mathfrak{a}_G \to \mathfrak{a}_{\mathrm{Res}_{F'/F} G}$ is injective, it boils down to checking that $H^{G'}(g_1) = H^{G'}(g_2)$, which is obvious.

For $f \in \mathcal{H}(G)$, define the following subsets of $\mathfrak{a}_G$:

$$\begin{aligned} \mathrm{supp}_{\mathfrak{a}_G}(f) &:= \{H^G(x) : x \in G(F)_{\mathrm{ss}} \text{ s.t. } f(x) \neq 0\}, \\ \mathrm{supp}^O_{\mathfrak{a}_G}(f) &:= \{H^G(x) : x \in G(F)_{\mathrm{ss}} \text{ s.t. } O_x(f) \neq 0\}, \\ \mathrm{supp}^{SO}_{\mathfrak{a}_G}(f) &:= \{H^G(x) : x \in G(F)_{\mathrm{ss}} \text{ s.t. } SO_x(f) \neq 0\}. \end{aligned} \tag{3.1.3}$$

Evidently $\mathrm{supp}^{SO}_{\mathfrak{a}_G}(f) \subset \mathrm{supp}^O_{\mathfrak{a}_G}(f) \subset \mathrm{supp}_{\mathfrak{a}_G}(f)$. Writing

$$\mathscr{P}(*) := \text{collection of subsets of } *,$$

we obtain a map $\mathrm{supp}_{\mathfrak{a}_G}$ (resp. $\mathrm{supp}^O_{\mathfrak{a}_G}$, $\mathrm{supp}^{SO}_{\mathfrak{a}_G}$) from $\mathcal{H}(G)$ (resp. $\mathcal{I}(G)$, $\mathcal{S}(G)$) to $\mathscr{P}(\mathfrak{a}_G)$.

We define analogous objects for M_ν in place of G. The injective restriction map $X^*_F(G)_{\mathbb{Q}} \to X^*_F(M_\nu)_{\mathbb{Q}}$ induces a canonical surjection

$$\mathrm{pr}_G : \mathfrak{a}_{M_\nu} \to \mathfrak{a}_G. \tag{3.1.4}$$

In fact we write $\mathfrak{a}_{P_\nu} := \mathfrak{a}_{M_\nu}$ and identify $X_*(A_{M_\nu})_{\mathbb{R}} \simeq \mathfrak{a}_{P_\nu}$ induced by $\mu \in X_*(A_{M_\nu}) \mapsto (\chi \mapsto \langle \chi, \mu \rangle)$. Via this identification, it is an easy exercise to describe pr_G as the average map along Weyl orbits. Namely if T is a maximal torus of M_ν (thus also of G) over F, and if the Weyl group is taken relative to T, then

$$\mathrm{pr}_G(\mu) = |\Omega^G|^{-1} \sum_{\omega \in \Omega^G} \omega(\mu) = |\overline{\Omega}^G|^{-1} \sum_{\omega \in \overline{\Omega}^G} \omega(\mu), \qquad \mu \in X_*(A_{M_\nu})_{\mathbb{R}}. \tag{3.1.5}$$

Define

$$\mathfrak{a}^{--}_{P_\nu} := \{y \in \mathfrak{a}_{P_\nu} : \langle \alpha, y \rangle < 0,\ \forall \alpha \in \Phi^+(P_\nu)\}.$$

By definition of P_ν, we have $\nu \in \mathfrak{a}^{--}_{P_\nu}$. For $y \in X_*(A_{M_\nu}) \subset X_*(A_{M_\nu})_{\mathbb{R}} = \mathfrak{a}_{M_\nu}$, notice that $y \in \mathfrak{a}^{--}_{P_\nu}$ if and only if $y(\varpi) \in A^{--}_{P^{\mathrm{op}}_\nu}$.

Lemma 3.1.6. *The sets* $\mathrm{supp}_{\mathfrak{a}_G}(f)$ *(resp.* $\mathrm{supp}^O_{\mathfrak{a}_G}(f)$*,* $\mathrm{supp}^{SO}_{\mathfrak{a}_G}(f)$*) remain unchanged if we restrict x in the definition* (3.1.3) *to a subset $D \subset G(F)_{\mathrm{reg}}$ that is open dense in $G(F)$.*

Proof. Since the map H^G is continuous with discrete image, for each y in $\mathrm{supp}_{\mathfrak{a}_G}(f)$ (resp. $\mathrm{supp}^O_{\mathfrak{a}_G}(f)$, $\mathrm{supp}^{SO}_{\mathfrak{a}_G}(f)$), the preimage $(H^G)^{-1}(y)$ is open and closed. If $y \in \mathrm{supp}_{\mathfrak{a}_G}(f)$ then $\mathrm{supp}(f) \cap (H^G)^{-1}(y)$ is nonempty open in $G(F)$ thus intersects D. This proves the assertion for $\mathrm{supp}_{\mathfrak{a}_G}(f)$.

Next let $y \in \mathrm{supp}^O_{\mathfrak{a}_G}(f)$. Then $(H^G)^{-1}(y) \cap D$ is open dense in $(H^G)^{-1}(y)$. If $O_x(f) = 0$ for every $x \in (H^G)^{-1}(y) \cap D$, we claim that

$$O_x(f) = 0, \qquad x \in (H^G)^{-1}(y) \cap G(F)_{\mathrm{ss}}.$$

Indeed, this follows from local constancy of $O_x(f)$ on regular elements if x is regular; a Shalika germ argument then proves $O_x(f) = 0$ for non-regular semisimple x. (Compare with the proof of Lemma 3.4.5 (1) below.) However, the claim contradicts $y \in \mathrm{supp}^O_{\mathfrak{a}_G}(f)$. The lemma for $\mathrm{supp}^O_{\mathfrak{a}_G}(f)$ follows. Finally the case of stable orbital integrals is proved likewise. $\square$

If $\phi \in \mathcal{H}_{\mathrm{acc}}(M_\nu)$ then its semisimple orbital integrals (resp. stable orbital integrals) are clearly supported on acceptable elements, so $\mathrm{supp}^O_{\mathfrak{a}_{M_\nu}}(\phi)$ and $\mathrm{supp}^{SO}_{\mathfrak{a}_{M_\nu}}(\phi)$ are contained in $\mathfrak{a}^{--}_{P_\nu}$.

Lemma 3.1.7. *The following diagrams commute.*

$$\begin{array}{ccc} \mathcal{I}_{\mathrm{acc}}(M_\nu) & \xrightarrow{\mathscr{J}_\nu} & \mathcal{I}(G) \\ \downarrow{\scriptstyle \mathrm{supp}^O_{\mathfrak{a}_{M_\nu}}} & & \downarrow{\scriptstyle \mathrm{supp}^O_{\mathfrak{a}_G}} \\ \mathscr{P}(\mathfrak{a}^{--}_{P_\nu}) & \xrightarrow{\mathrm{pr}_G} & \mathscr{P}(\mathfrak{a}_G) \end{array} \qquad \begin{array}{ccc} \mathcal{S}_{\mathrm{acc}}(M_\nu) & \xrightarrow{\mathscr{J}_\nu} & \mathcal{S}(G) \\ \downarrow{\scriptstyle \mathrm{supp}^{SO}_{\mathfrak{a}_{M_\nu}}} & & \downarrow{\scriptstyle \mathrm{supp}^{SO}_{\mathfrak{a}_G}} \\ \mathscr{P}(\mathfrak{a}^{--}_{P_\nu}) & \xrightarrow{\mathrm{pr}_G} & \mathscr{P}(\mathfrak{a}_G) \end{array}$$

Proof. This follows from Lemma 3.1.3 and Corollary 3.1.4 since, for each $m \in M_\nu(F)$, the canonical map $\mathfrak{a}_{M_\nu} \to \mathfrak{a}_G$ sends $H^{M_\nu}(m)$ to $H^G(m)$. □

Let $k \in \mathbb{Z}$ and $\phi \in \mathcal{H}(M_\nu)$. Define $\phi^{(k)}(l) := \phi(\nu(\varpi)^{-k}l)$ for $l \in M_\nu(F)$ so that $\phi^{(k)} \in \mathcal{H}(M_\nu)$. Since ν is central in M_ν, this induces a map

$$(\cdot)^{(k)} \colon \mathcal{I}(M_\nu) \to \mathcal{I}(M_\nu). \tag{3.1.6}$$

Lemma 3.1.8. *If $\phi \in \mathcal{I}_{\mathrm{acc}}(M_\nu)$ then $\phi^{(k)} \in \mathcal{I}_{\mathrm{acc}}(M_\nu)$ for all $k \geq 0$. Given $\phi \in \mathcal{I}(M_\nu)$, there exists $k_0 = k_0(\phi)$ such that $\phi^{(k)} \in \mathcal{I}_{\mathrm{acc}}(M_\nu)$ for all $k \geq k_0$. The analogue holds true with $\mathcal{H}$ in place of $\mathcal{I}$. Moreover, letting $f^{(k)}$ denote the ν-ascent of $\phi^{(k)}$ for $k \geq k_0$, we have*

$$\mathrm{supp}^\star_{\mathfrak{a}_G}\left(f^{(k)}\right) = \mathrm{pr}_G(\mathrm{supp}^\star_{\mathfrak{a}_{M_\nu}}(\phi^{(k)})) = k \cdot H^G(\nu(\varpi)) + \mathrm{pr}_G(\mathrm{supp}^\star_{\mathfrak{a}_{M_\nu}}(\phi)), \qquad \star \in \{O, SO\},$$

where $\mathrm{pr}_G : \mathfrak{a}_{M_\nu} \to \mathfrak{a}_G$ is the canonical surjection.

Proof. The second equality in the displayed formula is obvious, so we check the first equality. By Lemma 3.1.6 it is enough to verify firstly that if $O_g(f^{(k)}) \neq 0$ for $g \in G(F)_{\mathrm{reg}}$ then $H^G(g) \in \mathrm{pr}_G(\mathrm{supp}^O_{\mathfrak{a}_{M_\nu}}(\phi^{(k)}))$, and secondly that if $O_m(\phi^{(k)}) \neq 0$ for $m \in M(F)_{\mathrm{reg}}$ then $\mathrm{pr}_G(H^M(m)) \in \mathrm{supp}^O_{\mathfrak{a}_{M_\nu}}(\phi^{(k)})$. This follows from Lemma 3.1.3 (1) and Lemma 3.1.7. The case of stable orbital integrals is analogous. □

Lemma 3.1.9. *Let $\pi_1, \pi_2 \in \mathrm{Groth}(M_\nu(F))$. Assume that for each $\phi \in \mathcal{I}(M_\nu)$, there exists an integer $k_0(\phi)$ such that $\mathrm{Tr}\,\pi_1(\phi^{(k)}) = \mathrm{Tr}\,\pi_2(\phi^{(k)})$ for all $k \geq k_0(\phi)$. Then we have $\pi_1 = \pi_2$ in $\mathrm{Groth}(M_\nu(F))$.*

Proof. This is proved by the argument on p. 536 of [Shi09]. □

3.2. **ν-ascent and constant terms.** Fix an F-rational minimal parabolic subgroup $P_0 \subset P_\nu^{\mathrm{op}}$ of G with a Levi factor $M_0 \subset M_\nu$. Let P be another F-rational parabolic subgroup of G containing P_0, with a Levi factor M containing M_0. Henceforth we will often write $L := M_\nu$.

We have the constant term map (compare with (3.1.1))

$$\mathscr{C}^G_M : \mathcal{I}(G) \to \mathcal{I}(M), \qquad \mathcal{F} \mapsto \left((\pi_M \mapsto \mathcal{F}(\text{n-ind}^G_M(\pi_M))\right), \tag{3.2.1}$$

where $\text{n-ind}^G_M : \mathrm{Groth}(M(F)) \to \mathrm{Groth}(G(F))$ is the normalized parabolic induction (which does not change if P is replaced with a different parabolic with Levi factor M). On the level of functions, when $f \in \mathcal{H}(G)$, we can define $f_M \in \mathcal{H}(M)$ by integrals as in [Shi11, (3.5)] so that $O^G_g(f) = 0$ if $g \in G(F)_{\mathrm{reg}}$ is not conjugate to any element of $M(F)$, and

$$O^G_m(f) = D_{G/M}(m)^{1/2} O^M_m(f_M), \qquad \forall\ G\text{-regular } m \in M(F), \tag{3.2.2}$$

where $D_{G/M} : M(F) \to \mathbb{R}^\times_{>0}$ denotes the Weyl discriminant of G relative to M. This identity and parts (i) and (ii) of [Shi11, Lem. 3.3] tell us that $f \mapsto f_M$ exactly descends to the map $\mathscr{C}^G_M$ above. (Even though G is a general linear group in *loc. cit.*, everything applies to general reductive groups as that lemma is based on the general results of [vD72].)

Since n-ind^G_M induces a map $R(M)^{\mathrm{st}} \to R(G)^{\mathrm{st}}$ [KV16, Cor. 6.13], the map $\mathscr{C}^G_M$ descends to a map on the stable spaces, still denoted by the same symbol:

$$\mathscr{C}^G_M : \mathcal{S}(G) \to \mathcal{S}(M).$$

Define $\Omega^G_{M,L} := \{\omega \in \Omega^G : \omega(M \cap P_0) \subset P_0,\ \omega^{-1}(L \cap P_0) \subset P_0\}$, which is a set of representatives for $\Omega^L \backslash \Omega^G / \Omega^M$. For $\omega \in \Omega^G_{M,L}$, write $M_\omega := M \cap \omega^{-1}(L)$, $P_\omega := M \cap \omega^{-1}(P_\nu)$, and $L_\omega := \omega(M) \cap L$. Note that M_ω (resp. L_ω) is an F-rational Levi subgroup of M (resp. L) and that ω induces an isomorphism $M_\omega \xrightarrow{\sim} L_\omega$, which in turn induces $\omega : \mathcal{I}(M_\omega) \xrightarrow{\sim} \mathcal{I}(L_\omega)$ by $\phi \mapsto (g \mapsto \phi(\omega^{-1}g))$. Since

ν is central in L, its image lies in L_ω. Applying ω^{-1} we obtain a cocharacter $\nu_\omega := \omega^{-1}(\nu)$ of M_ω. Thus we have a chain of maps

$$\mathcal{I}(L) \overset{\mathscr{C}^L_{L_\omega}}{\longrightarrow} \mathcal{I}(L_\omega) \overset{\omega^{-1}}{\simeq} \mathcal{I}(M_\omega) \overset{\mathscr{J}_{\nu_\omega}}{\longrightarrow} \mathcal{I}(M).$$

Lemma 3.2.1. *We have the following commutative diagram.*

$$\begin{array}{ccccc} \mathcal{I}(L) & \xrightarrow{\mathscr{J}_\nu} & \mathcal{I}(G) & \xrightarrow{\mathscr{C}^G_M} & \mathcal{I}(M) \\ \Big\downarrow{\scriptstyle \oplus \mathscr{C}^L_{L_\omega}} & & & & \Big\uparrow{\scriptstyle \sum_\omega \mathscr{J}_{\nu_\omega}} \\ \bigoplus_{\omega\in\Omega^G_{M,L}} \mathcal{I}(L_\omega) & & \xrightarrow{\oplus\omega^{-1}} & & \bigoplus_{\omega\in\Omega^G_{M,L}} \mathcal{I}(M_\omega) \end{array}$$

The image of $\mathcal{I}_{\mathrm{acc}}(L)$ in $\mathcal{I}(M_\omega)$ is contained in $\mathcal{I}_{\nu_\omega\text{-acc}}(M_\omega)$. Finally, all this holds true with $\mathcal{S}(\cdot)$ in place of $\mathcal{I}(\cdot)$.

Proof. Let $\phi \in \mathcal{I}(L)$. We need to show that the images of ϕ in $\mathcal{I}(M)$ given in two different ways have the same trace against every $\pi_M \in \mathrm{Irr}(M(F))$. We compute

$$\mathrm{Tr}\,\left(\mathscr{C}^L_{L_\omega}(\mathscr{J}_\nu(\phi))|\pi_M\right) = \mathrm{Tr}\,\left(\mathscr{J}_\nu(\phi)|\text{n-ind}^G_L(\pi_M)\right) = \mathrm{Tr}\,\left(\phi|J_{P_\nu}(\text{n-ind}^G_L(\pi_M))\right)$$
$$= \sum_{\omega\in\Omega^G_{M,L}} \mathrm{Tr}\,\left(\phi|\text{n-ind}^L_{L_\omega}\left(\omega(J_{P_{\nu_\omega}}(\pi_M))\right)\right) = \sum_{\omega\in\Omega^G_{M,L}} \mathrm{Tr}\,\left(\mathscr{J}_{\nu_\omega}(\omega^{-1}(\mathscr{C}^L_{L_\omega}(\phi)))|\pi_M\right),$$

where the second last equality comes from Bernstein–Zelevinsky's geometric lemma [BZ77, 2.12]. The proof is complete. ☐

Lemma 3.2.2. *If $\phi \in \mathcal{I}_{\mathrm{acc}}(M_\nu)$ then $\mathscr{C}^G_M(\phi)$ is contained in $\mathcal{I}_{\mathrm{acc}}(G^{\mathfrak{e}}_\omega)$.*

Proof. The proof of Lemma 3.3.5 below works verbatim: just replace stable orbital integrals there with ordinary orbital integrals and use the identity (3.2.2) as well as the vanishing statement in the same sentence. (Since Lemma 3.3.5 is more general, we supply a detailed argument only for the latter.) ☐

3.3. **ν-ascent and endoscopic transfer.** In this subsection we assume that G is quasi-split over F. Let $\mathfrak{e} = (G^{\mathfrak{e}}, \mathcal{G}^{\mathfrak{e}}, s^{\mathfrak{e}}, \eta^{\mathfrak{e}})$ be an endoscopic datum for G such that $\mathcal{G}^{\mathfrak{e}} = {}^LG^{\mathfrak{e}}$. (As usual, the last condition will be removed via z-extensions.) Here we fix Γ_F-pinnings $(\mathcal{B}^{\mathfrak{e}}, \mathcal{T}^{\mathfrak{e}}, \{\mathcal{X}_{\alpha^{\mathfrak{e}}}\})$ and $(\mathcal{B}, \mathcal{T}, \{\mathcal{X}_\alpha\})$ for $G^{\mathfrak{e}}$ and G, respectively. Conjugating $\eta^{\mathfrak{e}}$ we may and will assume that $\eta^{\mathfrak{e}}(\mathcal{T}^{\mathfrak{e}}) = \mathcal{T}$ and $\eta^{\mathfrak{e}}(\mathcal{B}^{\mathfrak{e}}) \subset \mathcal{B}$.

We have a standard embedding ${}^LP_\nu \hookrightarrow {}^LG$ and a Levi subgroup ${}^LM_\nu \subset {}^LP_\nu$ as in [Bor79, 3.3, 3.4]. Choose a subtorus $S \subset \mathcal{T}$ such that $\mathrm{Cent}(S, {}^LG) = {}^LM_\nu$. (This is possible by [Bor79, Lem. 3.5].) Following [Xu17, §6], define

$$\Omega^G(\mathfrak{e}, \nu) := \{\omega \in \Omega^G | \mathrm{Cent}(\omega(S), {}^LG^{\mathfrak{e}}) \to W_F \text{ is surjective}\}$$

and $\Omega_{\mathfrak{e},\nu} := \Omega^{G^{\mathfrak{e}}} \backslash \Omega^G(\mathfrak{e}, M_\nu)/\Omega^{M_\nu}$. For each $\omega \in \Omega_{\mathfrak{e},\nu}$, we obtain an endoscopic datum $\mathfrak{e}_\omega = (G^{\mathfrak{e}}_\omega, {}^LG^{\mathfrak{e}}_\omega, s^{\mathfrak{e}}_\omega, \eta^{\mathfrak{e}}_\omega)$ for $L = M_\nu$ as follows. (Henceforth we view ${}^LG^{\mathfrak{e}}$ as a subgroup of ${}^LG$ via $\eta^{\mathfrak{e}}$.) Pick $g \in \widehat{G}$ such that $\mathrm{Int}(g)$ induces ω on S. Then $g^LP_\nu g^{-1} \cap {}^LG^{\mathfrak{e}}$ is a parabolic subgroup of ${}^LG^{\mathfrak{e}}$ with Levi subgroup $g^LM_\nu g^{-1}$, so there is a corresponding standard parabolic subgroup $P^{\mathfrak{e}}_\nu = M^{\mathfrak{e}}_\nu N^{\mathfrak{e}}_\nu$ such that the standard embedding ${}^LP^{\mathfrak{e}}_\nu \hookrightarrow {}^LG^{\mathfrak{e}}$ (resp. ${}^LM^{\mathfrak{e}}_\nu \hookrightarrow {}^LG^{\mathfrak{e}}$) becomes $g^LP_\nu g^{-1} \cap {}^LG^{\mathfrak{e}}$ (resp. $g^LM_\nu g^{-1} \cap {}^LG^{\mathfrak{e}}$) after composing with $\mathrm{Int}(g^{\mathfrak{e}})$ for some $g^{\mathfrak{e}} \in \widehat{G}^{\mathfrak{e}}$. Then there is a unique L-embedding $\eta^{\mathfrak{e}}_\omega : {}^LM^{\mathfrak{e}}_\nu \hookrightarrow {}^LM_\nu$ such that $\mathrm{Int}(g) \circ \eta^{\mathfrak{e}}_\omega = \eta^{\mathfrak{e}} \circ \mathrm{Int}(g^{\mathfrak{e}})$. Set $G^{\mathfrak{e}}_\omega := M^{\mathfrak{e}}_\nu$, and $s^{\mathfrak{e}}_\omega := g^{-1}sg \in \widehat{M}_\nu$. Then it is a routine exercise to check that $(G^{\mathfrak{e}}_\omega, {}^LG^{\mathfrak{e}}_\omega, s^{\mathfrak{e}}_\omega, \eta^{\mathfrak{e}}_\omega)$ is an endoscopic datum for M_ν.

There is a canonical embedding $A_{M_\nu} \hookrightarrow A_{M_\nu^{\mathfrak{e}}} = A_{G_\omega^{\mathfrak{e}}}$ (just like $Z_H \hookrightarrow Z$ in §2.6). Composing with $\nu : \mathbb{G}_m \to A_{M_\nu}$, we obtain

$$\nu_\omega : \mathbb{G}_m \to A_{G_\omega^{\mathfrak{e}}}.$$

By the above construction, $G_\omega^{\mathfrak{e}} = M_\nu^{\mathfrak{e}}$ is a Levi subgroup of $G^{\mathfrak{e}}$ that is the centralizer of ν_ω. In particular we have a map $\mathcal{J}_{\nu_\omega} : \mathscr{S}(G_\omega^{\mathfrak{e}}) \to \mathcal{S}(G^{\mathfrak{e}})$ as in (3.1.2). Consider the following commutative diagram

$$\begin{array}{ccccccc} W_F & \longrightarrow & {}^L G_\omega^{\mathfrak{e}} & \xrightarrow{\eta_\omega^{\mathfrak{e}}} & {}^L M_\nu & \longrightarrow & {}^L Z^0_{M_\nu} \\ & & \big\downarrow {\scriptstyle \mathrm{Int}(g^{\mathfrak{e}})} & & \big\downarrow {\scriptstyle \mathrm{Int}(g)} & & \big\downarrow \\ W_F & \longrightarrow & {}^L G^{\mathfrak{e}} & \xrightarrow{\eta^{\mathfrak{e}}} & {}^L G & \longrightarrow & {}^L Z^0_G, \end{array} \tag{3.3.1}$$

where the maps out of W_F come from canonical splittings for the L-groups, the two horizontal maps on the right are induced by $Z^0_{M_\nu} \subset M_\nu$ and $Z^0_G \subset G$, the first two vertical maps correspond to the Levi embeddings above followed by $\mathrm{Int}(g^{\mathfrak{e}})$ and $\mathrm{Int}(g)$ respectively, and the rightmost vertical map is induced by $Z^0_G \subset Z^0_{M_\nu}$. (The left square in (3.3.1) commutes by $\mathrm{Int}(g) \circ \eta_\omega^{\mathfrak{e}} = \eta^{\mathfrak{e}} \circ \mathrm{Int}(g^{\mathfrak{e}})$ above. The commutativity of the right square is obvious since $\mathrm{Int}(g)$ acts trivially on ${}^L Z^0_G$.) Denote by

$$\lambda_\omega^{\mathfrak{e}} : Z^0_{M_\nu}(F) \to \mathbb{C}^\times \qquad (\text{resp. } \lambda^{\mathfrak{e}} : Z^0_G(F) \to \mathbb{C}^\times)$$

the smooth character corresponding to the composite morphism from W_F to ${}^L Z^0_{M_\nu}$ (resp. ${}^L Z^0_G$) in the first (resp. second) row. The character $\lambda^{\mathfrak{e}}$ is the same as in §2.6 except that the domain is restricted to the split center and that we work in a setup without z-extensions; cf. §3.5 about the latter point. The commutativity of (3.3.1) implies that $\lambda_\omega^{\mathfrak{e}}|_{Z^0_G(F)} = \lambda^{\mathfrak{e}}$. The canonical splittings from W_F to ${}^L G_\omega^{\mathfrak{e}}$ and ${}^L G^{\mathfrak{e}}$ commute with the Levi embedding ${}^L G_\omega^{\mathfrak{e}} \hookrightarrow {}^L G^{\mathfrak{e}}$ without $\mathrm{Int}(g^{\mathfrak{e}})$, but the point is that $\mathrm{Int}(g^{\mathfrak{e}})$ on ${}^L G^{\mathfrak{e}}$ is equivariant with the trivial action on ${}^L Z^0_G$ via the horizontal maps in (3.3.1).

Lemma 3.3.1. *Assume that $\eta^{\mathfrak{e}}(W_F)$ is a bounded subgroup of ${}^L G$ in the sense above Lemma 2.6.2. (This condition can always be ensured by that lemma.) Then $\lambda_\omega^{\mathfrak{e}}$ is a unitary character.*

Proof. By assumption and commutativity of , $\eta_\omega^{\mathfrak{e}}(W_F)$ is a bounded subgroup of ${}^L M_\nu$, whose image in ${}^L Z^0_{M_\nu}$ is a bounded subgroup accordingly. Therefore $\lambda_\omega^{\mathfrak{e}}$ is a unitary character via the Langlands correspondence for tori. □

Proposition 3.3.2. *The following diagram commutes.*

$$\begin{array}{ccccc} \mathcal{I}(M_\nu) & \xrightarrow{\mathscr{J}_\nu} & \mathcal{I}(G) & \xrightarrow{\mathrm{LS}^{\mathfrak{e}}} & \mathcal{S}(G^{\mathfrak{e}}) \\ & {\scriptstyle \oplus \mathrm{LS}^{\mathfrak{e},\omega}} \searrow & & \nearrow {\scriptstyle \sum_\omega \mathscr{J}_{\nu_\omega}} & \\ & & \bigoplus_{\omega \in \Omega_{\mathfrak{e},\nu}} \mathcal{S}(G_\omega^{\mathfrak{e}}) & & \end{array}$$

Let $\phi \in C_c^\infty(M_\nu(F))$. If $f^{(k)} = \mathscr{J}_\nu(\phi^{(k)})$ then writing $\phi_\omega^{(k)} := \mathrm{LS}^{\mathfrak{e},\omega}(\phi)^{(k)}$, we have

$$\phi_\omega^{(k)} = \lambda_\omega^{\mathfrak{e}}(\nu(\varpi))^{-k} \mathrm{LS}^{\mathfrak{e},\omega}(\phi^{(k)}), \qquad \mathrm{LS}^{\mathfrak{e}}(f^{(k)}) = \sum_{\omega \in \Omega_{\mathfrak{e},\nu}} \lambda_\omega^{\mathfrak{e}}(\nu(\varpi))^k \mathscr{J}_{\nu_\omega}\left(\phi_\omega^{(k)}\right).$$

Remark 3.3.3. When $\mathfrak{e}$ is given by a Levi subgroup M as in §3.2 (so that $G^{\mathfrak{e}} = M$), we have $\mathrm{LS}^{\mathfrak{e}} = \mathscr{C}^G_M$, $\mathrm{LS}^{\mathfrak{e},\omega} = \mathscr{C}^L_{L_\omega}$, and the meaning of ν_ω is consistent between §3.2 and §3.3. The diagram of the proposition almost reduces to that of Lemma 3.2.1 but with the following differences. First, the diagram is contracted in the lower left corner. Second, we would have only stable distributions on M and M_ω.

Proof. The first equality follows from the equivariance property of transfer as discussed in the paragraph containing (2.6.3) (applied to $z = \nu(\varpi)^{-k}$, $G_1 = M_\nu$, $G_1^{\mathfrak{e}} = G_\omega^{\mathfrak{e}}$, and $f = \phi$). The commutative diagram comes from (C.4) in [Xu17] (when θ is trivial). This, together with the first equality, implies the last equality. □

Corollary 3.3.4. *Let $\phi^{(k)}$, $\phi_\omega^{(k)}$, and $f^{(k)}$ be as in Proposition 3.3.2. Then*[5]

$$\mathrm{supp}^{SO}_{\mathfrak{a}_{G^{\mathfrak{e}}}}\left(\mathscr{I}_{\nu_\omega}(\phi_\omega^{(k)})\right) = k \cdot H^{G^{\mathfrak{e}}}(\nu_\omega(\varpi)) + \mathrm{pr}_{G^{\mathfrak{e}}}\left(\mathrm{supp}^{SO}_{\mathfrak{a}_{L_\omega}}(\mathrm{LS}^{\mathfrak{e},\omega}(\phi))\right), \qquad \omega \in \Omega_{\mathfrak{e},\nu},$$

where $\mathrm{pr}_{G^{\mathfrak{e}}} : \mathfrak{a}_{G_\omega^{\mathfrak{e}}} \twoheadrightarrow \mathfrak{a}_{G^{\mathfrak{e}}}$ *is the natural projection.*

Proof. By Lemma 3.1.8 and Proposition 3.3.2,

$$\begin{aligned}\mathrm{supp}^{SO}_{\mathfrak{a}_{G^{\mathfrak{e}}}}\left(\mathscr{I}_{\nu_\omega}(\phi_\omega^{(k)})\right) &= \mathrm{pr}_{G^{\mathfrak{e}}}\left(\mathrm{supp}^{SO}_{\mathfrak{a}_{G_\omega^{\mathfrak{e}}}}(\phi_\omega^{(k)})\right) = \mathrm{pr}_{G^{\mathfrak{e}}}\left(\mathrm{supp}^{SO}_{\mathfrak{a}_{G_\omega^{\mathfrak{e}}}}(\mathrm{LS}^{\mathfrak{e},\omega}(\phi^{(k)})).\right)\\ &= \mathrm{pr}_{G^{\mathfrak{e}}}\left(k \cdot H^{G_\omega^{\mathfrak{e}}}(\nu_\omega(\varpi)) + \mathrm{supp}^{SO}_{\mathfrak{a}_{G_\omega^{\mathfrak{e}}}}(\mathrm{LS}^{\mathfrak{e},\omega}(\phi))\right).\end{aligned}$$

We finish by observing that $\mathrm{pr}_{G^{\mathfrak{e}}}(H^{G_\omega^{\mathfrak{e}}}(\nu_\omega(\varpi))) = H^{G^{\mathfrak{e}}}(\nu_\omega(\varpi))$. □

It is useful to know preservation of acceptability in the setting of Proposition 3.3.2 as this will allow an inductive argument in the proof of Corollary 4.2.3 below.

Lemma 3.3.5. *If $\phi \in \mathcal{I}_{\mathrm{acc}}(M_\nu)$ then $\phi_\omega := \mathrm{LS}^{\mathfrak{e},\omega}(\phi)$ is contained in $\mathcal{S}_{\mathrm{acc}}(G_\omega^{\mathfrak{e}})$.*

Proof. Suppose that $SO_{\gamma_\omega}(\phi_\omega) \neq 0$ for some strongly M_ν-regular element $\gamma_\omega \in G_\omega^{\mathfrak{e}}(F)$. We need to check that γ_ω is ν_ω-acceptable. (It is enough to consider strongly M_ν-regular elements thanks to Lemma 3.1.6.)

By the transfer identity for orbital integrals, there exists $\gamma \in M_\nu(F)_{\mathrm{reg}}$ whose stable conjugacy class matches that of γ_ω such that $O_\gamma(\phi) \neq 0$. The latter implies that γ is ν-acceptable. Write T, T_ω for the centralizers of γ, γ_ω in M_ν, $G_\omega^{\mathfrak{e}}$, respectively. The matching of conjugacy classes tells us that there is a canonical F-isomorphism $i : T \simeq T_\omega$ which carries γ to γ_ω, cf. [Kot86, §3.1]. (A priori i sends the stable conjugacy class of γ to that of γ_ω and is canonical up to a Weyl group orbit. But i is determined if required to map γ to γ_ω.) Since ν is central in M_ν, the map i necessarily carries ν to ν_ω. Regarding T and T_ω as maximal tori of G and $G^{\mathfrak{e}}$, respectively, we have an injection $i^* : R(G_\omega^{\mathfrak{e}}, T_\omega) \hookrightarrow R(G, T)$ between the sets of roots induced by i (again [Kot86, §3.1]) such that

$$\langle \alpha, \nu_\omega \rangle = \langle i^*(\alpha), \nu \rangle, \qquad \alpha \in R(G_\omega^{\mathfrak{e}}, T_\omega). \tag{3.3.2}$$

We are ready to complete the proof. Let $\alpha \in R(G_\omega^{\mathfrak{e}}, T_\omega)$ such that $\langle \alpha, \nu_\omega \rangle > 0$. Showing that γ_ω is ν_ω-acceptable amounts to verifying that $|\alpha(\gamma_\omega)| < 1$, cf. Definition 3.1.1. But $\langle i^*(\alpha), \nu \rangle > 0$ by (3.3.2), so the ν-acceptability of γ implies that $|i^*(\alpha)(\gamma)| < 1$. Since $i^*(\alpha)(\gamma) = \alpha(\gamma_\omega)$, the proof is finished. □

3.4. **C-regular functions and constant terms.** Assume that G is *split* over F and fix a reductive model over $\mathcal{O}_F$, still denoted by G. Let T be a split maximal torus of G over $\mathcal{O}_F$. Let $C \in \mathbb{R}_{>0}$.

Definition 3.4.1. A cocharacter $\mu : \mathbb{G}_m \to T$ is **C-regular** if the following two conditions hold.

(1) $|\langle \alpha, \mu \rangle| > C$ for every $\alpha \in \Phi(T, G)$,
(2) $|\langle \alpha|_{A_M}, \mathrm{pr}_M(\omega\mu) \rangle| > C$ for every proper Levi subgroup M of G containing T, every $\omega \in \Omega^G$, and every $\alpha \in \Phi(T, G) \backslash \Phi(T, M)$.

Write $X_*(T)_{C\text{-reg}}$ for the set of C-regular cocharacters.

Lemma 3.4.2. *The following are true.*

(1) The subset $X_*(T)_{C\text{-reg}}$ of $X_*(T)$ is nonempty, and stable under both $\mathbb{Z}$-multiples and the Ω^G-action.

[5] An inclusion $\subset$ should be enough for later use, but it seems $=$ holds.

(2) If $\mu, \mu_0 \in X_*(T)$ and μ is C-regular, then there exists $k_0 \in \mathbb{Z}_{>0}$ such that $\mu_0+k\mu$ is C-regular for all $k \geq k_0$.

Proof. (1) Let $X_*(T)_{\mathbb{R},C\text{-reg}}$ denote the subset of $X_*(T)_{\mathbb{R}}$ defined by the same inequalities as in Definition 3.4.1. We choose an inner product on $X_*(T)_{\mathbb{R}}$ invariant under the Weyl group action. Clearly $X_*(T)_{C\text{-reg}}$ and $X_*(T)_{\mathbb{R},C\text{-reg}}$ are stable under $\mathbb{Z}$-multiples and the Weyl group action, and the latter is open. It suffices to show that $X_*(T)_{\mathbb{R},C\text{-reg}}$ is nonempty. Indeed, if so, we choose an open ball $U \subset X_*(T)_{\mathbb{R},C\text{-reg}}$. For $k \in \mathbb{Z}_{>0}$ large enough, $k \cdot U$ contains a point of $X_*(T)$, which then also lies in $X_*(T)_{\mathbb{R},C\text{-reg}}$.

Let us verify that $X_*(T)_{\mathbb{R},C\text{-reg}} \neq \emptyset$. Identify $X_*(T)_{\mathbb{R}}$ with the standard inner product space $\mathbb{R}^n$ via a linear isomorphism. Say that a measurable subset $A \subset \mathbb{R}^n$ has density 0 if $\mathrm{vol}(A \cap B(0,r))/\mathrm{vol}(B(0,r)) \to 0$ as $r \to \infty$. We will show that the complement of $X_*(T)_{\mathbb{R},C\text{-reg}}$ in $X_*(T)_{\mathbb{R}}$ is a density 0 set. Since a finite union of density 0 sets still has density 0, it is enough to check that each of the conditions $|\langle \alpha, \mu\rangle| \leq C$ and $|\langle \alpha|_{A_M}, \mathrm{pr}_M(\omega\mu)\rangle| \leq C$ defines a density 0 subset in $X_*(T)_{\mathbb{R}}$. Either condition defines a subset of the form

$$|a_1x_1 + \cdots + a_nx_n| \leq C \tag{3.4.1}$$

in the standard coordinates of $\mathbb{R}^n$, with coefficients $a_1, ..., a_n \in \mathbb{R}$. Notice that neither $\langle \alpha, \mu\rangle$ nor $\langle \alpha|_{A_M}, \mathrm{pr}_M(\omega\mu)\rangle$ is trivially zero on all $\mu \in X_*(T)_{\mathbb{R}}$. (In the latter case, the reason is that $\alpha|_{A_M} \in X^*(A_M)$ is nontrivial since $\alpha \notin \Phi(T, M)$, and that $\mathrm{pr}_M : X_*(T)_{\mathbb{R}} \to X_*(A_M)$ is surjective.) Therefore not all a_i's are zero. Now it is elementary to see that (3.4.1) gives a density 0 subset. The proof of (1) is complete.

(2) Observing that the pairings in Definition 3.4.1 are linear in μ, one easily checks that it is enough to choose k_0 such that $(k_0 - 1)C$ is greater than $|\langle \alpha, \mu\rangle|$ and $|\langle \alpha|_{A_M}, \mathrm{pr}_M(\omega\mu)\rangle|$ for all α, M, ω as in that definition. □

Define $T(F)_{C\text{-reg}}$ to be the union of $\mu(\varpi_F)T(\mathcal{O}_F^\times)$ as μ runs over the set of C-regular cocharacters. For each $M \in \mathcal{L}(G)$ containing T, set

$$\mathfrak{a}_{M,C\text{-reg}} := \{a \in \mathfrak{a}_M : |\langle \alpha, a\rangle| > C|\log|\varpi||,\ \forall \alpha \in \Phi^+(T,G)\backslash\Phi^+(T,M)\}. \tag{3.4.2}$$

Here we use the pairing $X^*(T)_{\mathbb{R}} \times X_*(T)_{\mathbb{R}} \to \mathbb{R}$ to compute $\langle \alpha, a\rangle$, viewing a in $X_*(T)_{\mathbb{R}}$ via $\mathfrak{a}_M = X_*(A_M)_{\mathbb{R}} \subset X_*(T)_{\mathbb{R}}$. Recall that $X_*(A_M)_{\mathbb{R}} \simeq \mathrm{Hom}(X^*(M)_{\mathbb{R}}, \mathbb{R})$ via $a \mapsto (\chi \mapsto \langle \chi, a\rangle)$. Analogously $X_*(A_T)_{\mathbb{R}} \simeq \mathrm{Hom}(X^*(T)_{\mathbb{R}}, \mathbb{R})$. Write $\mathrm{pr}_M : X_*(A_T)_{\mathbb{R}} \twoheadrightarrow X_*(A_M)_{\mathbb{R}}$ for the map induced by the restriction $X^*_F(M) \to X^*_F(T)$. (This is the analogue of pr_G in §3.1.)

$$\mathrm{pr}_M(\mu) = |\overline{\Omega}^M|^{-1} \sum_{\omega \in \overline{\Omega}^M} \omega\mu. \tag{3.4.3}$$

This follows from the fact that $\langle \chi|_T, \mu\rangle = \langle \chi, \mu\rangle = \langle \chi, \omega\mu\rangle$ for every $\omega \in \overline{\Omega}^M$.

Lemma 3.4.3. *Let $M \subsetneq G$ be a Levi subgroup containing T over F. Then $H^M(T(F)_{C\text{-reg}}) \subset \mathfrak{a}_{M,C\text{-reg}}$.*

Proof. Consider $t := \mu(\varpi)$ with $\mu \in X_*(T)_{C\text{-reg}}$. Then $H^M(t) \in \mathrm{Hom}(X^*(M)_{\mathbb{R}}, \mathbb{R})$ is identified with the unique element $a \in X_*(A_M)_{\mathbb{R}}$ such that

$$\langle \chi, a\rangle = \log|\chi(\mu(\varpi))| = \langle \chi, \mu\rangle \log|\varpi|, \qquad \chi \in X^*(M)_{\mathbb{R}}.$$

Let $\alpha \in \Phi(T,G)\backslash\Phi(T,M)$. Since the composite of the restriction maps $X^*_F(M)_{\mathbb{R}} \to X^*_F(T)_{\mathbb{R}} \to X^*(A_M)_{\mathbb{R}}$ is an isomorphism, we can find $\chi \in X^*_F(M)_{\mathbb{R}}$ such that $\chi|_{A_M} = \alpha|_{A_M}$. Hence

$$\begin{aligned}\langle \alpha, a\rangle = \langle \alpha|_{A_M}, a\rangle = \langle \chi|_{A_M}, a\rangle = \langle \chi, a\rangle = \langle \chi, \mu\rangle \log|\varpi| = \langle \chi|_{A_T}, \mu\rangle \log|\varpi| \\ = \langle \chi, \mathrm{pr}_M(\mu)\rangle \log|\varpi| = \langle \alpha|_{A_M}, \mathrm{pr}_M(\mu)\rangle \log|\varpi|.\end{aligned}$$

Since μ is C-regular, $|\langle \alpha|_{A_M}, \mathrm{pr}_M(\mu)\rangle| > C$. Hence $|\langle \alpha, a\rangle| > C|\log|\varpi||$. □

The following definition is motivated by [FK88, p.195].

Definition 3.4.4. Let $C > 0$. We say $f \in \mathcal{H}(G)$ is **C-regular** if $\mathrm{supp}(f)$ is contained in the $G(F)$-conjugacy orbit of $T(F)_{C\text{-reg}}$.[6] Write $\mathcal{H}(G)_{C\text{-reg}}$ for the space of C-regular functions.

Lemma 3.4.5. *Let $f' \in \mathcal{H}(G)$. Assume that every $g \in G(F)_{\mathrm{reg}}$ such that $O_g(f') \neq 0$ (resp. $SO_g(f') \neq 0$) is $G(F)$-conjugate (resp. stably conjugate) to an element of $T(F)_{C\text{-reg}}$. Then*

(1) $O_g(f') = 0$ (resp. $SO_g(f') = 0$) if $g \in G(F)_{\mathrm{ss}}$ is not regular, and
(2) there exists $f \in \mathcal{H}(G)_{C\text{-reg}}$ such that f and f' have the same image in $\mathcal{I}(G)$ (resp. $\mathcal{S}(G)$).

Proof. (1) If $g \in G(F)_{\mathrm{ss}}$ is not regular then no regular element in a sufficiently small neighborhood of g intersects the $G(F)$-orbit of $T(F)_{C\text{-reg}}$. (Since every $t \in T(F)_{C\text{-reg}}$ satisfies $|1-\alpha(t)| = 1$ for $\alpha \in \Phi(T,G)$, no $\alpha(t)$ approaches 1.) Thus f' has vanishing regular orbital integrals in a neighborhood of g. This implies that $O_g(f') = 0$, by an argument as in the proof of [Rog83, Lem. 2.6] via the Shalika germ expansion around g. The case of stable orbital integrals is analogous.

(2) The point is that the $G(F)$-conjugacy orbit of $T(F)_{C\text{-reg}}$ is open and closed in $G(F)$. (Since $T(F)_{C\text{-reg}}$ is open and closed in $T(F)$, and the map $G(F)/T(F) \times T(F) \to G(F)$ induced by $(g,t) \mapsto gtg^{-1}$ is a local isomorphism.) Thus the product of f' and the characteristic function on the latter orbit is smooth and compactly supported, and thus belongs to $\mathcal{H}(G)_{C\text{-reg}}$. Denoting the product by f, we see that f and f' have equal orbital integrals (resp. stable orbital integrals) on regular semisimple elements. Therefore have the same image in $\mathcal{I}(G)$ (resp. $\mathcal{S}(G)$). □

Corollary 3.4.6. *Fix a C-regular cocharacter $\mu : \mathbb{G}_m \to T$. Let $\phi \in \mathcal{H}(T)$. Then there exists an integer $k_0 = k_0(\phi)$ such that for every integer $k \geq k_0$, the μ-ascent of $\phi^{(k)}$ is represented by a C-regular function on $G(F)$.*

Proof. There is a finite subset $\mu_0 \in X_*(T)$ such that $\mathrm{supp}(\phi) \subset \cup_{\mu_0 \in \mu_0} \mu_0(\varpi_F)T(\mathcal{O}_F)$. Applying Lemma 3.4.2 to each member of μ_0 and also Lemma 3.1.8, we can find $k_0 = k_0(\phi) \in \mathbb{Z}_{\geq 0}$ such that $\phi^{(k)}$ is μ-acceptable and $\mathrm{supp}(\phi^{(k)}) \subset T(F)_{C\text{-reg}}$ for all $k \geq k_0$. Write $f^{(k)}$ for a μ-ascent of $\phi^{(k)}$. By Lemma 3.4.5 it suffices to check for each $k \geq k_0$ and $g \in G(F)_{\mathrm{reg}}$ that if $O_g(f^{(k)}) \neq 0$ then g is in the $G(F)$-orbit of $T(F)_{C\text{-reg}}$. This follows from the observed properties of $\phi^{(k)}$ by Lemma 3.1.3. □

Lemma 3.4.7. *Let $f \in \mathcal{H}(G)_{C\text{-reg}}$, $M \in \mathcal{L}^{<}(G)$, and $\mathfrak{e} \in \mathcal{E}^{<}(G)$. The following are true.*

(1) $\mathscr{C}^G_M(f) \in \mathcal{I}(M)$ is represented by a function $f_M \in \mathcal{H}(M)$ whose support is contained in the $M(F)$-conjugacy orbit of $T(F)_{C\text{-reg}}$. (In particular f_M is a C-regular function on $M(F)$.)
(2) $\mathrm{LS}^{\mathfrak{e}}(f) \in \mathcal{S}(G^{\mathfrak{e}})$ is represented by a C-regular function on $G^{\mathfrak{e}}(F)$.

Proof. (1) We keep writing $T(F)_{C\text{-reg}}$ for the set of C-regular elements relative to G, which contain C-regular elements relative to M. Since $T(F)_{C\text{-reg}}$ is invariant under the Weyl group of G, an element $\gamma \in M(F)_{\mathrm{ss}}$ is conjugate to an element of $T(F)_{C\text{-reg}}$ in $G(F)$ if and only if it is so in $M(F)$. In light of Lemma 3.4.5, it suffices to show the following: if $O_\gamma(\mathscr{C}^G_M(f)) \neq 0$ for regular semisimple $\gamma \in M(F)$ then γ is $M(F)$-conjugate to an element of $T(F)_{C\text{-reg}}$.

If γ is G-regular then we have from §3.2 that $O_\gamma(f) = D_{G/M}(\gamma)O_\gamma(\mathscr{C}^G_M(f))$, which is nonzero only if γ is conjugate to an element of $T(F)_{C\text{-reg}}$. If γ is regular but outside the $M(F)$-orbit of $T(F)_{C\text{-reg}}$, then a sufficiently small neighborhood V of γ does not intersect the $M(F)$-orbit of $T(F)_{C\text{-reg}}$. On the other hand, G-regular elements are dense in V. Since an orbital integral is locally constant on the regular semisimple set, it follows that $O_\gamma(\mathscr{C}^G_M(f)) = 0$.

(2) If $G^{\mathfrak{e}}$ is ramified over F then T does not transfer to a maximal torus in $G^{\mathfrak{e}}$. So $\mathrm{LS}^{\mathfrak{e}}(f) = 0$ and the assertion holds. Otherwise T transfers to a maximal torus $T^{\mathfrak{e}} \subset G^{\mathfrak{e}}$, equipped with an F-isomorphism $T \simeq T^{\mathfrak{e}}$ (canonical up to the Weyl group action). Via the isomorphism we transport λ

[6] In practice it seems enough to impose the condition on $\mathrm{supp}^O(f)$. However when producing examples of C-regular f, often we have this condition satisfied.

to $\lambda^{\mathfrak{e}} : \mathbb{G}_m \to T^{\mathfrak{e}}$ and identify $\Phi(T^{\mathfrak{e}}, G^{\mathfrak{e}})$ as a subset of $\Phi(T, G)$. By abuse of notation, keep writing $T(F)_{C\text{-reg}}$ for its image in $T^{\mathfrak{e}}(F)$. Then C-regular elements of $T^{\mathfrak{e}}(F)$ are contained in $T(F)_{C\text{-reg}}$.

Now the rest of the proof of (2) similar to that of (1), based on Lemma 3.4.5. It suffices to check that if $SO_{\gamma^{\mathfrak{e}}}(\mathrm{LS}^{\mathfrak{e}}(f)) \neq 0$ for G-regular semisimple $\gamma^{\mathfrak{e}} \in G^{\mathfrak{e}}(F)$ then $\gamma^{\mathfrak{e}}$ is stably conjugate to an element of $T(F)_{C\text{-reg}}$. This is evident from the transfer of orbital integral identity. □

Corollary 3.4.8. *For $f \in \mathcal{H}(G)_{C\text{-reg}}$ and $M \in \mathcal{L}^{<}(G)$, we have* $\mathrm{supp}^{O}_{\mathfrak{a}_M}(\mathscr{C}^G_M(f)) \subset \mathfrak{a}_{M,C\text{-reg}}$.

Proof. Let f_M be as in the preceding lemma. Then

$$\mathrm{supp}^{O}_{\mathfrak{a}_M}(\mathscr{C}^G_M(f)) = \mathrm{supp}^{O}_{\mathfrak{a}_M}(f_M) \subset \mathrm{supp}_{\mathfrak{a}_M}(f_M) \subset H^M(T(F)_{C\text{-reg}}) \subset \mathfrak{a}_{M,C\text{-reg}},$$

where the last inclusion comes from Lemma 3.4.3. □

The following lemma, to be invoked in §7.5, sheds light on how much C-regular functions detect.

Lemma 3.4.9. *Let I be a finite set and let $C > 0$. Let $\pi_i \in \mathrm{Irr}(G(F))$ and $c_i \in \mathbb{C}$ for $i \in I$. If*

$$\sum_{i \in I} c_i \mathrm{Tr}\, \pi_i(f) = 0 \qquad \forall\ f \in \mathcal{H}(G)_{C\text{-reg}}$$

then $\sum_{i\in I} J_{P_0}(\pi_i) = 0$ in $\mathrm{Groth}(G(F)) \otimes_{\mathbb{Z}} \mathbb{C}$.

Proof. Fix a regular cocharacter $\mu : \mathbb{G}_m \to T$ over F such that $P_0 = P_\mu^{\mathrm{op}}$. For each $\phi \in \mathcal{H}(T)$, we have some integer k_0 such that $\phi^{(k)}$ are μ-acceptable for all $k \geq k_0$ and their μ-ascent $f^{(k)}$ are represented by C-regular functions by Corollary 3.4.6. Thanks to Lemma 3.1.3,

$$0 = \sum_{i \in I} c_i \mathrm{Tr}\,(f^{(k)}|\pi_i) = \sum_{i\in I} c_i \mathrm{Tr}\,(\phi^{(k)}|J_{P_0}(\pi_i)), \quad \forall k \geq k_0.$$

We conclude by Lemma 3.1.9. □

3.5. **Fixed central character.** We explain that the facts thus far in §3 hold in the setup with fixed central character. Let $\nu : \mathbb{G}_m \to G$ be a cocharacter over F and $(G^{\mathfrak{e}}, \mathcal{G}^{\mathfrak{e}}, s^{\mathfrak{e}}, \eta^{\mathfrak{e}})$ an endoscopic datum for G with $\mathcal{G}^{\mathfrak{e}} = {}^L G^{\mathfrak{e}}$. We can view $\mathfrak{X}$ as a closed subgroup of $M_\nu(F)$, $G^{\mathfrak{e}}(F)$, and $G^{\mathfrak{e}}_\omega(F)$ of the preceding sections via the canonical embeddings of $Z(F)$ into their centers.

As before, $\mathcal{H}_{\mathrm{acc}}(M_\nu, \chi^{-1}) \subset \mathcal{H}(M_\nu, \chi^{-1})$ is the subspace of functions which are supported on acceptable elements. Taking the image, we also have $\mathcal{I}_{\mathrm{acc}}(M_\nu, \chi^{-1})$ and $\mathcal{S}_{\mathrm{acc}}(M_\nu, \chi^{-1})$. Since the set of acceptable elements on $M_\nu(\overline{F})$ is invariant under translation by $Z(\overline{F})$, the χ-averaging map induces a surjection $\mathcal{H}_{\mathrm{acc}}(M_\nu) \to \mathcal{H}_{\mathrm{acc}}(M_\nu, \chi^{-1})$. The analogous surjectivity holds for $\mathcal{I}_{\mathrm{acc}}$ and $\mathcal{S}_{\mathrm{acc}}$.

The earlier results continue to hold with fixed central character, with the following minor modifications. We omit the proofs as no new ideas are required; the basic idea is to apply χ-averaging consistently.

To adapt §3.1. Averaging the ν-ascent map, we obtain

$$\mathscr{J}_\nu : \mathcal{I}(M_\nu, \chi^{-1}) \to \mathcal{I}(G, \chi^{-1}), \qquad \mathscr{J}_\nu : \mathcal{S}(M_\nu, \chi^{-1}) \to \mathcal{S}(G, \chi^{-1})$$

satisfying the orbital integral and trace identities in Lemma 3.1.3 (with central character of π equal to χ) and Corollary 3.1.4. The evident analogues of Lemmas 3.1.6 and 3.1.7 hold true (with no changes to the bottom rows in the latter lemma). The map $(\cdot)^{(k)}$ in (3.1.6) induces linear automorphisms on $\mathcal{I}(M_\nu, \chi^{-1})$ and $\mathcal{I}(G, \chi^{-1})$. With this, Lemmas 3.1.8 and 3.1.9 imply their natural analogues, restricting π_1, π_2 in the latter lemma to those with central character χ.

To adapt §3.2. Averaging the map $\mathcal{H}(G) \to \mathcal{H}(M)$ given by $f \mapsto f_M$, we obtain a map $\mathcal{H}(G, \chi^{-1}) \to \mathcal{H}(M, \chi^{-1})$, which induces

$$\mathscr{C}^G_M : \mathcal{I}(G, \chi^{-1}) \to \mathcal{I}(M, \chi^{-1})$$

satisfying the same orbital integral identity as in §3.2. We can also describe $\mathscr{C}^G_M$ by the same formula (3.2.1) from the space of linear functionals on $R(G, \chi)$ to that on $R(M, \chi)$. Lemmas 3.2.1 and 3.2.2 carry over as written, with χ^{-1}-equivariance imposed everywhere.

To adapt §3.3. The Langlands–Shelstad transfer with fixed central character was already considered in §2.6 by averaging the transfer without fixed central character. With this in mind, we deduce the obvious analogues of Proposition 3.3.2, Corollary 3.3.4, and Lemma 3.3.5. In particular, the diagram in that proposition yields the following analogue.

$$\begin{array}{ccccc} \mathcal{I}(M_\nu, \chi^{-1}) & \xrightarrow{\mathscr{J}_\nu} & \mathcal{I}(G, \chi^{-1}) & \xrightarrow{\mathrm{LS}^{\mathfrak{e}}} & \mathcal{S}(G^{\mathfrak{e}}, \chi^{\mathfrak{e},-1}) \\ & \searrow^{\oplus \mathrm{LS}^{\mathfrak{e},\omega}} & & \nearrow_{\sum_\omega \mathscr{J}_{\nu_\omega}} & \\ & & \bigoplus_{\omega \in \Omega_{\mathfrak{e},\nu}} \mathcal{S}(G^{\mathfrak{e}}_\omega, \chi^{\mathfrak{e},-1}) & & \end{array}$$

To adapt §3.4. Definition 3.4.4 extends obviously to $\mathcal{H}(G, \chi^{-1})$ by the same support condition. A key observation is that the notion of C-regularity is invariant under $Z(F)$-translation, so that the latter definition behaves well. More precisely, the χ-averaging map from $\mathcal{H}(G) \twoheadrightarrow \mathcal{H}(G, \chi^{-1})$ is still surjective when restricted to the respective subspaces of C-regular functions. Using this, we carry over all results in §3.4 to the setup with fixed central character, restricting to χ^{-1}-equivariant functions and representations with central character χ.

3.6. z**-extensions.** Throughout this section up to now, we assumed $\mathcal{G}^{\mathfrak{e}} = {}^L G^{\mathfrak{e}}$ on the endoscopic datum $\mathfrak{e}$. When the assumption is not guaranteed, we pass from $\mathfrak{e}$ and G to $\mathfrak{e}_1$ and G_1 via z-extensions and pull back the central character datum from $(\mathfrak{X}, \chi)$ to $(\mathfrak{X}_1, \chi_1)$ as explained in §2.7.

Lemma 3.6.1. *There exists a cocharacter $\nu_1 : \mathbb{G}_m \to G_1$ over F lifting $\nu : \mathbb{G}_m \to G$.*

Proof. It suffices to show the claim that if $1 \to T_1 \to T_2 \to T_3 \to 1$ is a short exact sequence of tori over F and if T_1 is a finite product of induced tori of the form $\mathrm{Res}_{F'/F}\mathbb{G}_m$, then $\mathrm{Hom}_F(\mathbb{G}_m, T_2) \to \mathrm{Hom}_F(\mathbb{G}_m, T_3)$ is onto. To see this, we apply X_* to obtain a short exact sequence of $\mathrm{Gal}(\overline{F}/F)$-modules $0 \to X_*(T_1) \to X_*(T_2) \to X_*(T_3) \to 0$. Since $H^1(\mathrm{Gal}(\overline{F}/F), X_*(T_1))$ vanishes by hypothesis via Shapiro's lemma, we deduce that $X_*(T_2)^{\mathrm{Gal}(\overline{F}/F)} \to X_*(T_3)^{\mathrm{Gal}(\overline{F}/F)}$ is onto, which is what we wanted to show. □

Let us fix ν_1 as above. By Definition 3.1.1, $\gamma_1 \in M_{\nu_1}(F)$ is ν_1-acceptable if and only if its image in $M_\nu(F)$ is ν-acceptable. Everything in this section goes through with $\mathfrak{e}_1, G_1, (\mathfrak{X}_1, \chi_1), \nu_1$ playing the roles of $\mathfrak{e}, G, (\mathfrak{X}, \chi), \nu$. We will write $\lambda^{\mathfrak{e}}_1, \lambda^{\mathfrak{e}}_{1,\omega}$ for the characters $\lambda^{\mathfrak{e}}, \lambda^{\mathfrak{e}}_\omega$ of §3.3 in the setup for $\mathfrak{e}_1$ and G_1.

4. Asymptotic analysis of the trace formula

We prove key trace formula estimates in this section, to be applied to identify leading terms in the trace formula for Igusa varieties in §7. The main estimate is Theorem 4.2.2, whose lengthy proof is presented in §4.4. We work in a purely group-theoretic setup, with no reference to Shimura or Igusa varieties, in order to enable an inductive argument on $\mathbb{Q}$-semisimple rank. The point is that the trace formula appearing in the intermediate steps need not be associated with any geometry.

4.1. **Setup and some basic lemmas.** Throughout Section 4, G is a connected *quasi-split* reductive group over $\mathbb{Q}$ which is cuspidal, i.e., Z_G^0 has the same $\mathbb{Q}$-split rank and $\mathbb{R}$-split rank. Let $(\mathfrak{X}, \chi)$ be a central character datum as in §2.8. Let ξ be an irreducible algebraic representation of $G_{\mathbb{C}}$ and $\zeta : G(\mathbb{R}) \to \mathbb{C}^\times$ be a continuous character such that $\xi \otimes \zeta$ has central character χ_∞^{-1} on $\mathfrak{X}_\infty$. The restriction $\chi_\infty|_{A_G(\mathbb{R})^0}$ via $A_G(\mathbb{R})^0 \subset \mathfrak{X}_\infty$ can be viewed as an element of $X^*(A_G)_{\mathbb{C}}$, which is again denoted χ_∞ by abuse of notation. Let λ_{χ_∞} denote the unique character making the following diagram commute. (The existence is obvious since the composition $A_G(\mathbb{R})^0 \to \mathfrak{a}_G$ is an isomorphism.)

$$A_G(\mathbb{R})^0 \hookrightarrow G(\mathbb{R}) \xrightarrow{H_\infty^G} \mathfrak{a}_G = X_*(A_G)_{\mathbb{R}} \xrightarrow{\lambda_{\chi_\infty}} \mathbb{C}^\times, \qquad \chi_\infty : A_G(\mathbb{R})^0 \to \mathbb{C}^\times$$

We have a canonical identification

$$\mathfrak{a}_G = X_*(A_G)_{\mathbb{R}} = \mathrm{Hom}(X^*(G)_{\mathbb{Q}}, \mathbb{R}), \qquad a \mapsto (\chi \mapsto \langle \chi|_{A_G}, a\rangle). \tag{4.1.1}$$

Fix distinct primes p, q. Let $\nu : \mathbb{G}_m \to G_{\mathbb{Q}_p}$ be a cocharacter over $\mathbb{Q}_p$. Let $\overline{\nu} \in \mathrm{Hom}(X^*_{\mathbb{Q}}(G), \mathbb{Q})$ denote the image of $\nu \in X_*(A_{M_\nu})_{\mathbb{Q}} = \mathrm{Hom}(X^*_{\mathbb{Q}}(M_\nu), \mathbb{Q})$ induced by $M_\nu \hookrightarrow G$.[7] By definition, $\overline{\nu}(\chi) = \nu(\chi)$ for $\chi \in X^*_{\mathbb{Q}}(G)$. Viewing $\overline{\nu}$ as a member of $\mathfrak{a}_G$, we can compute $\langle \chi_\infty, \overline{\nu}\rangle \in \mathbb{C}$ via the canonical pairing $X^*(A_G)_{\mathbb{C}} \times X_*(A_G)_{\mathbb{C}} \to \mathbb{C}$.

Lemma 4.1.1. $\lambda_{\chi_\infty}(H_p^G(\nu(p))) = p^{-\langle \chi_\infty, \overline{\nu}\rangle}$.

Proof. By definition, $H_p^G(\nu(p))$ sends $\chi \in X^*_{\mathbb{Q}}(G)$ to $\log|\chi(\nu(p))|_p$. Similarly for $a \in A_G(\mathbb{R})^0$, we have $H_\infty^G(a) = (\chi \mapsto \log|\chi(a)|_\infty)$. We claim that $H_p^G(\nu(p)) = H_\infty^G(\overline{\nu}(p)^{-1})$. To show this, choose $r \in \mathbb{Z}_{\geq 1}$ such that $r\overline{\nu} \in X_*(A_G)$. Since $\mathfrak{a}_G$ is torsion-free, it suffices to check that $H_p^G((r\nu)(p)) = H_\infty^G((r\overline{\nu})(p)^{-1})$, or equivalently that

$$|\chi((r\nu)(p))|_p = |\chi((r\overline{\nu})(p))|_\infty^{-1}, \qquad \chi \in X^*_{\mathbb{Q}}(G).$$

Since $\chi((r\overline{\nu})(p)) \in \mathbb{Q}$ is an integral power of p (as both χ and $r\overline{\nu}$ are algebraic), we have $|\chi((r\overline{\nu})(p))|_\infty = |\chi((r\overline{\nu})(p))|_p^{-1} = |\chi((r\nu)(p))|_p^{-1}$. This proves the claim. Now the claim implies that

$$\lambda_{\chi_\infty}(H_p^G(\nu(p))) = \lambda_{\chi_\infty}(H_\infty^G(\overline{\nu}(p)^{-1})) = \chi_\infty(\overline{\nu}(p)^{-1}) = p^{-\langle \chi_\infty, \overline{\nu}\rangle}.$$

□

If G is a connected reductive group over $\mathbb{Q}$ and S is a set of $\mathbb{Q}$-places, we write

$$H_S^G(\gamma) := \sum_{v \in S} H_v^G(\gamma) \in \mathfrak{a}_G.$$

If S^c is the complement of S, we write $H^{G,S^c} := H_S^G$.

Lemma 4.1.2. *Let G be a connected reductive group over $\mathbb{Q}$.*

(i) Let S be a set of $\mathbb{Q}$-places. Let $\gamma, \gamma' \in G(\mathbb{Q})$. If γ and γ' are conjugate in $G(\overline{\mathbb{Q}})$ then we have $H_S^G(\gamma) = H_S^G(\gamma') \in \mathfrak{a}_G$.

(ii) Let S be the set of all $\mathbb{Q}$-places. Let $\gamma \in G(\mathbb{Q})$, then $H_S^G(\gamma) = 0 \in \mathfrak{a}_G$.

Proof. (*i*) Let $F/\mathbb{Q}$ be a finite extension such that γ and γ' are conjugate in $G(F)$. Set $G' := \mathrm{Res}_{F/\mathbb{Q}} M$. The natural embedding $i : G \to G'$ allows to view γ, γ' as elements of $G'(\mathbb{Q})$, and induces an injection $\mathfrak{a}_G \hookrightarrow \mathfrak{a}_{G'}$. Thus it suffices to prove that $H_S^{G'}(\gamma) = H_S^{G'}(\gamma')$, since the map $H_S^G : G(\mathbb{A}_S) \to \mathfrak{a}_G$ is functorial with respect to i. By the reduction in the preceding paragraph,

[7] In the notation of the preceding section, $\overline{\nu} = \mathrm{pr}_G(\nu)$.

we may assume that γ and γ' are conjugate in $G(\mathbb{Q})$. Then the proof is trivial since H_S^G is a homomorphism into an abelian group.

(*ii*). Using the functoriality for $G \to Z'_G$ from step 1, we may replace G by its cocenter Z'_G, then Z'_G by the maximally split torus A inside Z'_G, and finally we may replace A by $\mathbb{G}_m$, in which case the statement boils down to the usual product formula. □

The following will be useful when studying Levi terms in the geometric side of the trace formula.

Lemma 4.1.3. *Let $M \in \mathcal{L}_{\mathrm{cusp}}(G)$ and let $\gamma \in \Gamma_{\mathrm{ell},\mathfrak{X}}(M)$ be a regular element. Let $P \subset G$ be a parabolic subgroup with Levi component M. Let $f^\infty \in \mathcal{H}(M(\mathbb{A}^\infty), \chi^{\infty,-1})$ and let ξ be an irreducible representation of $M_{\mathbb{C}}$. Then we have*

$$\begin{aligned}&\mathrm{vol}(\mathfrak{X}_{\mathbb{Q}}\backslash\mathfrak{X}/A_{G,\infty})^{-1}\chi(I_\gamma^M)\zeta(\gamma)\mathrm{Tr}\,(\gamma;\xi\delta_P^{1/2})O_\gamma^M(f^\infty)\\ &= \mathrm{vol}(I_\gamma^M(\mathbb{Q})A_{I_\gamma^M,\infty}\backslash I_\gamma^M(\mathbb{A})/\mathfrak{X})O_\gamma^M(f_{\xi,\zeta\delta_P^{1/2}}f^\infty).\end{aligned}$$

Proof. By Equation (2.8.3) we have

$$\chi(I_\gamma^M) = (-1)^{q(I_\gamma^M)}\tau(I_\gamma^M)\mathrm{vol}(A_{I^M_{\gamma,\infty}}\backslash I_\gamma^{M,\mathrm{cmpt}}(\mathbb{R}))^{-1}d(I_\gamma^M) = 1.$$

As γ is regular, I_γ^M is a torus and so $d(I_\gamma^M) = 1$. Additionally we have $q(I_\gamma^M) = 0$ (as γ is elliptic) and

$$\tau(I_\gamma^M) = \mathrm{vol}(I_\gamma^M(\mathbb{Q})A_{I_\gamma^M,\infty}\backslash I_\gamma^M(\mathbb{A})).$$

Thus we obtain

$$\chi(I_\gamma^M) = \mathrm{vol}(I_\gamma^M(\mathbb{Q})A_{I_\gamma^M,\infty}\backslash I_\gamma^M(\mathbb{A}))\mathrm{vol}(A_{I^M_{\gamma,\infty}}\backslash I_\gamma^{M,\mathrm{cmpt}}(\mathbb{R}))^{-1}.$$

By (2.4.1) and $e(I_\gamma^M) = 1$, we have

$$\zeta(\gamma)\mathrm{Tr}\,(\gamma;\xi\delta_P^{1/2}) = \mathrm{vol}(A_{M,\infty}\backslash I_\gamma^M(\mathbb{R}))O_\gamma^M(f_{\xi,\zeta\delta_P^{1/2}}).$$

We obtain

$$\begin{aligned}&\mathrm{vol}(\mathfrak{X}_{\mathbb{Q}}\backslash\mathfrak{X}/A_{G,\infty})^{-1}\chi(I_\gamma^M)\zeta(\gamma)\mathrm{Tr}\,(\gamma;\xi\delta_P^{1/2})O_\gamma^M(f^\infty) =\\ &= \frac{\mathrm{vol}(I_\gamma^M(\mathbb{Q})A_{I_\gamma^M,\infty}\backslash I_\gamma^M(\mathbb{A}))\mathrm{vol}(A_{M,\infty}\backslash I_\gamma^M(\mathbb{R}))}{\mathrm{vol}(\mathfrak{X}_{\mathbb{Q}}\backslash\mathfrak{X}/A_{G,\infty})\mathrm{vol}(A_{I_\gamma^M,\infty}\backslash I_\gamma^{M,\mathrm{cmpt}}(\mathbb{R}))}O_\gamma^M(f_{\xi,\zeta\delta_P^{1/2}}f^\infty)\\ &= \frac{\mathrm{vol}(I_\gamma^M(\mathbb{Q})A_{I_\gamma^M,\infty}\backslash I_\gamma^M(\mathbb{A}))}{\mathrm{vol}(\mathfrak{X}_{\mathbb{Q}}\backslash\mathfrak{X}/A_{G,\infty})}O_\gamma^M(f_{\xi,\zeta\delta_P^{1/2}}f^\infty)\\ &= \mathrm{vol}(I_\gamma^M(\mathbb{Q})A_{I_\gamma^M,\infty}\backslash I_\gamma^M(\mathbb{A})/\mathfrak{X}).O_\gamma^M(f_{\xi,\zeta\delta_P^{1/2}}f^\infty)\end{aligned}$$

□

4.2. **The main estimate and its consequences.** We prove the following bounds for elliptic endoscopic groups and Levi subgroups of G, to be applied in §7.

The notation $O(f(k))$ (resp. $o(k)$) for a nonzero $\mathbb{C}$-valued function $f(k)$ on $k \in \mathbb{Z}$ means that the quantity divided by $|f(k)|$ has bounded absolute value (resp. tends to 0) as $k \to +\infty$. In fact we only take $f(k)$ to be complex powers of p (but not necessarily real powers; that is why we take absolute values). As the reader will see, every instance of $o(f(k))$ represents a power-saving, namely it is bounded by a power of p with (the real part of) exponent strictly smaller than the exponent for $f(k)$.

Let us fix a $\mathbb{Q}$-rational Borel subgroup B with Levi component $T \subset B$ (which is a maximal torus in G). We fix a Levi decomposition $B = TN_0$. As before we write $A_T \subset T$ for the maximal $\mathbb{Q}$-split subtorus. Additionally we write $S_p \subset T_{\mathbb{Q}_p}$ for the maximal $\mathbb{Q}_p$-split subtorus.

Part of our setup is a cocharacter $\nu\colon \mathbb{G}_m \to G$ over $\mathbb{Q}_p$. By conjugating ν if necessary, we may and will assume that ν has image in T and that ν is B-dominant. Write $\rho \in X^*(T)_{\mathbb{Q}}$ for the half sum of all B-positive roots of T in G over $\overline{\mathbb{Q}}_p$. Thus we have $\langle \rho, \nu\rangle \in \frac{1}{2}\mathbb{Z}_{>0}$. We transport various data over $\mathbb{Q}_p$ or $\overline{\mathbb{Q}}_p$ to ones over $\mathbb{C}$ via a fixed isomorphism $\iota_p : \overline{\mathbb{Q}}_p \simeq \mathbb{C}$. (In the context of geometry, ι_p is fixed in §5.2.)

We also fix a prime q such that $G_{\mathbb{Q}_q}$ is a split group, so that the contents of §3.4 apply over $\mathbb{Q}_q$. (For the existence, choose a number field F over which G splits. Then any prime q that splits completely in F will do.)

Proposition 4.2.1. *Let $f^{\infty,p} = \prod_{v\neq\infty,p} f_v \in \mathcal{H}(G(\mathbb{A}^{\infty,p}), (\chi^{\infty,p})^{-1})$ and $\phi_p \in \mathcal{H}_{\mathrm{acc}}(M_\nu(\mathbb{Q}_p), \chi_p^{-1})$. For $k\in\mathbb{Z}$, write $f_p^{(k)} \in \mathcal{H}(G(\mathbb{Q}_p), \chi_p^{-1})$ for a ν-ascent of $\phi_p^{(k)}$ as in §3.2. Then*

$$T^G_{\mathrm{disc},\chi}(f_p^{(k)} f^{\infty,p} f_{\xi,\zeta}) = O\left(p^{k(\langle\rho,\nu\rangle + \langle \chi_\infty, \overline{\nu}\rangle)}\right).$$

Proof. The left hand side equals

$$\sum_{\pi\in\mathcal{A}_{\mathrm{disc},\chi}(G)} m(\pi)\mathrm{Tr}\,(f_p^{(k)}|\pi_p)\mathrm{Tr}\,(f^{\infty,p}|\pi^p)\mathrm{Tr}\,(f_{\xi,\zeta}|\pi_\infty).$$

Write $J_{P_\nu^{\mathrm{op}}}(\pi_p) = \sum_i c_i\tau_i$ in $\mathrm{Groth}(M_\nu(\mathbb{Q}_p))$ with $\tau_i \in \mathrm{Irr}(M_\nu(\mathbb{Q}_p))$. Let ω_{τ_i} denote the central character of τ_i. Then

$$\mathrm{Tr}\,(f_p^{(k)}|\pi_p) = \mathrm{Tr}\,\left(\phi_p^{(k)}|J_{P_\nu^{\mathrm{op}}}(\pi_p)\right) = \sum_i c_i \mathrm{Tr}\,(\phi_p^{(k)}|\tau_i) = \sum_i c_i\omega_{\tau_i}(\nu(p))^k\mathrm{Tr}\,(\phi_p|\tau_i).$$

We define a character $\lambda_{\mathbb{A}} : G(\mathbb{Q})\backslash G(\mathbb{A}) \to \mathbb{R}^\times_{>0}$ as the composite

$$\lambda_{\mathbb{A}}\colon G(\mathbb{Q})\backslash G(\mathbb{A}) \stackrel{H^G}{\to} \mathfrak{a}_G \stackrel{\lambda_{\chi_\infty}}{\to} \mathbb{R}^\times_{>0}.$$

Write λ_v for the restriction of $\lambda_{\mathbb{A}}$ to $G(\mathbb{Q}_v)$ for a place v of $\mathbb{Q}$. For each $\pi\in\mathcal{A}_{\mathrm{disc},\chi}(G)$ contributing to the sum, we see that $\pi\otimes\lambda_{\mathbb{A}}^{-1}$ is a unitary automorphic representation of $G(\mathbb{A})$ since $\pi_\infty\otimes\lambda_\infty^{-1}$ is unitary (by construction, $\pi_\infty\otimes\lambda_\infty^{-1}$ has trivial central character on $A_G(\mathbb{R})^0$). Thus $\pi_p\otimes\lambda_p^{-1}$ is unitary. Applying Corollary 2.5.2 to $\pi\otimes\lambda^{-1}$ at p, we have

$$\left|\omega_{\tau_i}(\nu(p))\lambda_p^{-1}(\nu(p))\right| \le \delta^{-1/2}_{P_\nu^{\mathrm{op}}}(\nu(p)) = p^{\langle\rho,\nu\rangle},$$

noting that $\nu(p)\in A^{--}_{P_\nu}$. We deduce via Lemma 4.1.1 that

$$|\omega_{\tau_i}(\nu(p))| \le p^{\langle\rho,\nu\rangle}|\lambda_p(\nu(p))|_p = p^{\langle\rho,\nu\rangle}|p^{-\langle\chi_\infty,\overline{\nu}\rangle}|_p = p^{\langle\rho,\nu\rangle+\langle\chi_\infty,\overline{\nu}\rangle}.$$

By [BZ77, Cor. 2.13] the length of $J_{P_\nu^{\mathrm{op}}}(\pi_p)$, namely $\sum_i c_i$, can be bounded only in terms of G. This completes the proof. □

We state the main trace formula estimate of this paper. The proof will be given in §4.4 below.

Theorem 4.2.2 (main estimate)**.** *Let $G, (\mathfrak{X},\chi), p, q, \xi, \zeta, \nu$ be as defined in the beginning of Section 4, and additionally assume that $\mathfrak{X} = Y(\mathbb{A})A_{G,\infty}$ for a central torus $Y\subset Z_G$ over $\mathbb{Q}$. Let*

- *$f^{\infty,p,q} = \prod_{v\neq\infty,p,q} f_v \in \mathcal{H}(G(\mathbb{A}^{\infty,p,q}), (\chi^{\infty,p,q})^{-1})$,*
- *$\phi_p\in\mathcal{H}_{\mathrm{acc}}(M_\nu(\mathbb{Q}_p),\chi_p^{-1})$, and*
- *$f_p^{(k)}\in\mathcal{H}(G(\mathbb{Q}_p),\chi_p^{-1})$ be a ν-ascent of $\phi_p^{(k)}$, for $k\in\mathbb{Z}_{\ge 0}$.*

Then there exists a constant $C = C(f^{\infty,q},\phi_p)\in\mathbb{R}_{>0}$ such that for each $f_q\in\mathcal{H}(G(\mathbb{Q}_q),\chi_q^{-1})_{C\text{-reg}}$,

$$T^G_{\mathrm{ell},\chi}(f_p^{(k)} f_q f^{\infty,p,q} f_{\xi,\zeta}) = T^G_{\mathrm{disc},\chi}(f_p^{(k)} f_q f^{\infty,p,q} f_{\xi,\zeta}) + o\left(p^{k(\langle\rho,\nu\rangle+\langle\chi_\infty,\overline{\nu}\rangle)}\right).$$

As a corollary, we derive the stable analogue of Theorem 4.2.2. We keep the setup of Theorem 4.2.2 and let $f^{\infty,p,q}, \phi_p, f_p^{(k)}$ be as in there. For each $\mathfrak{e} \in \mathcal{E}^{<}_{\mathrm{ell}}(G)$, we choose z-extensions to introduce $\mathfrak{e}_1 = (G_1^{\mathfrak{e}}, {}^L G_1^{\mathfrak{e}}, s_1^{\mathfrak{e}}, \eta_1^{\mathfrak{e}})$ and a central character datum $(\mathfrak{X}_1^{\mathfrak{e}}, \chi_1^{\mathfrak{e}})$ as in §2.6. Moreover we choose the representatives $\mathfrak{e}, \mathfrak{e}_1$ such that $\eta^{\mathfrak{e}}(W_F)$ and $\eta_1^{\mathfrak{e}}(W_F)$ have bounded images, as explained in Lemma 2.6.2 and §2.7. Let

$$f_1^{(k),\mathfrak{e}} = \prod_v f_{1,v}^{(k),\mathfrak{e}} \in \mathcal{H}(G_1^{\mathfrak{e}}(\mathbb{A}), (\chi_1^{\mathfrak{e}})^{-1})$$

be a transfer of $f_p^{(k)} f_q f^{\infty,p,q} f_{\xi,\zeta}$. Then we have the following bound.

Corollary 4.2.3. *In the setup of Theorem 4.2.2, there exists a constant* $C = C(f^{\infty,q}, \phi_p, \xi, \zeta) \in \mathbb{R}_{>0}$ *such that for every* $f_q \in \mathcal{H}(G(\mathbb{Q}_q))_{C\text{-reg}}$*, firstly*

$$ST^G_{\mathrm{ell},\chi}(f_p^{(k)} f_q f^{\infty,p,q} f_{\xi,\zeta}) = \begin{cases} T^G_{\mathrm{disc},\chi}(f_p^{(k)} f^{\infty,p,q} f_q f_{\xi,\zeta}) + o\left(p^{k(\langle \rho,\nu\rangle + \langle \chi_\infty, \overline{\nu}\rangle)}\right), \\ O\left(p^{k(\langle \rho,\nu\rangle + \langle \chi_\infty, \overline{\nu}\rangle)}\right), \end{cases}$$

and secondly for each $\mathfrak{e} \in \mathcal{E}^{<}_{\mathrm{ell}}(G)$ *(note that* $f_{1,q}^{(k),\mathfrak{e}}$ *inherits C-regularity from* f_q*),*

$$\mathrm{ST}^{G_1^{\mathfrak{e}}}_{\mathrm{ell},\chi_1^{\mathfrak{e}}}\left(f_1^{(k),\mathfrak{e}}\right) = o\left(p^{k(\langle \rho,\nu\rangle + \langle \chi_\infty, \overline{\nu}\rangle)}\right).$$

Remark 4.2.4. In the inductive proof of the last bound, we only use the fact that its q-component is C-regular, ∞-component is a Lefschetz function, and most importantly the p-component is an ascent for a suitable cocharacter. We do not rely on the fact that $f_1^{(k),\mathfrak{e}}$ is a transfer of a function on $G(\mathbb{A})$.

Proof. The second estimate is immediate from the first via Proposition 4.2.1. Let us prove the first and third asymptotic formulas, by reducing the former to the latter.

We induct on the semisimple rank of G. (For each G, we prove the corollary for all central character data and all ν.) The estimate is trivial when G is a torus, in which case $\mathrm{ST}^G_{\mathrm{ell},\chi} = T^G_{\mathrm{ell},\chi} = T^G_{\mathrm{disc},\chi}$. We assume that G is not a torus and that Corollary 4.2.3 is true for all groups which have lower semisimple rank than G. Write $f^{(k)} := f_p^{(k)} f_q f^{\infty,p,q} f_{\xi,\zeta}$. The stabilization (Proposition 2.9.1) tells us that

$$\mathrm{ST}^G_{\mathrm{ell},\chi}(f^{(k)}) = T^G_{\mathrm{ell},\chi}(f) - \sum_{\mathfrak{e} \in \mathcal{E}^{<}_{\mathrm{ell}}(G)} \iota(G, G^{\mathfrak{e}}) \mathrm{ST}^{G_1^{\mathfrak{e}}}_{\mathrm{ell},\chi_1^{\mathfrak{e}}}\left(f_1^{(k),\mathfrak{e}}\right).$$

In light of Theorem 4.2.2, since the summand is nonzero only for a finite set of $\mathfrak{e}$ by Lemma 2.9.3 (depending only on the finite set of primes v where either $G_{\mathbb{Q}_v}$ or f_v is ramified), it suffices to establish the last bound of the corollary. This task takes up the rest of the proof.

If $G^{\mathfrak{e}}_{\mathbb{R}}$ contains no elliptic maximal torus or if $A_{G^{\mathfrak{e}}} \neq A_G$ (equivalently if $A_{G_1^{\mathfrak{e}}} \neq A_{G_1}$), then $f^{\mathfrak{e}}_{1,\infty}$ is trivial as observed in [Kot90, p.182, p.189] so the desired estimate is trivially true. Henceforth, suppose that $G^{\mathfrak{e}}_{\mathbb{R}}$ contains an elliptic maximal torus. Then $f^{(k),\mathfrak{e}}_{1,\infty}$ is a finite linear combination of $f_{\eta_1^{\mathfrak{e}},\zeta_1^{\mathfrak{e}}}$ over the set of $(\eta_1^{\mathfrak{e}}, \zeta_1^{\mathfrak{e}})$ such that $\eta_1^{\mathfrak{e}} \circ \varpi_{\eta_1^{\mathfrak{e}},\zeta_1^{\mathfrak{e}}} \simeq \varpi_{\xi,\zeta}$. Proposition 3.3.2 and its adaptation to z-extensions according to §3.5 and §3.6 tell us that

$$f_{1,p}^{(k),\mathfrak{e}} = \sum_\omega \lambda^{\mathfrak{e}}_{1,\omega}(\nu_1(p))^k \mathscr{I}_{\nu_{1,\omega}}\left(\phi^{(k),\mathfrak{e}}_{1,p,\omega}\right) = f_{1,p}^{(k),\mathfrak{e}} = \sum_\omega \lambda^{\mathfrak{e}}_{1,\omega}(\nu_1(p))^k f^{(k),\mathfrak{e}}_{1,p,\omega},$$

where we have put $f^{(k),\mathfrak{e}}_{1,p,\omega} := \mathscr{I}_{\nu_{1,\omega}}\left(\phi^{(k),\mathfrak{e}}_{1,p,\omega}\right)$ for a $\nu_{1,\omega}$-ascent of $\phi^{(k),\mathfrak{e}}_{1,p,\omega} \in \mathcal{H}_{\mathrm{acc}}(G^{\mathfrak{e}}_{1,\nu}(\mathbb{Q}_p), (\chi^{\mathfrak{e}}_{1,p})^{-1})$. Here we applied Lemma 3.3.5 (keeping §3.5 and §3.6 in mind) to have the transfer $\phi^{(k),\mathfrak{e}}_{1,p,\omega}$ of $\phi^{(k)}_{1,p}$ supported on $\nu_{1,\omega}$-acceptable elements.

Recalling that $\eta_1^{\mathfrak{e}}(W_F) \subset {}^L G_1$ is a bounded subgroup, we see from Lemma 3.3.1 that $\lambda^{\mathfrak{e}}_{1,\omega}$ is a unitary character. Thus we are reduced to showing the existence of some $C_{\mathfrak{e}} > 0$ such that the following estimate holds for ω and $(\eta_1^{\mathfrak{e}}, \zeta_1^{\mathfrak{e}})$ as above whenever $f^{\mathfrak{e}}_{1,q}$ is $C_{\mathfrak{e}}$-regular:

$$\mathrm{ST}^{G_1^{\mathfrak{e}}}_{\mathrm{ell},\chi_1^{\mathfrak{e}}}\left(f_1^{\mathfrak{e},\infty,q,p} f^{\mathfrak{e}}_{1,q} f^{(k),\mathfrak{e}}_{1,p,\omega} f_{\eta_1^{\mathfrak{e}},\zeta_1^{\mathfrak{e}}}\right) \overset{?}{=} o\left(p^{k(\langle\rho,\nu\rangle + \langle\chi_\infty,\overline{\nu}\rangle)}\right), \qquad k \in \mathbb{Z}_{\geq 0}. \tag{4.2.1}$$

Indeed, take C to be the maximum of all $C_{\mathfrak{e}}$ over the finite set of $\mathfrak{e}$ contributing to the sum. Then for each C-regular f_q, Lemma 3.4.7 tells us that $f^{\mathfrak{e}}_{1,q}$ is C-regular (thus also $C_{\mathfrak{e}}$-regular). Thus the bound (4.2.1) applies, and we will be done.

By the induction hypothesis, there exists $C_{\mathfrak{e}} > 0$ such that whenever $f^{\mathfrak{e}}_{1,q}$ is $C_{\mathfrak{e}}$-regular, the left hand side of (4.2.1) is $O\left(p^{k(\langle\rho_1^{\mathfrak{e}},\nu_{1,\omega}\rangle + \langle\chi^{\mathfrak{e}}_{1,\infty},\overline{\nu}_{1,\omega}\rangle)}\right)$, with $\overline{\nu}_{1,\omega} \in X_*(A_{G_1^{\mathfrak{e}}})$ defined from $\nu_{1,\omega}$ in the same way $\overline{\nu}$ from ν, and where $\rho_1^{\mathfrak{e}}$ is the half sum of positive roots of $G_1^{\mathfrak{e}}$ for which $\nu_{1,\omega}$ is a dominant cocharacter. (In other words, $\rho_1^{\mathfrak{e}}$ is to $\nu_{1,\omega}$ as ρ is to ν.) Therefore it is enough to check that

(a) $\langle\rho_1^{\mathfrak{e}}, \nu_{1,\omega}\rangle < \langle\rho,\nu\rangle$ (in $\mathbb{Q}$).
(b) $\mathrm{Re}\langle\chi^{\mathfrak{e}}_\infty, \overline{\nu}\rangle = \mathrm{Re}\langle\chi_{1,\infty}, \overline{\nu}_{1,\omega}\rangle$,

Let us begin with (a). Since $\langle\rho,\nu\rangle = \langle\rho_1,\nu_1\rangle$, with ρ_1 defined for G_1 as ρ is for G (recall that $\nu_1 : \mathbb{G}_m \to G_1$ is a lift of ν), the proof of (a) is reduced to the case when $G_1 = G$ and $\nu_1 = \nu$. We have an embedding $\widehat{G^{\mathfrak{e}}} \hookrightarrow \widehat{G}$ coming from $\eta^{\mathfrak{e}}$, which restricts to $\widehat{G^{\mathfrak{e}}_\omega} \hookrightarrow \widehat{M_\nu}$. Here we have chosen Γ_F-invariant pinnings for the dual groups such that the restriction works as stated. We may and will arrange that the Borel subgroup of $\widehat{G}$ restricts to that of $\widehat{G^{\mathfrak{e}}}$. Fix a maximal torus $\widehat{T} \subset \widehat{G^{\mathfrak{e}}_\omega}$ that is part of the pinning for $\widehat{G^{\mathfrak{e}}_\omega}$. Viewing $\widehat{T}$ also as a maximal torus in each of $\widehat{G^{\mathfrak{e}}}$ and $\widehat{G}$, we write $\Phi^\vee(\widehat{T},\widehat{G})$ and $\Phi^\vee(\widehat{T},\widehat{G^{\mathfrak{e}}})$ for the corresponding sets of coroots. Write $\widehat{\nu} \in X^*(\widehat{T})$ for the dominant member in the Weyl orbit of characters determined by ν. Then

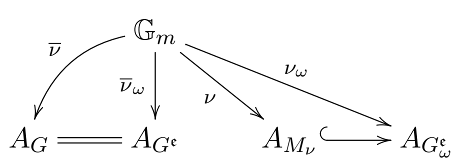
$$\langle\rho^{\mathfrak{e}},\nu_\omega\rangle = \sum_{\substack{\alpha^\vee\in\Phi^\vee(\widehat{T},\widehat{G^{\mathfrak{e}}_\omega}) \\ \langle\alpha^\vee,\nu\rangle>0}} \langle\alpha^\vee,\nu\rangle, \qquad \langle\rho,\nu\rangle = \sum_{\substack{\alpha^\vee\in\Phi^\vee(\widehat{T},\widehat{G}) \\ \langle\alpha^\vee,\nu\rangle>0}} \langle\alpha^\vee,\nu\rangle. \tag{4.2.2}$$

Thus it suffices to verify that there exists a coroot $\alpha^\vee \in \Phi^\vee(\widehat{T},\widehat{G})$ outside $\widehat{G^{\mathfrak{e}}}$ such that $\langle\alpha^\vee,\nu\rangle > 0$. The centralizer of $\widehat{\nu}$ in $\widehat{G}$ is identified with the dual group $\widehat{M_\nu}$ (namely $\langle\alpha^\vee,\nu\rangle = 0$ if and only if $\alpha^\vee$ is a coroot of $\widehat{M_\nu}$), so we will be done if $\mathrm{Lie}\,\widehat{M_\nu} + \mathrm{Lie}\,\widehat{G^{\mathfrak{e}}}$ is a proper subspace of $\mathrm{Lie}\,\widehat{G}$. This is exactly proved in [KST, Lem. 4.5 (ii)] applied to $\mathsf{G} = \widehat{G}$, $\mathsf{M} = \widehat{M_\nu}$, and $\delta = s^{\mathfrak{e}}$. (The proof of *loc. cit.* greatly simplifies. One reduces to the case when the Dynkin diagram of G is connected as in the first paragraph in the proof of that lemma. Then argue as in the fourth paragraph of that lemma, with $X_n = 0$ and with the role of X_{ss} played by the semisimple element $s^{\mathfrak{e}}$.)

Now we prove (b). Since $\langle\chi_\infty,\overline{\nu}\rangle = \langle\chi_{1,\infty},\overline{\nu}_1\rangle$, we reduce to showing (b) when $G_1 = G$ and $\mathfrak{e}e_1 = \mathfrak{e}e$ (with possibly nontrivial central character data). Thus we drop the 1's from the subscripts and check that

$$\mathrm{Re}\langle\chi^{\mathfrak{e}}_\infty,\overline{\nu}\rangle = \mathrm{Re}\langle\chi_\infty,\overline{\nu}_\omega\rangle.$$

We claim that $\overline{\nu} = \overline{\nu}_\omega$ in $X_*(A_G)_{\mathbb{R}} = X_*(A_{G^{\mathfrak{e}}})_{\mathbb{R}}$. In the diagram below, the triangle on the right commutes, and we want the triangle on the left commutes as well.



We choose maximal tori $T \subset M_\nu \subset G$ and $T^{\mathfrak{e}} \subset G^{\mathfrak{e}}_\omega \subset G^{\mathfrak{e}}$ with an isomorphism $T_{\overline{F}} \simeq T^{\mathfrak{e}}_{\overline{F}}$ to identify the absolute Weyl group $\overline{\Omega}^{G^{\mathfrak{e}}}$ as a subgroup of $\overline{\Omega}^G$. (This is done as in [Kot86, §3].) The isomorphism also identifies $\nu = \nu_\omega$. By (3.1.5), we have the equalities

$$\overline{\nu} = |\overline{\Omega}^G|^{-1} \sum_{\omega \in \overline{\Omega}^G} \omega(\nu), \qquad \overline{\nu}_\omega = |\overline{\Omega}^{G^\mathfrak{c}}|^{-1} \sum_{\omega \in \overline{\Omega}^{G^\mathfrak{c}}} \omega(\nu).$$

Hence $\overline{\nu} = |\overline{\Omega}^G/\overline{\Omega}^{G^\mathfrak{c}}|^{-1} \sum_{\omega \in \overline{\Omega}^G/\overline{\Omega}^{G^\mathfrak{c}}} \omega(\overline{\nu}_\omega) = \overline{\nu}_\omega$. Indeed, the last equality follows since $\overline{\nu}_\omega \in X_*(A_{G^\mathfrak{c}})_\mathbb{R} = X_*(A_G)_\mathbb{R}$, which tells us that $\omega(\overline{\nu}_\omega) = \overline{\nu}_\omega$ for $\omega \in \overline{\Omega}^G$.

Applying (2.6.3) at the archimedean place, we have $\chi_\infty = \lambda^\mathfrak{c}_\infty \chi^\mathfrak{c}_\infty$ as characters of $A_G(\mathbb{R})$. Since $\lambda^\mathfrak{c}_\infty$ is unitary, $|\chi_\infty| = |\chi^\mathfrak{c}_\infty|$. Since $\overline{\nu} \in X_*(A_G)_\mathbb{R}$ (not just in $X_*(A_G)_\mathbb{C}$), we conclude that $\mathrm{Re}\langle \chi^\mathfrak{c}_\infty, \overline{\nu}\rangle = \mathrm{Re}\langle \chi_\infty, \overline{\nu}\rangle$ as desired. This verifies (b). □

4.3. **Some facts and notation on Weyl groups and Weyl chambers.** In this subsection we fix some additional notation on Weyl groups, Weyl chambers, which will be needed in the proof of the main estimate in the next subsection.

Recall that we have fixed a maximal torus $T \subset G$ and that we wrote $A_T \subset T$ for the maximal $\mathbb{Q}$-split subtorus in T. We also write $S_p \subset T_{\mathbb{Q}_p}$ for the maximal $\mathbb{Q}_p$-split subtorus in $T_{\mathbb{Q}_p}$. We then have $A_{T,\mathbb{Q}_p} \subset S_p \subset T_{\mathbb{Q}_p}$. We will write

$$\Omega^G \subset \Omega^G_p \subset \overline{\Omega}^G$$

for the Weyl group of A_T, S_p, and T in G. Similar notation will be used for other objects related to Weyl groups, for instance if $M \subset G_{\mathbb{Q}_p}$ is a Levi subgroup defined over $\mathbb{Q}_p$, we write $\Omega^G_{M,p} \subset \Omega^G_p$ for the set of Kostant representatives for Ω^G_p/Ω^M_p.

Let M be the unique Levi of a standard parabolic $P \supset B$ such that $M \supset T$. Write $\Phi^G_M = \Phi^G_M(A_M; G)$ for the set of A_M roots that appear in $\mathrm{Lie}(G)$. We recall, an element $x \in \mathfrak{a}_M$ is called *regular* if $\langle \alpha, x\rangle \neq 0$ for all $\alpha \in \Phi^G_M$. We write $\mathfrak{a}^{\mathrm{reg}}_M$ for the subspace of all regular $x \in \mathfrak{a}_M$. We call the connected components of $\mathfrak{a}^{\mathrm{reg}}_M$ the (open) *Weyl chambers* of $\mathfrak{a}_M$. The subset

$$\mathcal{C}^+_M := \{x \in \mathfrak{a}^{\mathrm{reg}}_M \,|\, \forall \alpha \in \Phi^G_M : \langle \alpha, x\rangle > 0\} \subset \mathfrak{a}^{\mathrm{reg}}_M,$$

is the *dominant Weyl chamber*. Let $\Omega^G_M \subset \Omega^G$ be the set of Kostant representatives for the quotient Ω^G/Ω^M. The Weyl chambers $\mathcal{C} \in \pi_0(\mathfrak{a}^{\mathrm{reg}}_M)$ are parametrized via $\mathcal{C} = \mathcal{C}_\omega := \omega^{-1}(\mathcal{C}^+_M) \in \pi_0(\mathfrak{a}^{\mathrm{reg}}_M)$, where $\omega \in \Omega^G_M$. If $\mathcal{C} \subset \mathfrak{a}^{\mathrm{reg}}_M$ is a Weyl chamber, we write $\mathcal{C}^\vee \subset \mathfrak{a}^*_M$ for the *dual chamber*, *i.e.* the set of $t \in \mathfrak{a}^*_M$ such that $t(x) > 0$ for all $x \in \mathcal{C}$.

Remark 4.3.1. In Section 3 we defined similar notions, but using absolute roots in the $\mathbb{Q}_p$-relative space $X_*(S_p)_\mathbb{R}$. We explain here how the notions can be compared. Write $\Phi(A_T, B)$, $\Phi(S_p, B_{\mathbb{Q}_p})$, $\Phi(T, B)$ for the various sets of positive roots attached to A_T, S_p and T. The inclusions $A_{T,\overline{\mathbb{Q}}_p} \subset S_{p,\overline{\mathbb{Q}}_p} \subset T_{\overline{\mathbb{Q}}_p}$ induce surjections, $X^*(T_{\overline{\mathbb{Q}}_p}) \to X^*(S_{p,\overline{\mathbb{Q}}_p}) \to X^*(A_{T,\overline{\mathbb{Q}}_p})$, and the natural maps

$$\Phi(T_{\overline{\mathbb{Q}}_p}, B_{\overline{\mathbb{Q}}_p}) \to \Phi(S_{p,\overline{\mathbb{Q}}_p}, B_{\overline{\mathbb{Q}}_p}) \to \Phi(A_{T,\overline{\mathbb{Q}}_p}, B_{\overline{\mathbb{Q}}_p})$$

are surjective as well. Therefore in $X_*(A_T)_\mathbb{R}$ the following sets are all the same:

(1) The set of $x \in X_*(A_T)_\mathbb{R}$ such that for all $\alpha \in \Phi(T, B)$ we have $\langle \alpha, x\rangle \neq 0$;
(2) The set of $x \in X_*(A_T)_\mathbb{R}$ such that for all $\alpha \in \Phi(S_p, B_{\mathbb{Q}_p})$ we have $\langle \alpha, x\rangle \neq 0$;
(3) The set of $x \in X_*(A_T)_\mathbb{R}$ such that for all $\alpha \in \Phi(A_T, B_{\mathbb{Q}_p})$ we have $\langle \alpha, x\rangle \neq 0$.

Similarly, the various versions of C-regularity (defined by using $\mathbb{Q}$-relative, $\mathbb{Q}_p$-relative, or absolute roots) all coincide. In place of T the above discussion is also true for any Levi subgroup M and the space $\mathfrak{a}_M$. Moreover, the natural maps $\pi_0(\mathfrak{a}^{\mathrm{reg}}_M) \to \pi_0(X_*(S_p)^{\mathrm{reg}}_\mathbb{R}) \to \pi_0(X_*(T)^{\mathrm{reg}}_\mathbb{R})$ are injections. Finally, under the inclusion $\mathfrak{a}_T = X_*(A_T)_\mathbb{R} \to X_*(S_p)_\mathbb{R}$, the subset $(X_*(S_p)_\mathbb{R})_{C\text{-reg}} \subset X_*(S_p)_\mathbb{R}$ pulls back to the set of $x \in X_*(A_T)$ such that conditions (1) and (2) of Definition 3.4.1 hold for relative roots $\alpha \in \Phi(A_T, B)$.

4.4. **Proof of Theorem 4.2.2.** The rest of this section is devoted to establishing the main estimate in Theorem 4.2.2. We note that the argument spans until page 46 and will be interrupted by a lemma which establishes small facts that are needed along the way. Before diving into the technical details, we recommend the reader to first have a look at the outline that we sketched below (1.4.2) in the introduction.

Proof of 4.2.2. We argue by induction on the $\mathbb{Q}$-semisimple rank r_G of G. If $r_G = 0$, then we have $T^G_{\mathrm{ell},\chi} = T^G_{\mathrm{disc},\chi}$, and the statement follows. Assume now that the theorem is established for all groups of lower $\mathbb{Q}$-semisimple rank and the accompanying data.

We take the constant

$$C = C(f^{\infty,q}, \phi_p) := \frac{1}{\log q} \cdot \max_{M, x^{p,q,\infty}, \varepsilon_p, \alpha} |\langle \alpha, x^{p,q,\infty} + \varepsilon_p \rangle|, \tag{4.4.1}$$

where $M \in \mathcal{L}_{\mathrm{cusp}}(G)$, $x^{p,q,\infty} \in \mathrm{supp}^O_{\mathfrak{a}_M}(f^{\infty,p,q}_M)$, $\varepsilon_p \in \mathrm{pr}_M \mathrm{supp}_{\mathfrak{a}_{\omega_p(M)\cap M_\nu}}(\phi_{p,\omega_p(M)\cap M_\nu})$, and $\alpha \in \Phi^G_M$. Define the constants

$$v_{\mathfrak{X}} := \mathrm{vol}(\mathfrak{X}_{\mathbb{Q}} \backslash \mathfrak{X}/A_{G,\infty}) \quad \text{and} \quad c_M := (-1)^{\dim(A_M/A_G)} \frac{|\Omega^M|}{|\Omega^G|}, \quad M \in \mathcal{L}_{\mathrm{cusp}}(G).$$

Write $f^{\infty,(k)} := f^{\infty,p,q} f^{(k)}_p f_q$, to indicate the dependence on k at p. The running hypothesis on f_q is that it is C-regular for (4.4.1). By Proposition 2.8.1 we have

$$T^G_{\mathrm{disc},\chi}(f_{\xi,\zeta} f^{\infty,(k)}) = \sum_{M \in \mathcal{L}_{\mathrm{cusp}}} c_M v_{\mathfrak{X}}^{-1} \sum_{\gamma \in \Gamma_{\mathrm{ell},\mathfrak{X}}(M)} \frac{\chi(I^M_\gamma) \zeta(\gamma) \Phi_M(\gamma,\xi) O^M_\gamma(f^{\infty,(k)}_M)}{|\iota^M(\gamma)||\mathrm{Stab}^M_{\mathfrak{X}}(\gamma)|}, \tag{4.4.2}$$

(here we used that $\mathfrak{X}$ is of the form $Y(\mathbb{A})A_{G,\infty}$ for a central torus $Y \subset Z_G$ over $\mathbb{Q}$).

We first compare the term corresponding to $M = G \in \mathcal{L}_{\mathrm{cusp}}$ on the right hand side of Equation (4.4.2) with $T^G_{\mathrm{ell},\chi}$. In Lemma 4.1.3 we checked that in this case we have

$$c_G v_{\mathfrak{X}}^{-1} \frac{\chi(I^G_\gamma)\zeta(\gamma)\Phi^G_G(\gamma,\xi) O^G_\gamma(f_{\xi,\zeta} f^{\infty,(k)})}{|\iota^G(\gamma)||\mathrm{Stab}^G_{\mathfrak{X}}(\gamma)|} = \frac{\mathrm{vol}(I_\gamma(\mathbb{Q})\backslash I_\gamma(\mathbb{A})/\mathfrak{X}) O_\gamma(f_{\xi,\zeta} f^{\infty,(k)})}{\iota(\gamma)^{-1}|\mathrm{Stab}^G_{\mathfrak{X}}(\gamma)|^{-1}} \tag{4.4.3}$$

for γ regular elliptic. For $\gamma \in \Gamma_{\mathrm{ell},\mathfrak{X}}(G)$ non-regular, we have $O_\gamma(f_q) = 0$ since f_q is C-regular. Thus (4.4.2) (see also (2.8.2)) can be rearranged as

$$T^G_{\mathrm{ell},\chi}(f_{\xi,\zeta} f^{\infty,(k)}) = T^G_{\mathrm{disc},\chi}(f_{\xi,\zeta} f^{\infty,(k)}) - \sum_{M \in \mathcal{L}^<_{\mathrm{cusp}}} c_M v_{\mathfrak{X}}^{-1} \sum_{\gamma \in \Gamma_{\mathrm{ell},\mathfrak{X}}(M)} \frac{\chi(I^M_\gamma)\zeta(\gamma)\Phi_M(\gamma,\xi) O^M_\gamma(f^{\infty,(k)}_M)}{|\iota^M(\gamma)||\mathrm{Stab}^M_{\mathfrak{X}}(\gamma)|}. \tag{4.4.4}$$

By Lemma 3.2.1 we have

$$f^{(k)}_{p,M} = \sum_{\omega_p \in \Omega^{G_{\mathbb{Q}_p}}_{M,M_\nu}} f^{(k)}_{p,M,\omega_p} \in \mathcal{H}(M(\mathbb{Q}_p), \chi_p^{-1}), \tag{4.4.5}$$

where

$$f^{(k)}_{p,M,\omega_p} := \mathscr{I}_{\omega_p \nu}(\omega_p^{-1} \phi^{(k)}_{p,M_{\omega_p}}), \qquad M_{\omega_p} = \omega_p(M) \cap M_\nu. \tag{4.4.6}$$

By Lemma 3.4.7 (1) and Remark 4.3.1 we may arrange that the constant term $f_{q,M}$ is supported on C-regular elements. Thus $f_{q,M}$ is decomposed according to the chambers of $\mathfrak{a}^{\mathrm{reg}}_M$:

$$f_{q,M} = \sum_{\omega_q \in \Omega^G_M} f_{q,M,\omega_q} \in \mathcal{H}(M(\mathbb{Q}_q), \chi_q^{-1})_{C\text{-reg}},$$

where f_{q,M,ω_q} satisfies $\mathrm{supp}^O_{\mathfrak{a}_M}(f_{q,M,\omega_q}) \subset \mathcal{C}_{\omega_q}$. We define

$$f^{\infty,(k)}_{M,\omega_p,\omega_q} := f^{\infty,p,q}_M f^{(k)}_{p,M,\omega_p} f_{q,M,\omega_q} \in \mathcal{H}(M(\mathbb{A}), \chi^{-1}).$$

Changing the order of summation (each sum is finite), Equation (4.4.4) becomes

$$T^G_{\mathrm{ell},\chi}(f_{\xi,\zeta}f^{\infty,(k)}) = T^G_{\mathrm{disc},\chi}(f_{\xi,\zeta}f^{\infty,(k)}) - \sum_{M,\omega_p,\omega_q} c_M v_{\mathfrak{X}}^{-1} \sum_{\gamma\in\Gamma_{\mathrm{ell},\mathfrak{X}}(M)} \frac{\chi(I^M_\gamma)\zeta(\gamma)\Phi_M(\gamma,\xi)O^M_\gamma(f^{\infty,(k)}_{M,\omega_p,\omega_q})}{|\iota^M(\gamma)||\mathrm{Stab}^M_{\mathfrak{X}}(\gamma)|}. \tag{4.4.7}$$

To state the next lemma, we define a constant

$$k_1 = k_1(f^{\infty,q},\phi_p) := \max_{M,\omega_p,\omega_q,\alpha,\varepsilon_p,x^{p,\infty}} \left|\frac{\langle\alpha,\varepsilon_p + x^p\rangle}{\log(p)\langle\alpha,\mathrm{pr}_M(\omega_p^{-1}\nu)\rangle}\right| \in \mathbb{R}_{>0},$$

where the maximum is taken over $M\in\mathcal{L}^<_{\mathrm{cusp}}(G)$, $\omega_p\in\Omega^{G_{\mathbb{Q}_p}}_{M,M_\nu}$, $\omega_q\in\Omega^G_M$, $\varepsilon_p\in\mathrm{pr}_M\mathrm{supp}^O_{\mathfrak{a}_{M_{\omega_p}}}(\omega_p^{-1}\phi_{p,M_{\omega_p}})$, $x^{p,\infty}\in\mathrm{supp}^O_{\mathfrak{a}_M}(f^{\infty,p}_M)$, and α ranges over those $\alpha\in\Phi^G_M$ such that $\langle\alpha,\mathrm{pr}_M(\omega_p^{-1}\nu))\rangle\neq 0$. For each $M\in\mathcal{L}_{\mathrm{cusp}}(G)$, denote by $\mathcal{P}(M)$ the set of parabolic subgroups P of G of which M is a Levi component.

We have fixed a maximal torus T in $G_{\mathbb{C}}$ (we have $G_{\mathbb{C}}\simeq G_{\overline{\mathbb{Q}}_p}$ via ι_p), along with a Borel subgroup B. We write $\overline{\rho}=\overline{\rho}_G$ for the half sum of the B-positive roots of T in $\mathrm{Lie}(B)$. Note that we have $\overline{\rho}|_{A_T}=\rho$. We use similar definitions for $\overline{\rho}_M$ and ρ_M if $M\subset G$ is a Levi subgroup.

For each $\lambda_0\in X^*(T)^+$, write $\xi^M_{\lambda_0}$ for the irreducible $M_{\mathbb{C}}$-representation with highest weight λ_0, and define $\omega_\infty\star\lambda_0 := \omega_\infty(\lambda_0+\overline{\rho})-\overline{\rho}$ for each $\omega_\infty\in\overline{\Omega}^G_M$. Let $\lambda=\lambda_B,\lambda^*_B\in X^*(T)$ denote the highest weight of ξ and its dual representation ξ^*, respectively, relative to B. Write $\omega_0\in\overline{\Omega}^M$ for the longest Weyl group element, and

$$\lambda_{\omega_\infty} := -\omega_0(\omega_\infty\star\lambda^*_B) = \omega_0\omega_\infty\omega_0\lambda_B - \omega_0\omega_\infty\overline{\rho} - w_0\overline{\rho},$$

so that we have

$$\xi^M_{\lambda_{\omega_\infty}} = (\xi^M_{\omega_\infty\star\lambda^*_B})^*.$$

Lemma 4.4.1. *Assume that $k>k_1$. Consider $M,\omega_p,\omega_q,\gamma$ as in (4.4.7), such that*

$$O^M_\gamma(f^{\infty,(k)}_{M,\omega_p,\omega_q})\neq 0.$$

Let $x_\infty := H^M_\infty(\gamma)\in\mathfrak{a}_M$. Then the following are true.

(i) The element $x_\infty\in\mathfrak{a}_M$ is regular and lies in the chamber $\mathcal{C}_0=\mathcal{C}_0(M,\omega_p,\omega_q)\subset\mathfrak{a}^{\mathrm{reg}}_M$ which has the following set of positive roots

$$\{\alpha\in\Phi^G_M \mid \langle\alpha,\mathrm{pr}_M(\omega_p^{-1}\nu)\rangle<0\}\cup\{\alpha\in\Phi^G_M \mid \langle\alpha,\mathrm{pr}_M(\omega_p^{-1}\nu)\rangle=0 \text{ and } \alpha\in-\mathcal{C}^\vee_{\omega_q}\}. \tag{4.4.8}$$

(ii) There exists an explicit subset $\overline{\Omega}^{G\diamond}_M=\overline{\Omega}^{G\diamond}_M(M,\omega_p,\omega_q)\subset\overline{\Omega}^G_M$ (see (4.4.18)) and an explicit sign $\varepsilon^\diamond=\varepsilon^\diamond(M,\omega_p,\omega_q)$ (see below (4.4.19)) such that we have

$$\Phi_M(\gamma,\xi)=\varepsilon^\diamond\sum_{P\in\mathcal{P}(M)}\delta_P^{-1/2}(\gamma)\sum_{\omega_\infty\in\overline{\Omega}^{G\diamond}_M}\varepsilon(\omega_\infty)\mathrm{Tr}\,(\gamma;\xi^M_{\lambda_{\omega_\infty}}),$$

where $\varepsilon(w_\infty)\in\{\pm1\}$ denotes the sign as an element of the Weyl group $\overline{\Omega}^G$.

Proof. (i) In this proof, if S is a set of places of $\mathbb{Q}$, we write

$$x_S := H^M_S(\gamma)\in\mathfrak{a}_M,\quad x^S := H^{M,S}(\gamma)\in\mathfrak{a}_M.$$

We check that $\langle\alpha,x_\infty\rangle\neq 0$ for all $\alpha\in\Phi^G_M$ (*i.e.*, x is regular). By the product formula in Lemma 4.1.2 we have

$$-\langle\alpha,x_\infty\rangle = \langle\alpha,x^{p,\infty}\rangle+\langle\alpha,x_p\rangle. \tag{4.4.9}$$

By assumption $O_\gamma(f^{\infty,(k)}_{M,\omega_p}) \neq 0$, so $x^{p,\infty} \in \mathrm{supp}^O(f^{\infty,p}_M)$. Using Lemma 3.1.8 (and (4.4.6)) we find

$$\mathrm{supp}^O_{\mathfrak{a}_M}(f^{(k)}_{p,M,\omega_p}) = k \cdot H^M_p(\omega_p^{-1}\nu(p)) + \mathrm{pr}_M(\mathrm{supp}^O_{\mathfrak{a}_{M_{\omega_p}}}(\omega_p^{-1}\phi_{p,M_{\omega_p}})).$$

Therefore

$$x_p = k \cdot H^M_p(\omega_p^{-1}\nu(p)) + \varepsilon_p \tag{4.4.10}$$

for some $\varepsilon_p \in \mathrm{pr}_M \mathrm{supp}^O_{\mathfrak{a}_{M_{\omega_p}}}(\omega_p^{-1}\phi_{p,M_{\omega_p}})$. Thus

$$\langle \alpha, x_p\rangle = k \cdot \langle \alpha, H^M_p(\omega_p^{-1}\nu(p))\rangle + \langle \alpha, \varepsilon_p\rangle = -k(\log p) \cdot \langle \alpha, \mathrm{pr}_M(\omega_p^{-1}\nu))\rangle + \langle \alpha, \varepsilon_p\rangle. \tag{4.4.11}$$

We now distinguish cases. First consider $\alpha \in \Phi^G_M$ such that $\langle \alpha, \mathrm{pr}_M(\omega_p^{-1}\nu)\rangle \neq 0$. By (4.4.9),

$$-\langle \alpha, x_\infty\rangle = \langle \alpha, x^{p,\infty}\rangle + \langle \alpha, \varepsilon_p\rangle - k(\log p) \cdot \langle \alpha, \mathrm{pr}_M(\omega_p^{-1}\nu)\rangle.$$

As $k > k_1$ we have

$$k(\log p) \cdot |\langle \alpha, \mathrm{pr}_M(\omega_p^{-1}\nu)\rangle| > |\langle \alpha, \varepsilon_p + x^{p,\infty}\rangle|.$$

In particular $\langle \alpha, x_\infty\rangle \neq 0$.

The second case is when $\langle \alpha, \mathrm{pr}_M(\omega_p^{-1}\nu)\rangle = 0$. Then

$$-\langle \alpha, x_\infty\rangle = \langle \alpha, x^{p,\infty}\rangle + \langle \alpha, \varepsilon_p\rangle. \tag{4.4.12}$$

As f_q is C-regular, we have (see (3.4.2) and Remark 4.3.1

$$|\langle \alpha, x_q\rangle| > C \log q \geq |\langle \alpha, x^{p,q,\infty} + \varepsilon_p\rangle|.$$

for all $\alpha \in \Phi^G_M$. In particular

$$\langle \alpha, x_q\rangle + \langle \alpha, x^{p,q,\infty} + \varepsilon_p\rangle \neq 0.$$

Therefore each side of (4.4.12) does not vanish. Hence $\langle \alpha, x_\infty\rangle \neq 0$ for all $\alpha \in \Phi^G_M$.

We now determine for which $\alpha \in \Phi^G_M$ we have $\langle \alpha, x_\infty\rangle > 0$. If $\langle \alpha, \mathrm{pr}_M(\omega_p^{-1}\nu)\rangle \neq 0$, then

$$\mathrm{sign}(\langle \alpha, x_\infty\rangle) = -\mathrm{sign}(\langle \alpha, \mathrm{pr}_M(\omega_p^{-1}\nu)\rangle)$$

by the arguments following (4.4.11). If $\langle \alpha, \mathrm{pr}_M(\omega_p^{-1}\nu)\rangle = 0$, then

$$\mathrm{sign}(\langle \alpha, x_\infty\rangle) = -\mathrm{sign}(\langle \alpha, x_q\rangle)$$

by C-regularity (see (4.4.12)). We have $x_q \in \mathrm{supp}_{\mathfrak{a}_M}(f_{q,M,\omega_q})$. Statement (i) follows.

(ii) Let us start by recalling a result of Goresky–Kottwitz–MacPherson. We write $\mathrm{pr}^*_M \colon X^*(T)_\mathbb{R} \to X^*(A_M)_\mathbb{R}$ for the restriction mapping. Let $P = MN \in \mathcal{P}(M)$. Write ρ_N (resp. $\overline{\rho}_N$) for the half sum of the positive roots of A_M (resp. $\overline{T}$) that occur in the Lie algebra of the unipotent radical N of P.

Write ω_ξ for the central character of ξ. Write $\alpha_1, \dots, \alpha_n \in \mathfrak{a}^*_M$ for the simple roots of A_M in $\mathrm{Lie}\,(N)$, which form a basis of $(\mathfrak{a}_M/\mathfrak{a}_G)^*$. This determines the dual basis consisting of $t_1, \dots, t_n \in \mathfrak{a}_M/\mathfrak{a}_G$. Put $I := \{1, 2, \dots, n\}$. Define the subsets (cf. [GKM97, p.534])

$$\begin{aligned} I(\gamma) &:= \{i \in I \mid \langle \alpha_i, x\rangle < 0\}, \\ I(\omega_\infty) &:= \{i \in I \mid \langle \mathrm{pr}^*_M(-\omega_\infty \star \lambda^*_B) - \overline{\rho}_N - \omega_\xi, t_i\rangle > 0\}. \end{aligned} \tag{4.4.13}$$

By the discussion above Thm. 7.14.B in [GKM97] we have

$$\varphi_P(-x_\infty, \mathrm{pr}^*_M(\omega_\infty \star \lambda^*_B) + \overline{\rho}_N + \omega_\xi) = \begin{cases} (-1)^{\dim(A_G)}(-1)^{\dim(A_M/A_G) - |I(\gamma)|}, & \text{if } I(\omega_\infty) = I(\gamma), \\ 0, & \text{otherwise.} \end{cases} \tag{4.4.14}$$

We define $L_M(\gamma) \in \mathbb{C}$ following [GKM97, p. 511],[8] when x_∞ is regular:

$$L_M(\gamma) := (-1)^{\dim(A_G)} \sum_{P \in \mathcal{P}(M)} \delta_P^{-1/2}(\gamma) \sum_{\omega_\infty \in \overline{\Omega}_M^G} \varepsilon(\omega_\infty) \mathrm{Tr}\,(\gamma^{-1}; \xi^M_{\omega_\infty \star \lambda_B^*}) \cdot \\ \cdot \varphi_P(-x_\infty, \mathrm{pr}_M^*(\omega_\infty \star \lambda_B^*) + \overline{\rho}_N + \omega_\xi). \quad (4.4.15)$$

As γ is regular, Theorems 5.1 and 5.2 of [GKM97] imply the following identity[9]

$$\Phi_M(\gamma, \xi) = L_M(\gamma). \quad (4.4.16)$$

By (i), x_∞ is regular, so we may use Formula (4.4.15):

$$\Phi_M(\gamma, \xi) = L_M(\gamma) = (-1)^{\dim(A_G)} \sum_{P \in \mathcal{P}(M)} \delta_P^{-1/2}(\gamma) \sum_{\omega_\infty \in \overline{\Omega}_M^G} \varepsilon(\omega_\infty) \mathrm{Tr}\,(\gamma^{-1}; \xi^M_{\omega_\infty \star \lambda_B^*}) \cdot \\ \cdot \varphi_P(-x_\infty, \mathrm{pr}_M^*(\omega_\infty \star \lambda_B^*) + \overline{\rho}_N + \omega_\xi). \quad (4.4.17)$$

Since all contributing γ have $x_\infty = H_\infty^M(\gamma)$ lying inside the chamber $\mathcal{C}_0$ again by (i), the set $I(\gamma)$ does not depend on x_∞ (but does depend on (M, ω_p, ω_q)). Write $\mathcal{I}_0 = \mathcal{I}_0(M, \omega_p, \omega_q)$ for $I(\gamma)$, and

$$\overline{\Omega}_M^{G\diamond} = \overline{\Omega}_M^{G\diamond}(M, \omega_p, \omega_q) := \{\omega_\infty \in \overline{\Omega}_M^G \mid I(\omega_\infty) = \mathcal{I}_0\}, \quad (4.4.18)$$

in terms of (4.4.13). Then (4.4.17) simplifies thanks to (4.4.14):

$$L_M(\gamma) = (-1)^{\dim(A_M/A_G) - |\mathcal{I}_0|} \sum_{P \in \mathcal{P}(M)} \delta_P^{-1/2}(\gamma) \sum_{\omega_\infty \in \overline{\Omega}_M^{G\diamond}} \varepsilon(\omega_\infty) \mathrm{Tr}\,(\gamma^{-1}; \xi^M_{\omega_\infty \star \lambda_B^*}). \quad (4.4.19)$$

Statement (ii) follows by taking $\varepsilon^\diamond := (-1)^{\dim(A_M/A_G) - |\mathcal{I}_0|}$ and using $\mathrm{Tr}\,(\gamma^{-1}; \xi^M_{\omega_\infty \star \lambda_B^*}) = \mathrm{Tr}\,(\gamma, \xi^M_{\lambda_{\omega_\infty}})$. □

We continue assuming $k > k_1$. We write henceforward $c'_M := \varepsilon^\diamond \varepsilon(\omega_\infty) c_M$. We apply Lemma 4.4.1$(ii)$ to Equation (4.4.7) and change the order of summation to obtain

$$T^G_{\mathrm{ell},\chi}(f_{\xi,\zeta} f^{\infty,(k)}) = T^G_{\mathrm{disc},\chi}(f_{\xi,\zeta} f^{\infty,(k)}) + \\ \sum_{M, P, \omega_p, \omega_q, \omega_\infty} c'_M v_{\mathfrak{X}}^{-1} \sum_{\gamma \in \Gamma_{\mathrm{ell},\mathfrak{X}}(M)} \frac{\chi(I^M_\gamma) \mathrm{Tr}\,(\gamma; \xi^M_{\lambda_{\omega_\infty}} \otimes \zeta \delta_P^{-1/2}) O^M_\gamma(f^{\infty,(k)}_{M,\omega_p,\omega_q})}{|\iota^M(\gamma)| |\mathrm{Stab}^M_{\mathfrak{X}}(\gamma)|}, \quad (4.4.20)$$

where M, ω_p, ω_q run over the same sets as before, and P, ω_∞ range over $\mathcal{P}(M), \overline{\Omega}_M^{G\diamond}$, respectively. We apply Lemma 4.1.3 to equalize

$$v_{\mathfrak{X}}^{-1} \sum_{\gamma \in \Gamma_{\mathrm{ell},\mathfrak{X}}(M)} \frac{\chi(I^M_\gamma) \mathrm{Tr}\,(\gamma; \xi^M_{\lambda_{\omega_\infty}} \otimes \zeta \delta_P^{-1/2}) O^M_\gamma(f^{\infty,(k)}_{M,\omega_p,\omega_q})}{|\iota^M(\gamma)| |\mathrm{Stab}^M_{\mathfrak{X}}(\gamma)|} \\ = \sum_{\gamma \in \Gamma_{\mathrm{ell},\mathfrak{X}}(M)} \frac{\mathrm{vol}(I^M_\gamma(\mathbb{Q}) A_{I^M_\gamma,\infty} \backslash I^M_\gamma(\mathbb{A})/\mathfrak{X}) O^M_\gamma(f_{\lambda_{\omega_\infty}, \zeta \delta_P^{-1/2}} f^{\infty,(k)}_{M,\omega_p,\omega_q})}{|\iota^M(\gamma)| |\mathrm{Stab}^M_{\mathfrak{X}}(\gamma)|}, \quad (4.4.21)$$

using that every γ with $O^M_\gamma(f^{\infty,(k)}_{M,\omega_p,\omega_q}) \neq 0$ in (4.4.21) is regular since f_{q,M,ω_q} is supported on regular elements. Define

$$\mathfrak{X}_M := \mathfrak{X} \cdot A_{M,\infty}, \qquad v_M := \mathrm{vol}(\mathfrak{X}_{\mathbb{Q}} \backslash \mathfrak{X}/A_{G,\infty})^{-1} \mathrm{vol}(\mathfrak{X}_{M,\mathbb{Q}} \backslash \mathfrak{X}_M/A_{M,\infty}).$$

[8]We write $L_M(\gamma)$ where the authors of [GKM97] write $L^\nu_M(\gamma)$. This is because we only need to use the "middle weight profile", so there is no need to distinguish in our notation. In their formula the symbol ν_P appears, as they allow more general weight profiles. Since we use the middle weight profile, we have $\nu_P = -\overline{\rho}_N - \omega_\xi$.

[9]In [GKM97], they write $\Phi_M(\gamma, \Theta^{\xi^*})$ for $\Phi_M(\gamma, \xi^*)$. Their E corresponds to our ξ^*.

Write

$$z^M_{\omega_\infty} : A_{M,\infty} \to \mathbb{C}^\times$$

for the restriction of the central character of $\xi^M_{\lambda_{\omega_\infty}} \otimes \zeta\delta_P^{-1/2}$ to $A_{M,\infty}$. We observe that

$$z^M_{\omega_\infty}|_{A_{G,\infty}} = \chi_0^{-1}$$

as the restriction of the central character of $\xi^M_{\lambda_{\omega_\infty}}$ to Z_G coincides with the central character of the original representation ξ. Note that we have $Z_G(\mathbb{R}) \cap A_{M,\infty} = A_{G,\infty} \subset Z_M(\mathbb{R})$ and therefore

$$\mathfrak{X} \cap A_{M,\infty} = A_{G,\infty}.$$

Consequently, there exists a unique character

$$\chi^M_{\omega_\infty} : \mathfrak{X}_M \to \mathbb{C}^\times$$

such that

$$\chi^M_{\omega_\infty}|_{A_{M,\infty}} = (z^M_{\omega_\infty})^{-1} \quad \text{and} \quad \chi^M_{\omega_\infty}|_{\mathfrak{X}} = \chi.$$

The pair $(\mathfrak{X}_M, \chi^M_{\omega_\infty})$ is a central character datum for M as in §2.8, and $\mathfrak{X}_M$ is moreover of the form $\mathfrak{X}_M = Y(\mathbb{A})A_{M,\infty}$ as required by the statement of Theorem 4.2.2. Additionally, observe that

$$f_{\lambda_{\omega_\infty},\zeta\delta_P^{-1/2}} f^{\infty,(k)}_{M,\omega_p,\omega_q} \in \mathcal{H}(M(\mathbb{A}), \chi^{M,-1}_{\omega_\infty}).$$

The expression in (4.4.21) can be rewritten as

$$\sum_{\gamma\in\Gamma_{\mathrm{ell},\mathfrak{X}_M}(M)} \frac{v_M \cdot \mathrm{vol}(I^M_\gamma(\mathbb{Q})A_{I^M_\gamma,\infty}\backslash I^M_\gamma(\mathbb{A})/\mathfrak{X}_M) O^M_\gamma(f_{\lambda_{\omega_\infty},\zeta\delta_P^{-1/2}} f^{\infty,(k)}_{M,\omega_p,\omega_q})}{|\iota^M(\gamma)||\mathrm{Stab}^M_{\mathfrak{X}_M}(\gamma)|}$$
$$= v_M \cdot T^M_{\mathrm{ell},\chi^M_{\omega_\infty}}(f_{\lambda_{\omega_\infty},\zeta\delta_P^{-1/2}} f^{\infty,(k)}_{M,\omega_p,\omega_q}). \tag{4.4.22}$$

Put $c''_M := c'_M v_M$. Combining (4.4.20) and (4.4.22), we obtain

$$T^G_{\mathrm{ell},\chi}(f_{\xi,\zeta} f^{\infty,(k)}) = T^G_{\mathrm{disc},\chi}(f_{\xi,\zeta} f^{\infty,(k)}) + \sum_{M,P,\omega_p,\omega_q,\omega_\infty} c''_M \cdot T^M_{\mathrm{ell},\chi^M_{\omega_\infty}}(f_{\lambda_{\omega_\infty},\zeta\delta_P^{-1/2}} f^{\infty,(k)}_{M,\omega_p,\omega_q}). \tag{4.4.23}$$

We have $\omega_p(M \cap B) \subset B$ and $\omega_p^{-1}(M_\nu \cap B) \subset B$ since $\omega_p \in \Omega^{G_{\mathbb{Q}_p}}_{M,M_\nu}$. In particular, for any root α appearing in $\mathrm{Lie}(M \cap N_0)$, $\omega_p\alpha$ also appears in $\mathrm{Lie}(M \cap N_0)$. Consequently,

$$\langle \alpha, w_p^{-1}\nu \rangle = \langle w_p\alpha, \nu \rangle \geq 0,$$

and hence $w_p^{-1}\nu$ is dominant for $M \cap B$. (See the paragraph above Proposition 4.2.1 for dominance of ν relative to B.) By Proposition 4.2.1 and the induction hypothesis for $M \in \mathcal{L}^<_{\mathrm{cusp}}$, we have

$$T^M_{\mathrm{ell},\chi^M_{\omega_\infty}}(f_{\lambda_{\omega_\infty},\zeta\delta_P^{1/2}} f^{\infty,(k)}_{M,\omega_p,\omega_q}) = O\left(p^{k(\langle \rho_M, \omega_p^{-1}\nu\rangle + \langle (\chi^M_{\omega_\infty})_\infty, \mathrm{pr}_M(\omega_p^{-1}\nu)\rangle)}\right). \tag{4.4.24}$$

Again from Proposition 4.2.1 for G we have

$$T^G_{\mathrm{disc},\chi}(f^{(k)}_p f^{\infty,p} f_{\xi,\zeta}) = O\left(p^{k(\langle \rho,\nu\rangle + \langle \chi_\infty, \mathrm{pr}_G\nu\rangle)}\right). \tag{4.4.25}$$

Now assume that $(M, P, \omega_p, \omega_q, \omega_\infty)$ contribute to (4.4.23), in particular $M \in \mathcal{L}^<_{\mathrm{cusp}}$, and also assume that

$$O^M_\gamma(f_{\lambda_{\omega_\infty},\zeta\delta_P^{1/2}} f^{\infty,(k)}_{M,\omega_p,\omega_q}) \neq 0 \tag{4.4.26}$$

for some $\gamma \in \Gamma_{\mathrm{ell},\mathfrak{X}_M}(M)$. Then we claim that

$$\mathrm{Re}(\langle \rho, \nu \rangle + \langle \chi_\infty, \mathrm{pr}_G\nu \rangle) > \mathrm{Re}(\langle \rho_M, \omega_p^{-1}\nu \rangle + \langle (\chi^M_{\omega_\infty})_\infty, \mathrm{pr}_M(\omega_p^{-1}\nu)\rangle). \tag{4.4.27}$$

This claim, together with (4.4.24) and (4.4.25), tells us that the main term for G dominates the proper Levi terms in (4.4.23), thereby implies the theorem. Thus it is enough to prove the claim.

To this end, it is sufficient to show that

(a) $\langle\rho,\nu\rangle > \langle\rho_M,\omega_p^{-1}\nu\rangle$,

(b) $\mathrm{Re}\langle\chi_\infty,\mathrm{pr}_G\nu\rangle \geq \mathrm{Re}\langle(\chi^M_{\omega_\infty})_\infty,\mathrm{pr}_M(\omega_p^{-1}\nu)\rangle$.

Moreover, it is enough to prove that (a) and (b) are true for sufficiently large k (note that the set $\Omega_M^{G\diamond}$ and thus ω_∞ depends on k). Item (a) follows from

$$\langle\rho_M,\omega_p^{-1}\nu\rangle = \langle\rho_{\omega_p M},\nu\rangle < \langle\rho,\nu\rangle.$$

To see "<", first observe that the non-strict inequality "≤" follows from the fact that ω_p is a Kostant representative (cf. (4.4.5)). It remains to check that

$$\langle\rho_{\omega_p M},\nu\rangle \neq \langle\rho,\nu\rangle.$$

This argument is similar to the one in the paragraph below Equation (4.2.2): It suffices to find a root α in $\mathrm{Lie}(G)$ that is not in $\mathrm{Lie}(\omega_p M)$ such that $\langle\alpha,\nu\rangle\neq 0$. As ν is not central, the argument for Lemma 4.5(ii) of [KST] shows that $\mathrm{Lie}(M_\nu)+\mathrm{Lie}(M)\neq\mathrm{Lie}(G)$. Hence we can find a root α in $\mathrm{Lie}(G)$ which does not occur in either $\mathrm{Lie}(M)$ or $\mathrm{Lie}(M_\nu)$, *i.e.*, $\langle\alpha,\nu\rangle\neq 0$.

We now focus on (b). We first make the following temporary assumption

$$\langle\mathrm{pr}_M^*(-\omega_\infty\star\lambda_B^*)-\overline{\rho}_N-\omega_\xi,\mathrm{pr}_M(w_p^{-1}\nu)\rangle\neq 0. \tag{4.4.28}$$

From the equality $I(\gamma)=I(\omega_\infty)$ we obtain (cf. (4.4.13))

$$\langle\mathrm{pr}_M^*(-\omega_\infty\star\lambda_B^*)-\overline{\rho}_N-\omega_\xi,x\rangle\leq 0.$$

As $O_\gamma(f^{(k)}_{p,M,\omega_p})\neq 0$, we have $x=-k(\log p)\mathrm{pr}_M(w_p^{-1}\nu)+\varepsilon_p$, where $\varepsilon_p\in\mathrm{pr}_M\mathrm{supp}^O_{\mathfrak{a}_{M_{\omega_p}}}(\omega_p^{-1}\phi_{p,M_{\omega_p}})$ (cf. (4.4.10)). We obtain

$$\langle\mathrm{pr}_M^*(-\omega_\infty\star\lambda_B^*)-\overline{\rho}_N-\omega_\xi,-k(\log p)\mathrm{pr}_M(w_p^{-1}\nu)+\varepsilon_p\rangle\leq 0.$$

We may (and do) assume that k is sufficiently large so that

$$k|\langle\mathrm{pr}_M^*(-\omega_\infty\star\lambda_B^*)-\overline{\rho}_N-\omega_\xi,-(\log p)\mathrm{pr}_M(w_p^{-1}\nu)\rangle| > |\langle\mathrm{pr}_M^*(-\omega_\infty\star\lambda_B^*)-\overline{\rho}_N-\omega_\xi,\varepsilon_p'\rangle|$$

for all $\varepsilon_p'\in\mathrm{pr}_M\mathrm{supp}^O_{\mathfrak{a}_{M_{\omega_p}}}(\omega_p^{-1,\prime}\phi_{p,M_{\omega_p}})$ and all $\omega_p'\in\Omega^{G_{\mathbb{Q}_p}}_{M,M_\nu}$ (this amounts to a lower bound on k that depends only on the initial data). We deduce using the above temporary assumption and $k\ggg 0$ in the above sense that

$$\langle\mathrm{pr}_M^*(-\omega_\infty\star\lambda_B^*)-\overline{\rho}_N-\omega_\xi,\mathrm{pr}_M(w_p^{-1}\nu)\rangle\geq 0. \tag{4.4.29}$$

Observe that if (4.4.28) is false, then (4.4.29) is still true. Hence (4.4.29) is true unconditionally. We now conclude:

$$\begin{aligned}\langle(\chi^M_{\omega_\infty})_\infty,\mathrm{pr}_M(w_p^{-1}\nu)\rangle &= -\langle z^M_{\omega_\infty},\mathrm{pr}_M(w_p^{-1}\nu)\rangle\\ &= -\langle\mathrm{pr}_M^*(\lambda_{\omega_\infty}),\mathrm{pr}_M(w_p^{-1}\nu)\rangle-\langle\zeta\delta_P^{-1/2},\mathrm{pr}_M(w_p^{-1}\nu)\rangle\\ &= -\langle\mathrm{pr}_M^*(-w_0(\omega_\infty\star\lambda_B^*)),\mathrm{pr}_M(w_p^{-1}\nu)\rangle-\langle\zeta-\rho_N,\mathrm{pr}_M(w_p^{-1}\nu)\rangle\\ &= \underbrace{\langle\mathrm{pr}_M^*(\omega_\infty\star\lambda_B^*)+\overline{\rho}_N+\omega_\xi,\mathrm{pr}_M(w_p^{-1}\nu)\rangle}_{\leq 0}+\underbrace{\langle-\omega_\xi-\zeta,\mathrm{pr}_M(w_p^{-1}\nu)\rangle}_{=\langle\chi_\infty,\mathrm{pr}_M(w_p^{-1}\nu)\rangle}\end{aligned}$$

□

5. Shimura varieties of Hodge type

The goal of this section is to set up the scene for the mod p geometry of Shimura varieties and central leaves, paving the way for introducing Igusa varieties in the next section. We pay special attention to the connected components (synonymous to top-dimensional irreducible components in our setting) and phrase their description in the representation-theoretic language via H^0.

5.1. **Connected components in characteristic zero.** From here on, let (G,X) be a Shimura datum as in [Del79] satisfying axioms (2.1.1.1), (2.1.1.2), and (2.1.1.3) therein. (We assume that (G,X) is of Hodge type starting in the next subsection.) Write $E = E(G,X)$ for the reflex field [Del79, 2.2.1], which is a finite extension of $\mathbb{Q}$ in $\mathbb{C}$. We have the algebraic closure $\overline{E} \subset \mathbb{C}$. Let K be a neat open compact subgroup of $G(\mathbb{A}^\infty)$. We write $\mathrm{Sh}_K = \mathrm{Sh}_K(G,X)$ for the canonical model over E, which forms a projective system of quasi-projective varieties with finite étale transition maps as K varies. We have the E-scheme $\mathrm{Sh} := \varprojlim_K \mathrm{Sh}_K$. Put $d := \dim \mathrm{Sh}_K$ (which does not depend on K). Write $G(\mathbb{Q})_+$ for the preimage of $G(\mathbb{R})_+$ (defined in §2.4) in $G(\mathbb{Q})$. The closure of $G(\mathbb{Q})_+$ in $G(\mathbb{A}^\infty)$ is denoted by $G(\mathbb{Q})_+^-$.

Recall some facts about connected components from [Del79, 2.1]. We have a bijection

$$\pi_0(\mathrm{Sh}_{K,\overline{E}}) \xrightarrow{\sim} G(\mathbb{Q})\backslash G(\mathbb{A})/G(\mathbb{R})_+K. \tag{5.1.1}$$

This yields a $G(\mathbb{A}^\infty)$-equivariant bijection $\pi_0(\mathrm{Sh}_{\overline{E}}) \xrightarrow{\sim} G(\mathbb{A})/G(\mathbb{Q})\varrho(G_{\mathrm{sc}}(\mathbb{A}))G(\mathbb{R})_+$ upon taking limit over all K. Taking quotient by a particular K recovers the above bijection (5.1.1). Note that $G(\mathbb{A})/G(\mathbb{Q})\varrho(G_{\mathrm{sc}}(\mathbb{A}))G(\mathbb{R})_+$ is an abelian group quotient of $G(\mathbb{A})$, and $G(\mathbb{Q})\backslash G(\mathbb{A})/G(\mathbb{R})_+K$ is a finite abelian group quotient.

Fix a prime ℓ and a field isomorphism $\iota : \overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$. When V is a $\overline{\mathbb{Q}}_\ell$-vector space (possibly with a group or algebra action), write $\iota V := V \otimes_{\overline{\mathbb{Q}}_\ell,\iota} \mathbb{C}$. By convention, all instances of cohomology in this paper are étale cohomology. The description of $\pi_0(\mathrm{Sh}_{\overline{E}})$ translates into a $G(\mathbb{A}^\infty)$-module isomorphism

$$\iota H^0(\mathrm{Sh}_{\overline{E}}, \overline{\mathbb{Q}}_\ell) \simeq \bigoplus_\pi \pi^\infty, \tag{5.1.2}$$

where the sum runs over one-dimensional automorphic representations π such that π_∞ is trivial when restricted to $G(\mathbb{R})_+$. Indeed, at each prime p, we have $\dim \pi_p = 1$ since π_p factors through $G(\mathbb{Q}_p) \to G(\mathbb{Q}_p)^{\mathrm{ab}} = G(\mathbb{Q}_p)/\varrho(G_{\mathrm{sc}}(\mathbb{Q}_p))$, cf. Corollary 2.3.3. It is clear that each one-dimensional automorphic representation has multiplicity one in the space of automorphic forms.

Now fix a prime $p \neq \ell$ and an open compact subgroup $K_p \subset G(\mathbb{Q}_p)$. By taking limit of (5.1.1) over open compact subgroups $K^p \subset G(\mathbb{A}^{\infty,p})$, writing $\mathrm{Sh}_{K_p} := \varprojlim_{K^p} \mathrm{Sh}_{K_pK^p}$,

$$\pi_0(\mathrm{Sh}_{K_p,\overline{E}}) \xrightarrow{\sim} G(\mathbb{Q})_+^- \backslash G(\mathbb{A}^\infty)/K_p. \tag{5.1.3}$$

We have a $G(\mathbb{A}^{\infty,p})$-module[10]

$$H^i(\mathrm{Sh}_{K_p,\overline{E}}, \overline{\mathbb{Q}}_\ell) = \varinjlim_{K^p} H^i(\mathrm{Sh}_{K_pK^p,\overline{E}}, \overline{\mathbb{Q}}_\ell), \qquad i \geq 0,$$

where K^p runs over sufficiently small open compact subgroups of $G(\mathbb{A}^{\infty,p})$. The degree zero part admits an automorphic description similar to (5.1.2).

Lemma 5.1.1. *There is a $G(\mathbb{A}^{\infty,p})$-module isomorphism*

$$\iota H^0(\mathrm{Sh}_{K_p,\overline{E}}, \overline{\mathbb{Q}}_\ell) \simeq \bigoplus_\pi \pi^{\infty,p},$$

where the sum runs over discrete automorphic representations π of $G(\mathbb{A})$ such that (i) $\dim \pi = 1$, (ii) π_p is trivial on K_p, and (iii) π_∞ is trivial on $G(\mathbb{R})_+$.

Proof. This is clear from (5.1.2) by taking K_p-invariants. □

[10]See [Sta16, Tag 03Q4] for the canonical isomorphism, which is $G(\mathbb{A}^{\infty,p})$-equivariant by a routine check. Alternatively, it is harmless to think of the identity as a definition for the left hand side.

5.2. **Integral canonical models.** From now on, assume that (G, X) is a Shimura datum of *Hodge type*. This means that there is an embedding into the Siegel Shimura datum

$$i_{V,\psi} : (G, X) \hookrightarrow (\mathrm{GSp}(V, \psi), S^{\pm}_{V,\psi}),$$

where (V, ψ) is a symplectic space over $\mathbb{Q}$, and $S^{\pm}_{V,\psi}$ denotes the associated Siegel half spaces. For simplicity we write $\mathrm{GSp} = \mathrm{GSp}(V, \psi)$ and $S^{\pm} = S^{\pm}_{V,\psi}$).

We fix a prime p as earlier. To explain integral canonical models for $\mathrm{Sh} = \mathrm{Sh}(G, X)$ at p, we set things up following [Kis17, (1.3.3)], leaving the details to *loc. cit.* For the rest of this paper, we assume that G is unramified over $\mathbb{Q}_p$ and fix a reductive integral model $G_{\mathbb{Z}_{(p)}}$ over $\mathbb{Z}_{(p)}$. We still write G for this model if there is no danger of confusion. Thereby we have a fixed hyperspecial subgroup $K_p := G(\mathbb{Z}_p)$. We refer to this setup by $(\mathrm{Unr}(G, p, K_p))$.

- $(\mathbf{Unr}(G, p, K_p))$**:** G is unramified over $\mathbb{Q}_p$ with a fixed reductive integral model over $\mathbb{Z}_{(p)}$, and $K_p = G(\mathbb{Z}_p)$.

We may assume that $i_{V,\psi}$ is induced by an embedding $G_{\mathbb{Z}_{(p)}} \hookrightarrow \mathrm{GL}(V_{\mathbb{Z}_{(p)}})$ for a $\mathbb{Z}_{(p)}$-lattice $V_{\mathbb{Z}_{(p)}} \subset V$ and that ψ induces a perfect pairing on $V_{\mathbb{Z}_{(p)}}$. There exists a finite set of tensors $(s_\alpha) \subset V^{\otimes}_{\mathbb{Z}_{(p)}}$ such that $G_{\mathbb{Z}_{(p)}}$ is the scheme-theoretic stabilizer of (s_α) in $\mathrm{GL}(V_{\mathbb{Z}_{(p)}})$. We may assume that one of the tensors is given by $\psi \otimes \psi^\vee \in (V^\vee_{\mathbb{Z}_{(p)}})^{\otimes 2} \otimes V^{\otimes 2}_{\mathbb{Z}_{(p)}}$, whose stabilizer is $\mathrm{GSp}(V_{\mathbb{Z}_{(p)}}, \psi)$.[11] We fix the set (s_α). There is a hyperspecial subgroup $K'_p \subset \mathrm{GSp}(V, \psi)(\mathbb{Q}_p)$ extending K_p (i.e., $K'_p \cap G(\mathbb{Q}_p) = K_p$) such that $i_{V,\psi}$ induces an embedding of Shimura varieties over the reflex field E (so that the map is induced by $i_{V,\psi} : X \to S^{\pm}_{V,\psi}$ on $\mathbb{C}$-points)

$$\mathrm{Sh}_{K_p}(G, X) \hookrightarrow \mathrm{Sh}(\mathrm{GSp}, S^{\pm}) \otimes_{\mathbb{Q}} E. \tag{5.2.1}$$

It is implied by $(\mathrm{Unr}(G, p, K_p))$ that p is unramified in E. We fix an isomorphism $\iota_p : \mathbb{C} \simeq \overline{\mathbb{Q}}_p$, determining an embedding $E \hookrightarrow \overline{\mathbb{Q}}_p$ as well as a p-adic place $\mathfrak{p}$ of E. Thus we may identify $\overline{E}_{\mathfrak{p}} \simeq \overline{\mathbb{Q}}_p$. The integer ring $\mathcal{O}_E$ localized at $\mathfrak{p}$ is denoted by $\mathcal{O}_{E,(\mathfrak{p})}$, and its residue field by $k(\mathfrak{p})$. Identify the residue field of $\overline{\mathbb{Q}}_p$ with $\overline{\mathbb{F}}_p$, thus fixing an embedding $k(\mathfrak{p}) \hookrightarrow \overline{\mathbb{F}}_p$.

Kisin [Kis10, Thm. 2.3.8] (for $p > 2$) and Kim–Madapusi Pera [KMP16, Thm. 4.11] (for $p = 2$) constructed integral canonical models, namely a projective system of smooth quasi-projective schemes $\mathscr{S}_{K_pK^p}$ over $\mathcal{O}_{E,(\mathfrak{p})}$ for all sufficiently small open compact subgroups $K^p \subset G(\mathbb{A}^{\infty,p})$ with finite étale transition maps $\mathscr{S}_{K_pK^{p,\prime}} \to \mathscr{S}_{K_pK^p}$ for $K^{p,\prime} \subset K^p$. The projective system is equipped with an action of $G(\mathbb{A}^{\infty,p})$, given by the isomorphism

$$\mathscr{S}_{K_pK^p} \overset{\sim}{\to} \mathscr{S}_{K_pg^{-1}K^pg}, \qquad g \in G(\mathbb{A}^{\infty,p}), \quad K^p \subset G(\mathbb{A}^{\infty,p}),$$

extending the isomorphism $\mathrm{Sh}_{K_pK^p} \overset{\sim}{\to} \mathrm{Sh}_{K_pg^{-1}K^pg}$ giving the action of g on the generic fiber. The inverse limit $\mathscr{S}_{K_p} := \varprojlim_{K^p} \mathscr{S}_{K_pK^p}$ is a scheme over $\mathcal{O}_{E,(\mathfrak{p})}$ with a $G(\mathbb{A}^{\infty,p})$-action, characterized uniquely by an extension property [Kis10, Thm. (2.3.8), (2)]. The construction yields a map of $\mathcal{O}_{E,(\mathfrak{p})}$-schemes

$$\mathscr{S}_{K_p} \to \mathscr{S}_{K'_p}(\mathrm{GSp}, S^{\pm}) \otimes_{\mathbb{Z}_{(p)}} \mathcal{O}_{E,(\mathfrak{p})}, \tag{5.2.2}$$

whose fiber over E is identified with (5.2.1), where $\mathscr{S}_{K'_p}(\mathrm{GSp}, S^{\pm})$ is the integral model over $\mathbb{Z}_{(p)}$ for $\mathrm{Sh}(\mathrm{GSp}(V, \psi), S^{\pm}_{V,\psi})$ parametrizing polarized abelian schemes up to prime-to-p isogenies with prime-to-p level structure, as in [Kis10, (2.3.3)]. Moreover we have universal polarized abelian schemes $h : \mathcal{A}_{K_pK^p} \to \mathscr{S}_{K_pK^p}$ compatible with the transition maps in the projective system.

Let $\mathscr{S}_{K_pK^p,k(\mathfrak{p})} := \mathscr{S}_{K_pK^p} \otimes_{\mathcal{O}_{E,(\mathfrak{p})}} k(\mathfrak{p})$ denote the special fiber. Write Sh_{K_p} (resp. $\mathscr{S}_{K_p,k(\mathfrak{p})}$) for the inverse limit of $\mathrm{Sh}_{K_pK^p}$ (resp. $\mathscr{S}_{K_pK^p,k(\mathfrak{p})}$) over K^p. By base change to $\overline{E}_{\mathfrak{p}}$, $\mathcal{O}_{\overline{E}_{\mathfrak{p}}}$, and $\overline{k(\mathfrak{p})}$,

[11] This way the weak polarization in the sense of [Kis17] is remembered by a geometric incarnation of (s_α). So we need not keep track of polarizations on abelian varieties separately.

respectively, we obtain $\mathrm{Sh}_{K_p,\overline{E}_\mathfrak{p}}$, $\mathscr{S}_{K_p,\mathcal{O}_{\overline{E}_\mathfrak{p}}}$, and $\mathscr{S}_{K_p,\overline{k(\mathfrak{p})}}$ from Sh_{K_p}, $\mathscr{S}_{K_p}$, and $\mathscr{S}_{K_p,k(\mathfrak{p})}$. There are canonical $G(\mathbb{A}^{\infty,p})$-equivariant embeddings of generic and special fibers

$$\mathrm{Sh}_{K_p,\overline{E}_\mathfrak{p}} \hookrightarrow \mathscr{S}_{K_p,\mathcal{O}_{\overline{E}_\mathfrak{p}}} \hookleftarrow \mathscr{S}_{K_p,\overline{k(\mathfrak{p})}}.$$

These embeddings induce $G(\mathbb{A}^{\infty,p})$-equivariant bijections by means of arithmetic compactification as shown in [MP19, Cor. 4.1.11]:

$$\pi_0(\mathrm{Sh}_{K_p,\overline{E}_\mathfrak{p}}) \stackrel{\sim}{\to} \pi_0(\mathscr{S}_{K_p,\mathcal{O}_{\overline{E}_\mathfrak{p}}}) \stackrel{\sim}{\leftarrow} \pi_0(\mathscr{S}_{K_p,\overline{k(\mathfrak{p})}}).$$

Proposition 5.2.1. *The $G(\mathbb{A}^{\infty,p})$-action is transitive on $\pi_0(\mathrm{Sh}_{K_p,\overline{E}_\mathfrak{p}})$ and $\pi_0(\mathscr{S}_{K_p,\overline{k(\mathfrak{p})}})$.*

Proof. By the preceding proposition, it is enough to check the transitivity on $\pi_0(\mathrm{Sh}_{K_p,\overline{E}_\mathfrak{p}})$, which is [Kis10, Lem. 2.2.5] (using the fact that K_p is hyperspecial).[12] □

Let T be a $k(\mathfrak{p})$-scheme. At each point $x \in \mathscr{S}_{K_pK^p}(T)$ we have an abelian variety $\mathcal{A}_x$ over T (up to a prime-to-p isogeny) pulled back from $\mathcal{A}_{K_pK^p}$. As in [Kis17, (1.3.6)], we have $(\mathbf{s}_{\alpha,\ell}) \subset (R^1 h_{\text{ét}*}\mathbb{Q}_\ell)^\otimes$ for each prime $\ell \neq p$. By pullback, we equip the prime-to-p rational Tate module $V^p(\mathcal{A}_x)$ of $\mathcal{A}_x$ with $(\mathbf{s}_{\alpha,\ell,x})_{\ell\neq p}$.

When $T = \operatorname{Spec} k$ with $k/k(\mathfrak{p})$ an extension in $\overline{k(\mathfrak{p})}$, write $\mathbb{D}(\mathcal{A}_x[p^\infty])$ for the (integral) Dieudonné module of $\mathcal{A}_x[p^\infty]$, and Φ_x for the Frobenius operator acting on it. Following [Kis17, (1.3.10)] we have crystalline Tate tensors $(s_{\alpha,0,x}) \subset \mathbb{D}(\mathcal{A}_x[p^\infty])^\otimes$ coming from (s_α). Lovering [Lov17], and also Hamacher [Ham19, §2.2], have globalized $(s_{\alpha,0,x})$. Namely there exist crystalline Tate tensors $(\mathbf{s}_{\alpha,0})$ on the Dieudonné crystal $\mathbb{D}(\mathcal{A}_{K_pK^p}[p^\infty])$ associated with $\mathcal{A}_{K_pK^p}[p^\infty]$ over $\mathscr{S}_{K_pK^p,\overline{k(\mathfrak{p})}}$ such that $(\mathbf{s}_{\alpha,0})$ specializes to $(s_{\alpha,0,x})$ at every $x \in \mathscr{S}_{K_pK^p}(\overline{k(\mathfrak{p})})$.

5.3. **Central leaves.** A central leaf in the special fiber of a Shimura variety of Hodge type is the locus of points where the corresponding abelian varieties have p-divisible groups (with extra structure) in a fixed isomorphism class. This notion has been introduced by [Oor04, Man05, Ham19, Zha]. See also [HK19].

Let $B(G_{\mathbb{Q}_p})$ denote the set of $(G(\breve{\mathbb{Q}}_p),\sigma)$-conjugacy classes in $G(\breve{\mathbb{Q}}_p)$. Fix a Borel subgroup $B \subset G_{\mathbb{Z}_p}$ (here $G_{\mathbb{Z}_p}$ comes from the fixed model over $\mathbb{Z}_{(p)}$) and a maximal torus $T \subset B$. We have the set of dominant coweights $X_*(T_{\overline{\mathbb{Q}}_p})^+$ and $X_*(T_{\overline{\mathbb{Q}}_p})^+_{\mathbb{Q}}$. Via the fixed isomorphism $\iota_p : \overline{\mathbb{Q}}_p \simeq \mathbb{C}$, we obtain $T_\mathbb{C} \subset B_\mathbb{C} \subset G_\mathbb{C}$ as well as $X_*(T_\mathbb{C})^+$ and $X_*(T_\mathbb{C})^+_\mathbb{Q}$. Since the conjugacy class $\{\mu_X\}$ is defined over E and since $G_{\mathbb{Q}_p}$ is quasi-split, we have a cocharacter $\mu_p \in X_*(T_{\overline{\mathbb{Q}}_p})^+$ defined over $E_\mathfrak{p}$ in the conjugacy class $\{\iota_p\mu_X\}$. When there is no danger of confusion, we omit the subscripts $\overline{\mathbb{Q}}_p$ and $\mathbb{C}$. Write $\rho \in X^*(T)_\mathbb{Q}$ for the half sum of all positive roots, and $\langle\cdot,\cdot\rangle$ for the canonical pairing $X^*(T)_\mathbb{Q} \times X_*(T)_\mathbb{Q} \to \mathbb{Q}$ or its extension to $\mathbb{C}$-coefficients.

Each $b \in G(\breve{\mathbb{Q}}_p)$ gives rise to a Newton cocharacter $\nu_b \colon \mathbb{D} \to G_{\breve{\mathbb{Q}}_p}$ (so it is a "fractional" cocharacter of $G_{\breve{\mathbb{Q}}_p}$) and a connected reductive group J_b over $\mathbb{Q}_p$ given by

$$J_b(R) := \{g \in G(R \otimes_{\mathbb{Q}_p} \breve{\mathbb{Q}}_p) : g^{-1}b\sigma(g) = b\}, \quad R : \mathbb{Q}_p\text{-algebra}. \tag{5.3.1}$$

Lemma 5.3.1. *The Newton cocharacter ν_b factors through the center of J_b. The induced cocharacter $\mathbb{D} \to A_{J_b}$ is $\mathbb{Q}_p$-rational.*

Proof. The centrality follows from [Kot85, (4.4.2)]. The cocharacter $\mathbb{D} \to A_{J_b}$ is σ-invariant by the definition of J_b, thus $\mathbb{Q}_p$-rational. □

[12] Alternatively, this also follows from the weak approximation theorem, which tells us that $G(\mathbb{Q}) \hookrightarrow G(\mathbb{Q}_p) \times G(\mathbb{R})$ has dense image. For this, apply [PR94, Thm. 7.7] and notice that the set S_0 of the theorem can be taken away from p and ∞ from the discussion in §7.3 of *loc. cit.* since G is unramified at p.

We define an open compact subgroup of $J_b(\mathbb{Q}_p)$ (where "int" stands for integral):

$$J_b^{\mathrm{int}} := J_b(\mathbb{Q}_p) \cap G(\breve{\mathbb{Z}}_p) = \{g \in G(\breve{\mathbb{Z}}_p) : g^{-1} b \sigma(g) = b\}.$$

Given $b \in G(\breve{\mathbb{Q}}_p)$, we denote its $(G(\breve{\mathbb{Q}}_p), \sigma)$-conjugacy class by $[b]$ and $(G(\breve{\mathbb{Z}}_p), \sigma)$-conjugacy class by $[[b]]$. Recall that $b \in G(\breve{\mathbb{Q}}_p)$, or $[b] \in B(G_{\mathbb{Q}_p})$, is **basic** if $\nu_b : \mathbb{D} \to G_{\breve{\mathbb{Q}}_p}$ has image in $Z(G_{\breve{\mathbb{Q}}_p})$, or equivalently if J_b is an inner form of G [RR96, Prop. 1.12]. The following condition will appear in our irreducibility results later. The definition makes a difference only when G^{ad} is not $\mathbb{Q}$-simple. See Lemma 5.3.7 below for a relation to §2.5.

Definition 5.3.2. Let $G^{\mathrm{ad}} = \prod_{i \in I} G_i^{\mathrm{ad}}$ be a decomposition into $\mathbb{Q}$-simple factors. An element $b \in G(\breve{\mathbb{Q}}_p)$, or $[b] \in B(G_{\mathbb{Q}_p})$, is said to be **$\mathbb{Q}$-non-basic** if its image in $B(G_{i,\mathbb{Q}_p})$ via the natural composite map $G \to G^{\mathrm{ad}} \to G_i$ is non-basic for every $i \in I$.

Remark 5.3.3. The definition is not purely local in that it depends on not only $G_{\mathbb{Q}_p}$ but also G. Compare $G = \mathrm{GL}_2 \times \mathrm{GL}_2$ with $G = \mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_2$, where F is a real quadratic field in which p splits.

Let $\overline{x} : \operatorname{Spec} k \hookrightarrow \mathscr{S}_{K_p K^p, k(\mathfrak{p})}$ be a geometric point supported at $x \in \mathscr{S}_{K_p K^p, k(\mathfrak{p})}$, with $k/k(\mathfrak{p})$ be an algebraically closed field extension. Write $W := W(k)$ and $L := \mathrm{Frac} W$ (so that $W = \breve{\mathbb{Z}}_p$ and $L = \breve{\mathbb{Q}}_p$ if $k = \overline{\mathbb{F}}_p$). We will still write σ for the canonical Frobenius on W. There exists a W-linear isomorphism

$$V^*_{\mathbb{Z}_{(p)}} \otimes_{\mathbb{Z}_{(p)}} W \simeq \mathbb{D}(\mathcal{A}_{\overline{x}}[p^\infty]) \tag{5.3.2}$$

carrying (s_α) to $(s_{\alpha,0,x})$. We transport the Frobenius operator $\Phi_{\overline{x}}$ on the right hand side to $b_{\overline{x}}(1 \otimes \sigma)$ on the left hand side to define $b_{\overline{x}} \in G(L)$. Then $[[b_{\overline{x}}]]$ (thus also $[b_{\overline{x}}]$) is independent of the choice of isomorphism.

The central leaf is defined to be the set

$$C_{b,K^p} := \{x \in \mathscr{S}_{K_p K^p, k(\mathfrak{p})} : \exists \text{ isom. of crystals } V^*_{\mathbb{Z}_{(p)}} \otimes_{\mathbb{Z}_{(p)}} W \simeq \mathbb{D}(\mathcal{A}_{\overline{x}}[p^\infty]) \text{ s.t. } (s_\alpha) \mapsto (\mathbf{s}_{\alpha,0,\overline{x}})\},$$

where the isomorphism of crystals means that the σ-linear map $b(1 \otimes \sigma)$ is carried to $\Phi_{\overline{x}}$ via the W-linear isomorphism. The definition of C_{b,K^p} depends only on $[[b]]$ since $[[b]]$ determines the crystal $V^*_{\mathbb{Z}_{(p)}} \otimes_{\mathbb{Z}_{(p)}} \breve{\mathbb{Z}}_p$ with Frobenius operator $b(1 \otimes \sigma)$ and crystalline Tate tensors (s_α).

By [Ham17, Prop. 2, p.1262] and since Newton strata are locally closed ([RR96, Thm. 3.6]), C_{b,K^p} is a locally closed subset of $\mathscr{S}_{K_p K^p, k(\mathfrak{p})}$. (The analogous assertion for Kisin–Pappas models is shown in [HK19, Cor. 4.12].) We promote C_{b,K^p} to a locally closed $k(\mathfrak{p})$-subscheme of $\mathscr{S}_{K_p K^p, k(\mathfrak{p})}$ equipped with reduced subscheme structure. We still write C_{b,K^p} for the scheme and call it the **central leaf** associated with b. The projection maps between $\mathscr{S}_{K_p K^p, k(\mathfrak{p})}$ as K^p varies, which are finite étale ([Kis10, Thm. 2.3.8]), induce finite étale projection maps between C_{b,K^p}. Put $C_b := \varprojlim_{K^p} C_{b,K^p}$. The following proposition is due to C. Zhang and Hamacher [Zha, Ham19] independently.

Proposition 5.3.4. *The $k(\mathfrak{p})$-scheme C_{b,K^p} is smooth. If nonempty, its dimension is $\langle 2\rho, \nu_b \rangle$.*

Proof. These properties can be checked after extending base to $\overline{k(\mathfrak{p})}$. Since C_{b,K^p} is reduced, it is still reduced over $\overline{k}(\mathfrak{p})$. Thus the proposition follows from [Ham19, Prop. 2.6]. □

A finite subset $B(G_{\mathbb{Q}_p}, \mu_p^{-1}) \subset B(G_{\mathbb{Q}_p})$ has been defined in [Kot97, §6] by a group-theoretic generalization of Mazur's inequality. The set $B(G_{\mathbb{Q}_p}, \mu_p^{-1})$ contains exactly one basic element, but may contain several elements that are not $\mathbb{Q}$-non-basic. The significance of $B(G_{\mathbb{Q}_p}, \mu_p^{-1})$ is clear from the following.

Proposition 5.3.5. *The central leaf C_{b,K^p} is nonempty if and only if $[b] \in B(G_{\mathbb{Q}_p}, \mu_p^{-1})$.*

Proof. It follows from [KMPS, Prop. 1.3.9] that the Newton stratum for b is nonempty if and only if $[b] \in B(G_{\mathbb{Q}_p}, \mu_p^{-1})$. Thus the "only if" part of the proposition holds. The "if" part follows from this via [HK19, Rem. 5.6], noting that Axiom A there is proven by Kisin; see the paragraph above Remark 5.6 therein. $\square$

Henceforth we always assume that $[b] \in B(G_{\mathbb{Q}_p}, \mu_p^{-1})$ in light of the proposition. Other than this, we have not imposed conditions on $b \in G(\breve{\mathbb{Q}}_p)$ but now we replace b with a better representative in its $(G(\breve{\mathbb{Z}}_p), \sigma)$-conjugacy class (so that C_{b,K^p} does not change). Since C_{b,K^p} is a variety of finite type over $k(\mathfrak{p})$, there exists $x \in C_{b,K^p}(\mathbb{F}_{p^r})$ (with $\mathbb{F}_{p^r} \supset k(\mathfrak{p})$); in particular $[[b_x]] = [[b]]$. We may increase r to be divisible by $[E_\mathfrak{p} : \mathbb{Q}_p]$ such that $\sigma^r(\mu_p) = \mu_p$ and that $(s_{\alpha,0,x})$ are all defined over $\mathbb{Q}_{p^r}$. As in [Kis17, (1.4.1)], we have a $\mathbb{Z}_{p^r}$-linear isomorphism (5.3.2)

$$V^*_{\mathbb{Z}_{(p)}} \simeq \mathbb{D}(\mathcal{A}_x[p^\infty]) \tag{5.3.3}$$

carrying (s_α) to $(s_{\alpha,0,x})$. We fix such an isomorphism. Let Φ_x denote the (absolute) Frobenius operator on the right hand side. As before, Φ_x is transported to $b_x(1 \otimes \sigma)$ so that

$$b_x \in G(\mathbb{Z}_{p^r})\sigma\mu_p(p)^{-1}G(\mathbb{Z}_{p^r})$$

by [Kis17, (1.4.1)]. We see from [Kis17, (1.4.4)] that $b_x\sigma(b_x)\cdots\sigma^{r-1}(b_x)\sigma^r$ acting on the left hand side of (5.3.3) is transported to the geometric p^r-Frobenius on the right hand side. The latter is a semisimple $\mathbb{Q}_{p^r}$-linear map by Tate's theorem. After making r further divisible, we may assume that the geometric p^r-Frobenius is diagonalizable and that $r\nu_{b_x} : \mathbb{G}_m \to G_{\breve{\mathbb{Q}}_p}$ is a cocharacter (as opposed to a fractional cocharacter). Since $b_x \in G(\mathbb{Q}_{p^r})$, [Kot85, (4.4.1)] tells us that ν_{b_x} is defined over $\mathbb{Q}_{p^r}$. In particular the slope decomposition of $\mathbb{D}(\mathcal{A}_x[p^\infty]) \otimes_{\mathbb{Z}_{p^r}} \mathbb{Q}_{p^r}$ is defined over $\mathbb{Q}_{p^r}$. The map $\iota r\nu_{b_x} : \mathbb{G}_m \to \mathrm{GSp}_{\breve{\mathbb{Q}}_p}$ maps p to the scalar $p^{r\lambda}$ on each slope λ-component (with $r\lambda \in \mathbb{Z}$); so in fact $r\nu_{b_x}$ is defined over $\mathbb{Q}_{p^r}$. The matching of the p^r-Frobenius now gives

$$b_x\sigma(b_x)\cdots\sigma^{r-1}(b_x) = r\nu_{b_x}(p)$$

(This is called the decency equation, cf. [RZ96, Def. 1.8].)

In summary, replacing b with b_x and making r more divisible as above, we may and will assume from now on that

- (br1) $\mathbb{F}_{p^r} \supset k(\mathfrak{p})$ and $C_{b,K^p}(\mathbb{F}_{p^r}) \neq \emptyset$,
- (br2) $b \in G(\mathbb{Z}_{p^r})\sigma\mu_p(p)^{-1}G(\mathbb{Z}_{p^r})$,
- (br3) $b\sigma(b)\cdots\sigma^{r-1}(b) = r\nu_b(p)$.

These conditions have the following implications. (So the running assumption that $[b] \in B(G_{\mathbb{Q}_p}, \mu_p^{-1})$ is subsumed by (br2).)

- (br1)' μ_p is defined over $\mathbb{Q}_{p^r}$, by (br1).
- (br2)' $[b] \in B(G_{\mathbb{Q}_p}, \mu_p^{-1})$ and ν_b is defined over $\mathbb{Q}_{p^r}$, by (br2).

Since μ_p is defined over $E_\mathfrak{p}$, which is unramified over $\mathbb{Q}_p$, (br1)' is easy to see. In (br2)', $[b] \in B(G_{\mathbb{Q}_p}, \mu_p^{-1})$ comes from [RR96, Thm. 4.2]; we already explained above that ν_b is defined over $\mathbb{Q}_{p^r}$ if $b \in G(\mathbb{Q}_{p^r})$.

Since the $G(\mathbb{Q}_{p^r})$-conjugacy class of $r\nu_b$ is defined over $\mathbb{Q}_p$ [Kot85, (4.4.3)], and since $G_{\mathbb{Q}_p}$ is quasi-split, there exists $h \in G(\mathbb{Q}_{p^r})$ such that $h^{-1}(r\nu_b)h$ is defined over $\mathbb{Q}_p$. Multiplying h on the right by an element of $G(\mathbb{Q}_p)$, we can ensure that $h^{-1}(r\nu_b)h$ factors through $\mathbb{G}_m \to T$ and is B-dominant, namely $h^{-1}(r\nu_b)h \in X_*(T)^+$. Fix such a h and put $b^\circ := h^{-1}b\sigma(h)$ so that $\nu_{b^\circ} = h^{-1}(\nu_b)h$ from [Kot85, (4.4.2)]. We also have a $\mathbb{Q}_p$-isomorphism

$$J_b \xrightarrow{\sim} J_{b^\circ}, \qquad g \mapsto h^{-1}gh$$

determined by h, which carries $r\nu_b$ to $r\nu_{b^\circ}$.

Starting from $\nu_{b^\circ} \in X_*(T)^+_{\mathbb{Q}}$ defined over $\mathbb{Q}_p$ as above, we put $P_{b^\circ} := P_{\nu_{b^\circ}}$ in the notation of §3.1, and similarly define $P^{\rm op}_{b^\circ}$, N_{b°, $N^{\rm op}_{b^\circ}$, and M_{b°. In particular $P^{\rm op}_{b^\circ}$ (resp. M_{b°) is a standard $\mathbb{Q}_p$-rational parabolic (resp. Levi) subgroup of $G_{\mathbb{Q}_p}$, and M_{b° is the centralizer of ν_{b° in $G_{\mathbb{Q}_p}$. There is an inner twist [RZ96, Cor. 1.14]

$$J_{b^\circ} \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^r} \simeq M_{b^\circ} \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^r}$$

given by the cocycle $\mathrm{Gal}(\mathbb{Q}_{p^n}/\mathbb{Q}_p) \to M_{b^\circ}(\mathbb{Q}_{p^r})$, $\sigma \mapsto b^\circ$. Thus M_{b° is also an inner twist of J_b over $\mathbb{Q}_p$ (which is independent of the choice of b° up to isomorphism of inner twists by routine check). Under the canonical $\mathbb{Q}_p$-isomorphisms $Z(M_{b^\circ}) \simeq Z(J_b)$ and $A_{M_{b^\circ}} = A_{J_b}$, it is readily checked that ν_{b° is carried to ν_b.

Example 5.3.6. We have the following for the ordinary strata of modular curves, when $G_{\mathbb{Q}_p} = \mathrm{GL}_2$. Take B and T to the subgroup of upper triangular (resp. diagonal) matrices. Then μ is the cocharacter $z \mapsto \mathrm{diag}(z, 1)$ up to conjugacy. We can take $b = b^\circ$ such that $\nu_b(z) = \mathrm{diag}(1, z^{-1})$, which is visibly B-dominant. Then $P^{\rm op}_b = B = P_{-\nu_b}$, $M_b = T$, and $\delta_{P_b}(\nu_b(p)) = |p^{-1}| = p$.

Lemma 5.3.7. *The element $b \in G(\breve{\mathbb{Q}}_p)$ as above is $\mathbb{Q}$-non-basic if and only if $(\mathbb{Q}\text{-nb}(P_b))$ of §2.5 holds.*

Proof. Write $G^{\rm ad} = \prod_{i\in I} G^{\rm ad}_i$ as in Definition 5.3.2 and $b_i \in G^{\rm ad}_i(\breve{\mathbb{Q}}_p)$ for the image of b. By functoriality of Newton cocharacters, the composition of ν_b with the natural map $G \to G^{\rm ad}_i$ is ν_{b_i}, which is $\mathbb{Q}_p$-rational since ν_b is. This implies that the image of P_b in $G^{\rm ad}_i$ is P_{b_i}, where $P_{b_i} \subset G^{\rm ad}_i$ is defined analogously as P_b in G over $\mathbb{Q}_p$. Each $b_i \in G^{\rm ad}_i(\breve{\mathbb{Q}}_p)$ is basic if and only if ν_{b_i} is central in $G^{\rm ad}_i$ (i.e. trivial) if and only if $P_{b_i} = G^{\rm ad}_i$. Therefore $(\mathbb{Q}\text{-nb}(P_b))$ holds if and only if b_i is non-basic for every $i \in I$, and the latter is the definition for b to be $\mathbb{Q}$-non-basic. □

Lemma 5.3.8. *Let $1 \to Z_1 \to G_1 \to G \to 1$ be a z-extension over $\mathbb{Q}_p$. Let $\mu_p : \mathbb{G}_m \to G$ be a cocharacter over $\mathbb{Q}_{p^r}$ as above, and suppose that b and r satisfy (br1)–(br3). Then there exist the following data:*

- *a cocharacter $\mu_{p,1} : \mathbb{G}_m \to G_1$ over $\mathbb{Q}_{p^r}$ lifting μ together with $b_1 \in G_1(\mathbb{Q}_{p^r})$ lifting b such that $b_1 \in G_1(\mathbb{Z}_{p^r})\sigma\mu_{p,1}(p)^{-1}G_1(\mathbb{Z}_{p^r})$ (thus the analogues of (br1)', (br2), and (br2)' for G_1 hold true),*
- *$b_1^\circ \in G_1(\mathbb{Q}_{p^r})$ in the σ-conjugacy class of b_1 such that ν_{b° is defined over $\mathbb{Q}_p$ and lifts ν_b°.*

Moreover, we can make r more divisible, while keeping the same μ_p, b, $\mu_{p,1}$, and b_1°, such that $r\nu_{b_1}$ is a cocharacter of G_1.[13] *(Thus the analogues of (br1)', (br2), and (br2)' for G_1 continue to hold for the new r.)*

Proof. By the proof of Lemma 3.6.1, there exists $\mu_{p,1}$ over $\mathbb{Q}_{p^r}$ lifting μ; we fix any $\mu_{p,1}$. Since $G_1(\mathbb{Z}_{p^r}) \to G(\mathbb{Z}_{p^r})$ is onto (by the surjectivity on $\mathbb{F}_{p^r}$-points and the smoothness of $G_1 \to G$), the map $G_1(\mathbb{Q}_{p^r}) \to G(\mathbb{Q}_{p^r})$ induces a surjection

$$G_1(\mathbb{Z}_{p^r})\sigma\mu_{p,1}(p)^{-1}G_1(\mathbb{Z}_{p^r}) \twoheadrightarrow G(\mathbb{Z}_{p^r})\sigma\mu(p)^{-1}G(\mathbb{Z}_{p^r}).$$

Take $b_1 \in G_1(\mathbb{Q}_{p^r})$ to be any preimage of b under this map. This takes care of the first point in the lemma. As for the second point, since the G_1 is quasi-split over $\mathbb{Q}_p$, there exists $b_1^\circ \in G_1(\mathbb{Q}_{p^r})$ σ-conjugate to b_1 such that $\nu_{b_1^\circ}$ is defined over $\mathbb{Q}_p$, and also such that $\nu_{b_1^\circ}$ factors through $T \subset G$, as explained for ν_{b° above. Then the composite of $\nu_{b_1^\circ}$ with $G_1 \twoheadrightarrow G$ is conjugate to ν_{b° in G, so one is carried to the other by an element of the $\mathbb{Q}_p$-rational Weyl group of G [Kot84a, Lem. 1.1.3 (a)]. Identifying the latter group with the $\mathbb{Q}_p$-rational Weyl group of G_1, we can use the same element to modify $\nu_{b_1^\circ}$ so that $\nu_{b_1^\circ}$ maps to ν_{b° under $G_1 \twoheadrightarrow G$. Finally, the last point on r in the lemma is trivial. □

[13] A priori we only know that $r\nu_{b_1}$ is a fractional cocharacter, even though $r\nu_b$ is an (integral) cocharacter of G.

In the setup of the lemma, we introduce $\mathbb{Q}_p$-algebraic groups J_{b_1}, $J_{b_1^\circ}$, $P_{b_1^\circ}$, $M_{b_1^\circ}$, etc. for G_1 by mimicking the definition for G. Let T_1, B_1 denote the preimages of T, B in G_1. Since $\nu_{b_1^\circ}$ maps to ν_{b°, it is clear that $\nu_{b_1}^\circ \in X_*(T_1)^+$, where $+$ means B_1-dominance, and that $P_{b_1^\circ}$, $M_{b_1^\circ}$ map to P_{b°, M_{b°. As before, we can identify $Z(M_{b_1^\circ}) = Z(J_{b_1^\circ})$, which carries $\nu_{b_1^\circ}$ to ν_{b_1}. With this understanding, we will abuse notation to write $M_b, P_b, M_{b_1}, P_{b_1}$ etc. for $M_{b^\circ}, P_{b^\circ}, M_{b_1^\circ}, P_{b_1^\circ}$ etc. to simplify notation, and write ν_b, ν_{b_1} for $\nu_{b^\circ}, \nu_{b_1^\circ}$ if there is little danger of confusion.

5.4. **The Hecke orbit conjecture.** We state Oort's Hecke orbit conjecture for Shimura varieties of Hodge type with hyperspecial level at p, breaking it down into discrete and continuous parts following [Cha06]. The reader is reminded of the setup ($\mathrm{Unr}(G, p, K_p)$) that $G_{\mathbb{Q}_p}$ is unramified and that $K_p \subset G(\mathbb{Q}_p)$ is hyperspecial.

Let $x \in \mathscr{S}_{K^pK_p,k(\mathfrak{p})}(\overline{\mathbb{F}}_p)$. Denote by $\tilde{x} \subset |\mathscr{S}_{K_p,k(\mathfrak{p})}|$ the preimage of x in the topological space $|\mathscr{S}_{K_p,k(\mathfrak{p})}|$ via the projection map $\mathscr{S}_{K_p,k(\mathfrak{p})} \to \mathscr{S}_{K^pK_p,k(\mathfrak{p})}$. Define the prime-to-$p$ **Hecke orbit**

$$H(x) := \tilde{x} \cdot G(\mathbb{A}^{\infty,p}) \ \subset\ |\mathscr{S}_{K_p,k(\mathfrak{p})}|.$$

Write $H_{K^p}(x)$ for the image of $H(x)$ in $|\mathscr{S}_{K^pK_p,k(\mathfrak{p})}|$. By $C_{K^p}(x)$ we mean the central leaf through x, namely $C_{\mathcal{A}_x[p^\infty],K^p}$. Since the action of $G(\mathbb{A}^{\infty,p})$ does not interfere with the isomorphism class of $\mathcal{A}_x[p^\infty]$ with tensors $(s_{\alpha,0,x})$, we see that

$$H_{K^p}(x) \subset |C_{K^p}(x)|.$$

One of the main concerns of this paper is the following conjecture.

Conjecture 5.4.1 (Hecke Orbit Conjecture)**.** *Let $x \in \mathscr{S}_{K^pK_p,k(\mathfrak{p})}(\overline{\mathbb{F}}_p)$ such that $[b_x]$ is $\mathbb{Q}$-non-basic. Then the subset $H_{K^p}(x)$ enjoys the following properties.*

(HO) *$H_{K^p}(x)$ is Zariski dense in the central leaf $C_{K^p}(x)$.*

($\mathrm{HO}_{\mathrm{disc}}$) *$H_{K^p}(x)$ meets every irreducible component of $C_{K^p}(x)$.*

($\mathrm{HO}_{\mathrm{cont}}$) *The Zariski closure of $H_{K^p}(x)$ in $C_{K^p}(x)$ is a union of irreducible components of $C_{K^p}(x)$.*

Remark 5.4.2. The hypothesis on $[b_x]$ cannot be weaken to only requiring that $[b_x]$ be basic. For example, for Shimura varieties arising from $(G \times \cdots \times G, X \times \cdots \times X)$, with (G, X) a Shimura datum, we see the necessity to assume $[b_x]$ to be basic in every copy of G (which is a $\mathbb{Q}$-factor).

We introduce a conjecture closely related to ($\mathrm{HO}_{\mathrm{disc}}$). If true, the conjecture would yield a canonical $G(\mathbb{A}^{\infty,p})$-equivariant bijection $\pi_0(C(x)) \xrightarrow{\sim} \pi_0\left(\mathscr{S}_{K_p,\overline{k(\mathfrak{p})}}\right)$ via limit over K^p.

Conjecture 5.4.3 ($\mathrm{HO}'_{\mathrm{disc}}$)**.** *For every $x \in \mathscr{S}_{K^pK_p}(\overline{k(\mathfrak{p})})$ such that $[b_x]$ is $\mathbb{Q}$-non-basic, the immersion $C_{K^p}(x) \hookrightarrow \mathscr{S}_{K^pK_p,k(\mathfrak{p})}$ induces a bijection*

$$\pi_0(C_{K^p}(x)) \xrightarrow{\sim} \pi_0(\mathscr{S}_{K_pK^p,\overline{k(\mathfrak{p})}}).$$

The logical relationship between the conjectures is as follows. Obviously ($\mathrm{HO}_{\mathrm{disc}}$) and ($\mathrm{HO}_{\mathrm{cont}}$) together yield (HO). We will see shortly in the proof of Corollary 5.4.5 that ($\mathrm{HO}_{\mathrm{disc}}$) follows from ($\mathrm{HO}'_{\mathrm{disc}}$).

$$(\mathrm{HO}) \underset{+\,(\mathrm{HO}_{\mathrm{cont}})}{\overset{\Longrightarrow}{\longleftarrow}} (\mathrm{HO}_{\mathrm{disc}}) \Longleftarrow (\mathrm{HO}'_{\mathrm{disc}})$$

A representation-theoretic reformulation of ($\mathrm{HO}'_{\mathrm{disc}}$) is the following theorem, to be implied by the main theorem of this paper on Igusa varieties. Recall μ_p from §5.3.

Theorem 5.4.4. *Let b be $\mathbb{Q}$-non-basic with $[b] \in B(G_{\mathbb{Q}_p}, \mu_p^{-1})$. As $G(\mathbb{A}^{\infty,p})$-modules,*

$$H^0(C_b, \overline{\mathbb{Q}}_\ell) \simeq H^0(\mathrm{Sh}_{K_p}, \overline{\mathbb{Q}}_\ell).$$

Proof. See §8.1 below. □

Theorem 5.4.4 ensures the existence of an isomorphism between the two $G(\mathbb{A}^{\infty,p})$-modules. This is easily promoted to the assertion that the natural map between them induced by the immersions $C_{b,K^p} \to \mathrm{Sh}_{K_pK^p}$ (as K^p varies) is an isomorphism, as the next corollary shows.

Corollary 5.4.5. *Conjectures* $(\mathrm{HO}'_{\mathrm{disc}})$ *and* $(\mathrm{HO}_{\mathrm{disc}})$ *are true for* $\mathbb{Q}$*-non-basic* b *with* $[b] \in B(G_{\mathbb{Q}_p}, \mu_p^{-1})$.

Proof. Let us prove $(\mathrm{HO}'_{\mathrm{disc}})$. The immersions $C_{K^p}(x) \hookrightarrow \mathscr{S}_{K_pK^p,\overline{k(\mathfrak{p})}}$ as K^p varies induce a $G(\mathbb{A}^{\infty,p})$-equivariant map

$$\pi_0(C(x)) \to \pi_0(\mathscr{S}_{K_p,\overline{k(\mathfrak{p})}}),$$

which is a surjection since $G(\mathbb{A}^{\infty,p})$ acts transitively on the target (Proposition 5.2.1). Given this, the injectivity follows from the isomorphism of Theorem 5.4.4.

Next we verify $(\mathrm{HO}_{\mathrm{disc}})$. If $[b_x]$ is $\mathbb{Q}$-non-basic, then the $G(\mathbb{A}^{\infty,p})$-action is transitive on $\pi_0(C(x))$ by Proposition 5.2.1 and $(\mathrm{HO}'_{\mathrm{disc}})$. Since $H(x)$ is invariant under the $G(\mathbb{A}^{\infty,p})$-action, we deduce $(\mathrm{HO}_{\mathrm{disc}})$. □

Thanks to Xiao's work on $(\mathrm{HO}_{\mathrm{cont}})$, we verify some special cases of Conjecture (HO). Just for the next corollary, suppose that (G,X) comes from a PEL datum $(B,\mathcal{O}_B,*,V,\langle,\rangle,h)$ of type A or C as in *loc. cit.* or [Kot92b]. Write F for the center of B (which is a field), and F_0 for the fixed field of $*$ in F. We say that $x \in \mathscr{S}_{K_pK^p}(\overline{k(\mathfrak{p})})$ is *B-hypersymmetric* if $\mathrm{End}^0_B(\mathcal{A}_x) \otimes_{\mathbb{Q}} \mathbb{Q}_p \simeq \mathrm{End}^0_{B\otimes\mathbb{Q}_p}(\mathcal{A}_x[p^\infty])$ via the natural map, where End^0 means the space of morphisms in the corresponding isogeny categories.

Corollary 5.4.6. *In the setup* $(\mathrm{Unr}(G,p,K_p))$*, assume that* (G,X) *comes from a PEL datum of type A or C as above, and that every prime of* F_0 *above* p *is inert in* F*. (The last condition is vacuous for type C.) If* $x \in \mathscr{S}_{K_pK^p}(\overline{k(\mathfrak{p})})$ *lies in the connected component of a non-basic Newton stratum such that the component contains a* B*-hypersymmetric point, then Conjecture* (HO) *is true for* x*, i.e.,* $H_{K^p}(x)$ *is Zariski dense in* $C_{K^p}(x)$.

Proof. This corollary follows immediately from the preceding one and [Xia, Thm. 7.1], as observed in Theorem 1.8 of *loc. cit.* □

Remark 5.4.7. For simplicity, we used only case (ii) in part (2) of [Xia, Thm. 7.1]. Case (i) therein extends the above corollary to another case for PEL type A when p splits completely between F and F_0.

6. Igusa varieties

In this section the proof of the main theorem is reduced to a statement about the cohomology of Igusa varieties. We will also see that it suffices to consider the completely slope divisible case, when the tower of Igusa varieties admits models over a fixed finite field. This allows us to apply Fujiwara's formula to compute the cohomology in the next section.

6.1. **Infinite-level Igusa varieties.** We continue in the setup of §5.3, with $b \in G(\breve{\mathbb{Q}}_p)$ satisfying (br1)–(br3). Let $b' \in \mathrm{GSp}(\breve{\mathbb{Q}}_p)$ denote the image of b. By Dieudonné theory, we have a polarized p-divisible group $\Sigma_{b'}$ over $\mathbb{F}_{p^r}$ such that $\mathbb{D}(\Sigma_{b'}) = V^*_{\mathbb{Z}_{(p)}} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_{p^r}$ with Frobenius operator $b'(1\otimes\sigma)$. By Σ_b we mean the p-divisible group $\Sigma_{b'}$ equipped with crystalline Tate tensors (t_α) on $\mathbb{D}(\Sigma_{b'})$ corresponding to (s_α) on $V_{\mathbb{Z}_{(p)}}$. When there is no danger of confusion, we still write Σ_b and $\Sigma_{b'}$ for their base changes to $\overline{\mathbb{F}}_p$.

Applying the construction of §5.3 to $\mathscr{S}_{K'_pK'^{,p}}(\mathrm{GSp},S^\pm)$ and b', we obtain a central leaf $C_{b',K'^{,p}} \subset \mathscr{S}_{K'_pK'^{,p}}(\mathrm{GSp},S^\pm)$. Let R be an $\overline{\mathbb{F}}_p$-algebra. Following [CS17, Sect. 4.3] we have the Igusa variety

$\mathfrak{Ig}_{b',K'^{,p}} \to C_{b',K'^{,p},\overline{\mathbb{F}}_p}$ whose R-points parametrize isomorphisms

$$\Sigma_{b'} \times_{\overline{\mathbb{F}}_p} R \simeq \mathcal{A}_R[p^\infty] \tag{6.1.1}$$

compatible with polarizations up to $\mathbb{Z}_{(p)}^\times$-multiples, where $\mathcal{A}_R$ denotes the pullback of the universal abelian scheme via $\operatorname{Spec} R \to C_{b',K'^{,p}\overline{\mathbb{F}}_p}$. Then $\mathfrak{Ig}_{b',K'^{,p}}$ is a perfect scheme, which is an $\underline{\mathrm{Aut}}(\Sigma_{b'})$-torsor over $C_{b',K'^{,p}\overline{\mathbb{F}}_p}$ by [CS, Cor. 2.3.2], where $\underline{\mathrm{Aut}}(\Sigma_{b'})$ denotes the group scheme of automorphisms of $\Sigma_{b'}$ (preserving the polarization up to $\mathbb{Z}_p^\times$-multiples).

The map $\mathscr{S}_{K_pK^p,\overline{\mathbb{F}}_p} \to \mathscr{S}_{K'_pK'^{,p},\overline{\mathbb{F}}_p}$ clearly induces a map $C_{b,K^p,\overline{\mathbb{F}}_p} \to C_{b',K'^{,p},\overline{\mathbb{F}}_p}$. We define the subscheme

$$\mathfrak{Ig}_{b,K^p} \subset (\mathfrak{Ig}_{b',K'^{,p}} \times_{C_{b',K'^{,p},\overline{\mathbb{F}}_p}} C_{b,K^p,\overline{\mathbb{F}}_p})^{\mathrm{perf}} = \mathfrak{Ig}_{b',K'^{,p}} \times_{C^{\mathrm{perf}}_{b',K'^{,p},\overline{\mathbb{F}}_p}} C^{\mathrm{perf}}_{b,K^p,\overline{\mathbb{F}}_p} \tag{6.1.2}$$

to be the locus where (6.1.1) carries (s_α) to $(\mathbf{s}_{\alpha,0})$ on the Dieudonné modules. Composing with projection maps, we have $\overline{\mathbb{F}}_p$-morphisms $\mathfrak{Ig}_{b,K^p} \to \mathfrak{Ig}_{b',K'^{,p}}$ and $\mathfrak{Ig}_{b,K^p} \to C^{\mathrm{perf}}_{b,K^p,\overline{\mathbb{F}}_p}$. The latter gives rise to the composite map

$$\mathfrak{Ig}_{b,K^p} \to C^{\mathrm{perf}}_{b,K^p,\overline{\mathbb{F}}_p} \to C_{b,K^p,\overline{\mathbb{F}}_p} \to \mathscr{S}_{K_pK^p,\overline{\mathbb{F}}_p}.$$

The Hecke action of $G(\mathbb{A}^{\infty,p})$ on $\mathscr{S}_{K_pK^p,\overline{\mathbb{F}}_p}$ as K^p varies is similarly defined on the tower of $C_{b,K^p,\overline{\mathbb{F}}_p}$ (resp. $\mathfrak{Ig}_{b,K^p}$).

Lemma 6.1.1. *The following are true.*

1. *The $\overline{\mathbb{F}}_p$-scheme $\mathfrak{Ig}_{b,K^p}$ is perfect and a pro-étale J_b^{int}-torsor over $C^{\mathrm{perf}}_{b,K^p,\overline{\mathbb{F}}_p}$.* [14]
2. *The map $\mathfrak{Ig}_{b,K^p} \to \mathfrak{Ig}_{b',K'^{,p}}$ is a closed embedding, under which the $J_{b'}(\mathbb{Q}_p)$-action on $\mathfrak{Ig}_{b',K'^{,p}}$ restricts to an action of $J_b(\mathbb{Q}_p)$ on $\mathfrak{Ig}_{b,K^p}$ (via the embedding $J_b(\mathbb{Q}_p) \hookrightarrow J_{b'}(\mathbb{Q}_p)$).*

Proof. This follows from [Ham19, Prop. 4.1, 4.10], noting that our $\mathfrak{Ig}_{b,K^p}$ is his $\mathscr{J}_\infty^{(p^{-\infty})}$ (the perfection of his $\mathscr{J}_\infty$) and that our J_b^{int} is his Γ_b. Two points require some further explanation. Firstly, we see that $J_b^{\mathrm{int}} = \Gamma_b$ as follows. Observe that $J_b^{\mathrm{int}} \subset J_b(\mathbb{Q}_p) \subset J_{b'}(\mathbb{Q}_p)$ and $J_b^{\mathrm{int}} \subset G(\breve{\mathbb{Z}}_p) \subset \mathrm{GSp}(\breve{\mathbb{Z}}_p)$. Thus J_b^{int} consists of automorphisms of $\Sigma_{b'}$ which are exactly the stabilizers of (t_α) via Dieudonné theory. Secondly, [Ham19, Prop. 4.1] tells us that $\mathscr{J}_\infty \to C_{b,K^p,\overline{\mathbb{F}}_p}$ is a pro-étale J_b^{int}-torsor. Since every perfection map (as a limit of absolute Frobenius) is a universal homeomorphism, which preserves the pro-étale topology [BS15, Lem. 5.4.2], it follows that the perfection $\mathscr{J}_\infty^{(p^{-\infty})} \to C^{\mathrm{perf}}_{b,K^p,\overline{\mathbb{F}}_p}$ is also a pro-étale J_b^{int}-torsor. □

Lemma 6.1.2. *Let R be a perfect $\overline{\mathbb{F}}_p$-algebra. Then $\mathfrak{Ig}_{b,K^p}(R)$ is identified with the set of equivalence classes of (x, j), where*

- *$x \in \mathscr{S}_{K_pK^p}(R)$ is an abelian scheme over $\operatorname{Spec} R$ and*
- *$j : \Sigma_b \times_{\overline{\mathbb{F}}_p} R \to \mathcal{A}_x[p^\infty]$ is a quasi-isogeny carrying (s_α) to $(\mathbf{s}_{\alpha,0,x})$,*

and $\mathcal{A}_x$ denotes the pullback of the universal abelian scheme along x. Here (x, j) and (x', j') are considered equivalent if, in the notation of §5.2, there exists a p-power isogeny $i : \mathcal{A}_x \to \mathcal{A}_{x'}$ carrying $(\mathbf{s}_{\alpha,\ell,x})_{\ell\neq p}$ to $(\mathbf{s}_{\alpha,\ell,x'})_{\ell\neq p}$ and $(\mathbf{s}_{\alpha,0,x})$ to $(\mathbf{s}_{\alpha,0,x'})$ such that $i \circ j = j'$. Each $\rho \in J_b(\mathbb{Q}_p)$ acts on the R-points of $\mathfrak{Ig}_{b,K^p}$ by sending j to $j \circ \rho$. [15]

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[14] It can be shown that $\mathfrak{Ig}_{b,K^p} \to C_{b,K^p}$ is an $\underline{\mathrm{Aut}}(\Sigma_b)$-torsor by [CS, Cor. 2.3.2] and adapting the argument there, but we do not need it.

[15] We make a right action of $J_b(\mathbb{Q}_p)$ on $\mathfrak{Ig}_{b,K^p}$ so that it becomes a left action on the cohomology. In [CS17, §4.3], their arrow j is reverse to ours, from $A[p^\infty]$ to $\Sigma_b \times_{\overline{\mathbb{F}}_p} R$. The two conventions are identified via taking the inverse of j (with the understanding that the authors of *loc. cit.* are also using the right action of $J_b(\mathbb{Q}_p)$, though this does not appear there explicitly).

Proof. This is the Hodge-type analogue of [CS17, Lem. 4.3.4] proven in the PEL case. By *loc. cit.*, $\mathfrak{Ig}_{b',K',p}(R)$ is the set of p-power isogeny classes of (A,j) with $A \in \mathscr{S}_{K'_pK',p}(R)$ and $j : \Sigma_b \times_{\overline{\mathbb{F}}_p} R \to A[p^\infty]$ a quasi-isogeny compatible with polarizations up to $\mathbb{Z}_p^\times$. Now we have a commutative diagram from the construction of central leaves and Igusa varieties:

$$\begin{array}{ccccc} \mathfrak{Ig}_{b,K^p} & \longrightarrow & C_{b,K^p,\overline{\mathbb{F}}_p} & \xhookrightarrow{\text{loc. closed}} & \mathscr{S}_{K_pK^p,\overline{\mathbb{F}}_p} \\ \text{closed}\big\downarrow & & \big\downarrow & & \big\downarrow \\ \mathfrak{Ig}_{b',K',p} & \longrightarrow & C_{b',K',p,\overline{\mathbb{F}}_p} & \xhookrightarrow{\text{loc. closed}} & \mathscr{S}_{K'_pK',p,\overline{\mathbb{F}}_p} \end{array}$$

Now we prove the first assertion by constructing the maps in both directions, which are easily seen to be inverses of each other. Given $y \in \mathfrak{Ig}_{b,K^p}(R)$, its image gives $x \in \mathscr{S}_{K_pK^p}(R)$. The j comes from the image of y in $\mathfrak{Ig}_{b',K',p}(R)$. The compatibility of j with crystalline Tate tensors follows from the very definition of $\mathfrak{Ig}_{b,K^p}$. Conversely, let (x,j) be as in the lemma. Modifying by a quasi-isogeny, we may assume that j is an isomorphism. Then (A,j) comes from a point $y' \in \mathfrak{Ig}_{b',K',p}(R)$ as observed above. Since $\operatorname{Spec} R$ and C_{b,K^p} are reduced, $x \in \mathscr{S}_{K_pK^p}(R)$ comes from a point in $x \in C_{b,K^p}(R)$. Then y' and x have the same image in $C_{b',K',p,\overline{\mathbb{F}}_p}(R)$, so determine a point

$$y \in \left(\mathfrak{Ig}_{b',K',p} \times_{C_{b',K',p,\overline{\mathbb{F}}_p}} C_{b,K^p,\overline{\mathbb{F}}_p}\right)^{\text{perf}}(R) = \left(\mathfrak{Ig}_{b',K',p} \times_{C_{b',K',p,\overline{\mathbb{F}}_p}} C_{b,K^p,\overline{\mathbb{F}}_p}\right)(R).$$

The compatibility of j with crystalline Tate tensors exactly tells us that $y \in \mathfrak{Ig}_{b,K^p}(R)$.

It remains to show the last assertion. In light of Lemma 6.1.1 (2), the assertion on the $J_b(\mathbb{Q}_p)$-action follows from the analogue description for $J_{b'}(\mathbb{Q}_p)$-action on $\mathfrak{Ig}_{b',K',p}$ as in [CS17, Lem. 4.3.4, Cor. 4.3.5]. □

The $J_b(\mathbb{Q}_p)$-action on $\mathfrak{Ig}_{b,K^p}$ commutes with the Hecke action of $G(\mathbb{A}^{\infty,p})$ (as K^p varies) as it is clear on the moduli description. Now we would like to understand the $G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$-representation

$$H^i(\mathfrak{Ig}_b, \overline{\mathbb{Q}}_\ell) := \varinjlim_{K^p} H^i(\mathfrak{Ig}_{b,K^p}, \overline{\mathbb{Q}}_\ell), \qquad i \geq 0,$$

where the limit is over sufficiently small open compact subgroups of $G(\mathbb{A}^{\infty,p})$.

From §2.3 we obtain the following commutative diagram. Indeed, all maps and the commutativity are obvious except possibly the map $J_b(\mathbb{Q}_p)^{\text{ab}} \twoheadrightarrow M_b(\mathbb{Q}_p)^{\text{ab}}$, which comes from the proof of Corollary 2.3.3. (The latter also tells us that $M_b(\mathbb{Q}_p)^{\text{ab}} = M_b(\mathbb{Q}_p)^\flat$ and $G(\mathbb{Q}_p)^{\text{ab}} = G(\mathbb{Q}_p)^\flat$.)

$$\begin{array}{ccccc} J_b(\mathbb{Q}_p) & & M_b(\mathbb{Q}_p) & \longrightarrow & G(\mathbb{Q}_p) \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ J_b(\mathbb{Q}_p)^{\text{ab}} & \twoheadrightarrow & M_b(\mathbb{Q}_p)^{\text{ab}} & \longrightarrow & G(\mathbb{Q}_p)^{\text{ab}} \end{array} \tag{6.1.3}$$

Thus every one-dimensional smooth representation of $G(\mathbb{Q}_p)$ (necessarily factoring through $G(\mathbb{Q}_p)^{\text{ab}}$) can be viewed as a one-dimensional representation of $M_b(\mathbb{Q}_p)$ or $J_b(\mathbb{Q}_p)$ via the above diagram. We are ready to state the main theorem on the cohomology of Igusa varieties in this paper.

Theorem 6.1.3 (Main Theorem)**.** *In the setup of* (Unr(G,p,K_p))*, let* $b \in G(\breve{\mathbb{Q}}_p)$ *be a* $\mathbb{Q}$*-non-basic element such that* $[b] \in B(G_{\mathbb{Q}_p}, \mu_p^{-1})$*. Then there is a* $G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$*-module isomorphism*

$$\iota H^0(\mathfrak{Ig}_b, \overline{\mathbb{Q}}_\ell) \simeq \bigoplus_\pi \pi^{\infty,p} \otimes \pi_p,$$

where the sum runs over discrete automorphic representations π *of* $G(\mathbb{A})$ *such that (i)* $\dim \pi = 1$*, (ii)* π_∞ *is trivial on* $G(\mathbb{R})_+$*.*

Proof. This theorem will be reduced to the completely slope divisible case by Corollary 6.2.3 and established in §7 below. □

Remark 6.1.4. Since $\dim \pi_p = 1$, we have $\pi_p \otimes \delta_{P_b} = J_{P_b^{\mathrm{op}}}(\pi_p) \otimes \delta_{P_b}^{1/2}$. (The point is that the unipotent radical N_b^{op} acts trivially on π_p.) The latter appears in Lemma 3.1.3, and naturally occurs when describing the cohomology of Igusa varieties, as in [HT01, Thm. V.5.4] and [Shi12, Thm. 6.7].

6.2. **Finite-level Igusa varieties in the completely slope divisible case.** Let us assume that the p-divisible group Σ_b is completely slope divisible. (This condition has nothing to do with G-structures.) We will recall the definition of finite-level Igusa varieties following [Man05, CS17, Ham19]. We continue to fix a sufficiently divisible $r \in \mathbb{Z}_{\geq 1}$ as in §5.3 such that conditions (br1)–(br3) there hold. We see from Dieudonné theory that Σ_b with its slope decomposition is defined over $\mathbb{F}_{p^r}$.

We start from the Siegel case. Write $\mathrm{Ig}_{b',m,K'^{,p}} \to C_{b',K'^{,p}}$ for Igusa varieties of level $m \in \mathbb{Z}_{\geq 1}$ as in [Man05, §4] or [Ham19, §3.1] (but over $\mathbb{F}_{p^r}$ rather than over $\overline{\mathbb{F}}_p$), defined to parametrize liftable isomorphisms on the p^m-torsion subgroup of each slope component. As shown in *loc. cit.*, $\mathrm{Ig}_{b',m,K'^{,p}} \to C_{b',K'^{,p}}$ is a finite étale morphism, forming a projective system for varying m via the obvious projection maps. Write $\mathrm{Ig}_{b',K'^{,p}}$ for the projective limit of $\mathrm{Ig}_{b',m,K'^{,p}}$ over m. There are maps $\mathfrak{Ig}_{b',K'^{,p}} \to \mathrm{Ig}_{b',m,K'^{,p},\overline{\mathbb{F}}_p}$ for $m \geq 1$ compatible with each other, since the isomorphism (6.1.1) induces isomorphisms on isoclinic components. This induces an isomorphism $\mathfrak{Ig}_{b',K'^{,p}} \to \mathrm{Ig}^{\mathrm{perf}}_{b',K'^{,p},\overline{\mathbb{F}}_p}$. See [CS17, Prop. 4.3.8] and the preceding paragraph for details.

Following [Ham19, §4.1] (but working over $\mathbb{F}_{p^r}$ rather than over $\overline{\mathbb{F}}_p$), define the $\mathbb{F}_{p^r}$-subscheme

$$\tilde{\mathrm{Ig}}_{b,K^p} \subset \left(\mathrm{Ig}_{b',K'^{,p}} \times_{C_{b',K'^{,p}}} C_{b,K^p}\right)^{\mathrm{perf}}$$

to be the locus given by the same condition as in (6.1.2). Define Ig_{b,m,K^p} as the image of the composite map

$$\tilde{\mathrm{Ig}}_{b,K^p} \to \mathrm{Ig}_{b',K'^{,p}} \times_{C_{b',K'^{,p}}} C_{b,K^p} \to \mathrm{Ig}_{b',m,K'^{,p}} \times_{C_{b',K'^{,p}}} C_{b,K^p}.$$

The projection onto the second component gives an $\mathbb{F}_{p^r}$-morphism $\mathrm{Ig}_{b,m,K^p} \to C_{b,K^p}$, which is finite étale by [Ham19, Prop. 4.1]. Via the canonical projection $\mathrm{Ig}_{b,m+1,K^p} \to \mathrm{Ig}_{b,m,K^p}$ commuting with the maps to C_{b,K^p}, we take the projective limit and denote it by Ig_{b,K^p}.

Besides the Hecke action of $G(\mathbb{A}^{\infty,p})$ on the tower of Ig_{b,K^p} as K^p varies, we also have $\mathrm{Ig}_{b,K^p,\overline{\mathbb{F}}_p}$ acted on by a submonoid $S_b \subset J_b(\mathbb{Q}_p)$ defined in [Man05, p.586]. (The latter action is defined only over $\overline{\mathbb{F}}_p$ in general since self quasi-isogenies of Σ_b are not always defined over finite fields.) The precise definition is unimportant, but it suffices to know two facts. Firstly, S_b generates $J_b(\mathbb{Q}_p)$ as a group. Secondly, S_b contains p^{-1} (the inverse of the multiplication by p map on Σ_b) and [16]

$$\mathrm{fr}^{-r} := r\nu_b(p) \in J_b(\mathbb{Q}_p).$$

By Lemma 5.3.1, $\mathrm{fr}^{-r} \in A_{J_b}(\mathbb{Q}_p)$. Let Fr denote the absolute Frobenius morphism on an $\mathbb{F}_p$-scheme.

Lemma 6.2.1. *The following hold true.*

(1) $\mathrm{fr}^{-r} \in A^{--}_{P_b^{\mathrm{op}}} \subset A_{M_b}(\mathbb{Q}_p) = A_{J_b}(\mathbb{Q}_p)$. *As an element of* $M_b(\mathbb{Q}_p)$*, we have* $\mathrm{fr}^{-r} \in A^{--}_{P_b^{\mathrm{op}}}$. *(Recall that* $A^{--}_{P_b^{\mathrm{op}}}$ *was defined in §2.1.)*

(2) The action of $\mathrm{Fr}^r \times 1$ *on* $\mathrm{Ig}_{b,K^p} \times_{\mathbb{F}_{p^r}} \overline{\mathbb{F}}_p$ *induces the same action on* $\mathfrak{Ig}_{b,K^p}$ *as the action of* $\mathrm{fr}^{-r} \in J_b(\mathbb{Q}_p)$.

[16] Here is a note on the sign. On slope $0 \leq \lambda \leq 1$ component, the action of fr^r is p^λ, but ν_b records slope $-\lambda$ since we use the covariant Dieudonné theory.

(3) There is a canonical isomorphism $\mathfrak{Ig}_{b,K^p} \simeq \mathrm{Ig}^{\mathrm{perf}}_{b,K^p,\overline{\mathbb{F}}_p}$ over $C_{b,K^p,\overline{\mathbb{F}}_p}$, compatible with the $G(\mathbb{A}^{\infty,p}) \times S_b$-actions as K^p varies.

Proof. (1) We already know $\mathrm{fr}^{-r} \in A_{J_b}(\mathbb{Q}_p) = A_{M_b}(\mathbb{Q}_p)$. (See §5.3 for the equality.) Since $r\nu_b$ is B-dominant (§5.3), we have $r\nu_b(p) \in A^{-}_{P_b^{\mathrm{op}}}$. Moreover $r\nu_b(p) \in A^{--}_{P_b^{\mathrm{op}}}$ as the centralizer of $r\nu_b(p)$ in G is exactly M_b.

(2) Write Fr_Σ for the absolute Frobenius action on $\Sigma_b/\mathbb{F}_{p^r}$. Recall that $b\sigma(b)\cdots\sigma^{r-1}(b) = r\nu_b(p)$ from §5.3. Thus $\mathrm{fr}^{-r} = r\nu_b(p)$ acts on $\Sigma_b/\mathbb{F}_{p^r}$ as $(\mathrm{Fr}_\Sigma)^r$. Thus fr^{-r} sends (x,j) to $(x, j\circ \mathrm{Fr}^r_\Sigma)$ in the description of R-points in Lemma 6.1.2. On the other hand, $\mathrm{Fr}^r \times 1$ on $\mathfrak{Ig}_{b,K^p}$ sends (x,j) to $(x^{(r)}, j^{(r)})$, where $x^{(r)}$ corresponds to the p^r-th power Frobenius twist of x (so that $\mathcal{A}_{x^{(r)}} = (\mathcal{A}_x)^{(r)}$), and $j^{(r)}$ is the p^r-th power twist of j. Finally we observe that $(x^{(r)}, j^{(r)})$ is equivalent to $(x, j\circ \mathrm{Fr}^r_\Sigma)$ via the p^r-power relative Frobenius $\mathcal{A}_x \to \mathcal{A}_{x^{(r)}}$.

(3) We have the map $\mathfrak{Ig}_{b,K^p} \to \mathrm{Ig}_{b,K^p,\overline{\mathbb{F}}_p}$ over $C_{b,K^p,\overline{\mathbb{F}}_p}$ from the definition, which factors through $\mathfrak{Ig}_{b,K^p} \to \mathrm{Ig}^{\mathrm{perf}}_{b,K^p,\overline{\mathbb{F}}_p}$ since $\mathfrak{Ig}_{b,K^p}$ is perfect. This is shown to be an isomorphism exactly as in the proof of [CS17, Prop. 4.3.8], the point being a canonical splitting of the slope decomposition over the perfect scheme $\mathrm{Ig}^{\mathrm{perf}}_{b,K^p,\overline{\mathbb{F}}_p}$. □

Now we compare Igusa varieties arising from two central leaves in the same Newton stratum. Let $b, b_0 \in G(\breve{\mathbb{Q}}_p)$. Assume that b and b_0 satisfy the conditions at the end of §5.3. We have an isomorphism $J_b(\mathbb{Q}_p) \simeq J_{b_0}(\mathbb{Q}_p)$ (induced by a conjugation in the ambient group $G(\breve{\mathbb{Q}}_p)$), canonical up to $J_b(\mathbb{Q}_p)$-conjugacy.

Proposition 6.2.2. *There exists a $G(\mathbb{A}^{\infty,p})$-equivariant isomorphism*

$$\mathfrak{Ig}_b \stackrel{\sim}{\to} \mathfrak{Ig}_{b_0},$$

which is also equivariant for the actions of $J_b(\mathbb{Q}_p)$ and $J_{b_0}(\mathbb{Q}_p)$ through a suitable isomorphism $J_b(\mathbb{Q}_p) \simeq J_{b_0}(\mathbb{Q}_p)$ in its canonical $J_b(\mathbb{Q}_p)$-conjugacy class.

Proof. Since $[b_x] = [b_{x_0}]$, there exists a quasi-isogeny $f : \Sigma_{b_0} \to \Sigma_b$ compatible with G-structures. Using the description of Lemma 6.1.2, we can give an isomorphism $\mathfrak{Ig}_b \stackrel{\sim}{\to} \mathfrak{Ig}_{b_0}$ on R-points by $(A,j) \mapsto (A, j\circ f)$. The equivariance property is evident. □

Corollary 6.2.3. *In the setting of Theorem 6.1.3, if the theorem is true for all b such that Σ_b is completely slope divisible, then the theorem is true without the restriction on b.*

Proof. Let b_0 be arbitrary. The argument of [Zha, Lem. 4.2.8] shows the existence of $g \in G(\breve{\mathbb{Q}}_p)$ such that, setting $b := g^{-1} b_0 \sigma(g)$,

- $b \in G(\breve{\mathbb{Z}}_p)\sigma\mu(p)^{-1}G(\breve{\mathbb{Z}}_p)$,
- Σ_b is completely slope divisible.

(We can further multiply g on the right by an element of $G(\breve{\mathbb{Z}}_p)$, thus not affecting complete slope divisibility, while ensuring that (br1)–(br3) are satisfied as in §5.3.) By Proposition 6.2.2, fixing an isomorphism $J_b(\mathbb{Q}_p) \simeq J_{b_0}(\mathbb{Q}_p)$ as in there, we have

$$H^0(\mathfrak{Ig}_b, \overline{\mathbb{Q}}_\ell) \simeq H^0(\mathfrak{Ig}_{b_0}, \overline{\mathbb{Q}}_\ell) \quad \text{as } G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)\text{-modules.}$$

We are assuming Theorem 6.1.3 for b, so it follows that the same theorem holds for b_0. (Observe that the transfer of one-dimensional representations via $J_b(\mathbb{Q}_p) \simeq J_{b_0}(\mathbb{Q}_p)$ do not depend on the choice of isomorphism.) □

7. Cohomology of Igusa varieties

The main purpose of this section is to prove Theorem 6.1.3. Throughout we are in the completely slope divisible case of §6.2, as this is sufficient in light of Corollary 6.2.3. We will convert to a problem about compactly supported cohomology and apply Mack-Crane's Langlands–Kottwitz type formula to bring in techniques from the trace formula and harmonic analysis. All ingredients will be combined together in §7.6 to identify the leading term in the Lang–Weil estimate.

7.1. **Compactly supported cohomology in top degree.** In 6.2 we have constructed Ig_{b,K^p} such that $\mathfrak{Ig}_{b,K^p}$ is the perfection of $\mathrm{Ig}_{b,K^p,\overline{\mathbb{F}}_p}$ (with compatible transition maps as K^p varies). Recall that $\dim \mathrm{Ig}_b = \langle 2\rho, \nu_b\rangle$. Define for $i \in \mathbb{Z}_{\geq 0}$,

$$H^i_c(\mathrm{Ig}_{b,m,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell) := \varinjlim_{K^p} H^i_c(\mathrm{Ig}_{b,m,K^p,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell), \qquad H^i_c(\mathrm{Ig}_{b,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell) := \varinjlim_{m\geq 0} H^i_c(\mathrm{Ig}_{b,m,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell).$$

As for $H^i_c(\mathrm{Ig}_{b,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell)$, we have a $G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$-module structure on $H^i_c(\mathrm{Ig}_{b,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell)$. This is an admissible $G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$-module as the cohomology is finite-dimensional at each finite level. It is convenient to prove the following dual version of Theorem 6.1.3.

Theorem 7.1.1. *In the setup of* (Unr(G,p,K_p))*, assume that b is $\mathbb{Q}$-non-basic, and that Σ_b is completely slope divisible. Then there is a $G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$-module isomorphism*

$$\iota H^{\langle 4\rho,\nu_b\rangle}_c(\mathrm{Ig}_{b,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell) \simeq \bigoplus_{\pi} \pi^{\infty,p} \otimes (\pi_p \otimes \delta_{P_b}),$$

where the sum runs over discrete automorphic representations π of $G(\mathbb{A})$ such that (i) $\dim \pi = 1$, (ii) π_∞ is trivial on $G(\mathbb{R})_+$.

Proof. The proof will be carried out in §7.6 after setting up a stabilized trace formula (Theorem 7.5.1), by employing the estimates in §4. □

Theorem 7.1.1 implies Theorem 6.1.3. We may put ourselves in the completely slope divisible case by Corollary 6.2.3. Write $d := \langle 2\rho, \nu_b\rangle$. Applying Poincaré duality to finite-level Igusa varieties Ig_{b,m,K^p} and taking direct limit over m and K^p, we obtain a pairing

$$H^0(\mathrm{Ig}_{b,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell) \times H^{2d}_c(\mathrm{Ig}_{b,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(d)) \to \overline{\mathbb{Q}}_\ell,$$

where $\overline{\mathbb{Q}}_\ell(d)$ denotes the d-th power Tate twist. The construction of duality (Exp.XVIII,§3 in [SGA73]) goes through a family of canonical isomorphisms $Rf^!_{m,K^p}\overline{\mathbb{Q}}_\ell \simeq \overline{\mathbb{Q}}_\ell(d)[-2d]$ (concentrated in degree $2d$) over m and K^p, where $f_{m,K^p} : \mathrm{Ig}_{b,m,K^p} \to \operatorname{Spec} \mathbb{F}_{p^r}$ denotes the structure map. Thus the action of $G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$ on $\mathrm{Ig}_b = \{\mathrm{Ig}_{b,m,K^p}\}$ induces an action on $\overline{\mathbb{Q}}_\ell(d)[-2d]$, through a character $\varsigma : G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p) \to \overline{\mathbb{Q}}_\ell^\times$. (As in §6.2.1, the action of $J_b(\mathbb{Q}_p)$ is defined a priori on a submonoid S_b and then extended to $J_b(\mathbb{Q}_p)$. Alternatively, this action can be defined directly after perfectifying Ig_b.) Together with the $G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$-action on Ig_b, this yields an action of $G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$ on $H^{2d}_c(\mathrm{Ig}_{b,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(d))$ and $H^0(\mathrm{Ig}_{b,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell)$, respectively. It follows from the functoriality of Poincaré duality that the above pairing is $G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$-equivariant. Thus $H^0(\mathrm{Ig}_{b,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell)$ is isomorphic to the (smooth) contragredient of $H^{2d}_c(\mathrm{Ig}_{b,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(d))$, which is isomorphic to $H^{2d}_c(\mathrm{Ig}_{b,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell) \otimes \varsigma$. Therefore Theorem 7.1.1 implies that

$$\iota H^0(\mathrm{Ig}_{b,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell) \simeq \bigoplus_{\pi\in\Pi_1(G)} ((\pi^{\infty,p}) \otimes (\pi_p \otimes \delta_{P_b}))^\vee \otimes \varsigma^{-1}, \tag{7.1.1}$$

where $\Pi_1(G)$ denotes the range of π in the summation of that theorem. (Since $\dim \pi = 1$, the dual sign in (7.1.1) simply means inverse.)

On the other hand, $H^0(\mathrm{Ig}_{b,\overline{\mathbb{F}}_p},\overline{\mathbb{Q}}_\ell)$ is the space of smooth $\overline{\mathbb{Q}}_\ell$-valued functions on $\pi_0(\mathrm{Ig}_{b,\overline{\mathbb{F}}_p})$, on which $G(\mathbb{A}^{\infty,p})\times J_b(\mathbb{Q}_p)$ acts through right translation. (Here smoothness means invariance under an open compact subgroup of $G(\mathbb{A}^{\infty,p})\times J_b(\mathbb{Q}_p)$.) In particular $H^0(\mathrm{Ig}_{b,\overline{\mathbb{F}}_p},\overline{\mathbb{Q}}_\ell)$ contains the trivial representation, which consists of constant functions on $\pi_0(\mathrm{Ig}_{b,\overline{\mathbb{F}}_p})$. Hence $\varsigma^{-1}=(\pi_0^{\infty,p})\otimes(\pi_{0,p}\otimes\delta_{P_b})$ for some $\pi_0\in\Pi_1(G)$. Since $\Pi_1(G)$ is invariant under twist by π_0, we see that (7.1.1) is still true without ς. Moreover $\Pi_1(G)$ is invariant under taking dual, so we can rewrite (7.1.1) as

$$\iota H^0(\mathrm{Ig}_{b,\overline{\mathbb{F}}_p},\overline{\mathbb{Q}}_\ell)\simeq\bigoplus_\pi \pi^{\infty,p}\otimes(\pi_p\otimes\delta_{P_b}).$$

Finally, the same holds with $\mathfrak{Ig}_b$ in place of $\mathrm{Ig}_{b,\overline{\mathbb{F}}_p}$ thanks to Lemma 6.2.1 (3). □

Remark 7.1.2. It may be possible to compute the character ς in the proof, but we have got around it. As we know the Frobenius action on $\overline{\mathbb{Q}}_\ell(d)[-2d]$, Lemma 6.2.1 (2) tells us that $\mathrm{fr}^{-r}\in J_b(\mathbb{Q}_p)$ acts by p^{rd}. We guess that ς is trivial on $G(\mathbb{A}^{\infty,p})$ and equal to $\delta_{P_b}^{-1}$ on $J_b(\mathbb{Q}_p)$.

7.2. **The basic setup for harmonic analysis.** Let $\phi^{\infty,p}=\otimes_{v\neq\infty,p}\phi_v\in\mathcal{H}(G(\mathbb{A}^{\infty,p}))$ and $\phi_p\in\mathcal{H}(J_b(\mathbb{Q}_p))$. With a view towards Theorem 7.1.1, we want to compute

$$\mathrm{Tr}\left(\phi^{\infty,p}\phi_p^{(j)}\middle|\iota H_c(\mathrm{Ig}_{b,\overline{\mathbb{F}}_p},\overline{\mathbb{Q}}_\ell)\right):=\sum_{i\geq0}(-1)^i\mathrm{Tr}\left(\phi^{\infty,p}\phi_p^{(j)}\middle|\iota H_c^i(\mathrm{Ig}_{b,\overline{\mathbb{F}}_p},\overline{\mathbb{Q}}_\ell)\right).$$

We keep T, B, $b\in G(\breve{\mathbb{Q}}_p)$, and $r\in\mathbb{Z}_{\geq1}$ as before, so that $r\nu_b\in X_*(T)_+$. Recall that $r\nu_b(p)\in A_{J_b}$. Given $\phi_p\in\mathcal{H}(J_b(\mathbb{Q}_p))$, define

$$\phi_p^{(j)}\in\mathcal{H}(J_b(\mathbb{Q}_p))\quad\text{by}\quad\phi_p^{(j)}(\delta):=\phi_p(j\nu_b(p)^{-1}\delta),\qquad j\in r\mathbb{Z}_{\geq1}.$$

This coincides with the analogous definition of $\phi_p^{(k)}$ in §3.1, namely $\phi_p^{(j)}=\phi_p^{(k)}$ via $k=j/r$ and $\nu=r\nu_b$. (The difference is that ν is a cocharacter but ν_b is only a fractional cocharacter.)

An element $\delta\in J_b(\overline{\mathbb{Q}}_p)$ is **acceptable** if its image in $M_b(\overline{\mathbb{Q}}_p)$ is acceptable (Definition 3.1.1) under the isomorphism $J_b(\overline{\mathbb{Q}}_p)\simeq M_b(\overline{\mathbb{Q}}_p)$ induced by some (thus any) inner twist at the end of §5.3. As in §3.1, let $\mathcal{H}_{\mathrm{acc}}(J_b(\mathbb{Q}_p))\subset\mathcal{H}(J_b(\mathbb{Q}_p))$ denote the subspace of functions supported on acceptable elements. Choose $j_0\in\mathbb{Z}_{\geq0}$ such that

$$\phi_p^{(j)}\in\mathcal{H}_{\mathrm{acc}}(J_b(\mathbb{Q}_p)),\qquad j\in r\mathbb{Z},\ j\geq j_0.$$

Such a j_0 exists by the argument of Lemma 3.1.8. By Lemma 6.2.1 and the definition of $\phi_p^{(j)}$, we have

$$\mathrm{Tr}\left(\phi^{\infty,p}\phi_p^{(j)}\middle|\iota H_c(\mathrm{Ig}_{b,\overline{\mathbb{F}}_p},\overline{\mathbb{Q}}_\ell)\right)=\mathrm{Tr}\left(\phi^{\infty,p}\phi_p\times(\mathrm{Fr}^j\times1)\middle|\iota H_c(\mathrm{Ig}_{b,\overline{\mathbb{F}}_p},\overline{\mathbb{Q}}_\ell)\right),\tag{7.2.1}$$

where Fr^j is the j/r-th power of the relative Frobenius of Ig_b over $\mathbb{F}_{p^r}$. Since the action of Fr^j is the same as the action of a central element of $J_b(\mathbb{Q}_p)$, it commutes with the action of $\phi^{\infty,p}\phi_p$. Thus (7.2.1) and the Lang–Weil bound tell us that the top degree compactly supported cohomology in Theorem 7.1.1 is captured by the leading term as $j\to\infty$. This will be the basic idea underlying the proof of the theorem in §7.6 below.

We fix the global central character datum $(\mathfrak{X},\chi_0)=(A_{G,\infty},\mathbf{1})$ for G, which can also be viewed as a central character datum for G^* via $Z(G)=Z(G^*)$. (Since we compute the cohomology with constant coefficients, we do not need to consider nontrivial χ_0.)

Also fixed is a z-extension $1\to Z_1\to G_1\to G\to1$ over $\mathbb{Q}$ once and for all as at the start of §2.7. Moreover we choose

$$\mu_{p,1}:\mathbb{G}_m\to G_1\text{ over }\mathbb{Q}_{p^r},\qquad b_1\in G_1(\mathbb{Q}_{p^r}),$$

and possibly make r more divisible so that $r\nu_{b_1}$ is a cocharacter as in Lemma 5.3.8. Via $\iota_p : \overline{\mathbb{Q}}_p \simeq \mathbb{C}$, we transport $\mu_{p,1}$ to $\mu_1 : \mathbb{G}_m \to G_1$ over $\mathbb{C}$. The choice of $\mu_{p,1}$ and b_1 will affect the test functions at ∞ and p, respectively.

For each $\mathfrak{e} \in \mathcal{E}_{\mathrm{ell}}(G)$, fix a $\mathbb{Q}$-rational minimal parabolic subgroup of $G^{\mathfrak{e}}$ and its Levi component as at the start of §4.2 (with $G^{\mathfrak{e}}$ in place of G there). Call them P_0^* and M_0^* in the case $\mathfrak{e} = \mathfrak{e}^*$. On the other hand, we have $G_{\mathbb{Q}_p} \supset B_{\mathbb{Q}_p} \supset T_{\mathbb{Q}_p}$ from §5.3. Since $G_{\mathbb{Q}_p}$ is quasi-split, there is a canonical $G^{\mathrm{ad}}(\mathbb{Q}_p)$-conjugacy class of isomorphisms $G_{\mathbb{Q}_p} \simeq G^*_{\mathbb{Q}_p}$. We fix one such isomorphism such that $B_{\mathbb{Q}_p}$ (resp. $T_{\mathbb{Q}_p}$) is carried into $P^*_{0,\mathbb{Q}_p}$ (resp. $M^*_{0,\mathbb{Q}_p}$). The images of $B_{\mathbb{Q}_p}$ and $T_{\mathbb{Q}_p}$ in $G^*_{\mathbb{Q}_p}$ will play the roles of B and T in §4.2.

For each $\mathfrak{e} \in \mathcal{E}^{<}_{\mathrm{ell}}(G)$, we have a z-extension $1 \to Z_1 \to G_1^{\mathfrak{e}} \to G^{\mathfrak{e}} \to 1$ over $\mathbb{Q}$ and an endoscopic datum $\mathfrak{e}_1 = (G_1^{\mathfrak{e}}, \mathcal{G}_1^{\mathfrak{e}}, s_1^{\mathfrak{e}}, \eta_1^{\mathfrak{e}})$ for G_1 as in §2.7. This determines a central character datum $(\mathfrak{X}_1^{\mathfrak{e}}, \chi_1^{\mathfrak{e}})$ for $G_1^{\mathfrak{e}}$.

7.3. **The test functions away from p.** For each $\mathfrak{e} \in \mathcal{E}_{\mathrm{ell}}(G)$, Let us introduce the test functions to enter the statement of Theorem 7.5.1 below. Here we consider the places away from p. The place p will be treated in the next subsection.

The first case is away from p and ∞. When $\mathfrak{e} = \mathfrak{e}^*$, we have $(f^{\mathrm{Ig},*})^{\infty,p} = \otimes'_{v\neq\infty,p} f_v^*$, where $f_v^* \in \mathcal{H}(G^*(\mathbb{Q}_v))$ is a transfer of ϕ_v as in §2.5. In case $\mathfrak{e} \in \mathcal{E}^{<}_{\mathrm{ell}}(G)$, at each $v \neq \infty, p$, the function ϕ_v admits a transfer $f^{\mathfrak{e}}_{1,v} \in \mathcal{H}(G_1^{\mathfrak{e}}(\mathbb{Q}_v), (\chi^{\mathfrak{e}}_{1,v})^{-1})$. Then we take

$$(f^{\mathrm{Ig},\mathfrak{e}})_1^{\infty,p} = \otimes'_{v\neq\infty,p} f^{\mathfrak{e}}_{1,v} \in \mathcal{H}(G_1^{\mathfrak{e}}(\mathbb{A}^{\infty,p}), (\chi_1^{\mathfrak{e},\infty,p})^{-1}).$$

The next case is the real place. The construction of the test function $f^{\mathrm{Ig},\mathfrak{e}}_{1,\infty} \in \mathcal{H}(G_1^{\mathfrak{e}}(\mathbb{R}), (\chi^{\mathfrak{e}}_{1,\infty})^{-1})$ follows [Kot90, §7] based on Shelstad's real endoscopy and Clozel-Delorme's pseudocoefficients. We adapt it to the case with central characters. In the easier case of $\mathfrak{e} = \mathfrak{e}^* = (G^*, {}^L G^*, 1, \mathrm{id})$, we take $f^{\mathrm{Ig},*}_\infty := e(G_\infty) f_{\mathbf{1}}$ in the notation of §2.4. Now let $\mathfrak{e} \in \mathcal{E}^{<}_{\mathrm{ell}}(G)$. In the notation of §2.4, both ξ and ζ are trivial in the current setup (since we are focusing on the constant coefficient case). Write ξ_1 and ζ_1 for the pullbacks of ξ and ζ from G to G_1; they are again trivial. We obtain a discrete L-packet $\Pi(\xi_1, \zeta_1)$ for $G_1(\mathbb{R})$ along with an L-parameter $\phi_{\xi_1,\zeta_1} : W_{\mathbb{R}} \to {}^L G_1$ as in §2.4. Let $\Phi_2(G^{\mathfrak{e}}_{1,\mathbb{R}}, \phi_{\xi_1,\zeta_1})$ denote the set of discrete L-parameters $\phi' \in \Phi(G_1^{\mathfrak{e}}(\mathbb{R}))$ such that $\eta_1^{\mathfrak{e}} \phi' \simeq \phi_{\xi_1,\zeta_1}$. Then define (cf. [Kot90, p.186])

$$f^{\mathrm{Ig},\mathfrak{e}}_{1,\infty} := (-1)^{q(G_1)} \langle \mu_1, s_1^{\mathfrak{e}} \rangle \sum_{\phi'} \det(\omega_*(\phi')) f_{\phi'},$$

where $f_{\phi'}$ is the averaged Lefschetz function for the L-packet of ϕ' defined in §2.4, and the sum runs over $\phi' \in \Phi(G^{\mathfrak{e}}_{1,\mathbb{R}}, \phi_{\xi_1,\zeta_1})$. As in [KSZ, §11.2.3][17] we check that $f^{\mathrm{Ig},\mathfrak{e}}_{1,\infty}$ is $(\chi^{\mathfrak{e}}_{1,\infty})^{-1}$-equivariant and compactly supported modulo $\mathfrak{X}^{\mathfrak{e}}_{1,\infty}$.

7.4. **The test functions at p.** We apply the contents of §3 to the cocharacter $\nu := r\nu_b$ over $\mathbb{Q}_p$. In particular, we have $P_b := P_\nu$ whose Levi factor is $M_b = M_\nu$.

Consider the case $\mathfrak{e} = \mathfrak{e}^*$. Each function $\phi_p \in \mathcal{H}(J_b(\mathbb{Q}_p))$ admits a transfer $\phi_p^* \in \mathcal{H}(M_b(\mathbb{Q}_p))$ as explained in §2.3. When $\phi_p \in \mathcal{H}_{\mathrm{acc}}(J_b(\mathbb{Q}_p))$, we can arrange that $\phi_p^* \in \mathcal{H}_{\mathrm{acc}}(M_b(\mathbb{Q}_p))$ by multiplying the indicator function on the set of acceptable elements in $M_b(\mathbb{Q}_p)$. (This is possible as the subset of acceptable elements is nonempty, open, and stable under $J_b(\mathbb{Q}_p)$-conjugacy.) The image of ϕ_p^* in $\mathcal{S}(M_b)$ depends only on ϕ_p (as an element of $\mathcal{S}(J_b)$). In the notation of §3.1, define

$$f_p^{\mathrm{Ig},*,(j)} := \mathscr{J}_\nu \left(\delta_{P_\nu}^{1/2} \cdot \phi_p^{*,(j)} \right) \in \mathcal{S}(G), \qquad j \in \mathbb{Z}_{\geq 0},$$

[17] The reference numbering is subject to change.

As before, we still write $f_p^{\mathrm{Ig},*,(j)}$ for a representative in $\mathcal{H}(G(\mathbb{Q}_p))$. Lemma 3.1.3 implies that

$$\mathrm{Tr}\left(f_p^{\mathrm{Ig},*,(j)}|\pi_p\right)=\mathrm{Tr}\left(\phi_p^{*,(j)}|J_{P_\nu^{\mathrm{op}}}(\pi_p)\otimes\delta_{P_\nu}^{1/2}\right),\qquad\forall\pi_p\in\mathrm{Irr}(G(\mathbb{Q}_p)).$$

Now let $\mathfrak{e}\in\mathcal{E}^<_{\mathrm{ell}}(G)$. Recall that $b_1\in G_1(\mathbb{Q}_{p^r})$ was chosen. Take $\nu_1:=r\nu_{b_1}$. The z-extension $1\to Z_1\to G_1\to G\to 1$ gives rise to z-extensions of M_b and J_b over $\mathbb{Q}_p$. By pulling back via $M_b\hookrightarrow G$, and from the definition of J_b and J_{b_1}, we indeed obtain exact sequences of $\mathbb{Q}_p$-groups

$$1\to Z_1\to M_{b_1}\to M_b\to 1,\qquad 1\to Z_1\to J_{b_1}\to J_b\to 1.$$

(For J_b, the point is that the σ-stabilizer subgroup of $\mathrm{Res}_{\breve{\mathbb{Q}}_p/\mathbb{Q}_p}\mathbb{G}_m$ is simply $\mathbb{G}_m$.) We pull back $\phi_p^{(j)}\in\mathcal{H}(J_b(\mathbb{Q}_p))$ to obtain $\phi_{1,p}^{(j)}\in\mathcal{H}(J_{b_1}(\mathbb{Q}_p),\chi_{1,p})$. (Recall that $\chi_1=\prod_v\chi_{1,v}$ is the trivial character on $\mathfrak{X}_1=Z_1(\mathbb{A})$.) Write $\phi_p^*\in\mathcal{H}(M_b(\mathbb{Q}_p))$ for a transfer of ϕ_p, and $\phi_{1,p}^*\in\mathcal{H}(M_{b_1}(\mathbb{Q}_p),\chi_{1,p})$ for the pullback of ϕ_p^*. Then $\phi_p^{*,(j)}$ (defined in §3.1) is a transfer of $\phi_p^{(j)}$ (namely $\phi^{*,(j)}=(\phi_p^{(j)})^*$ in $\mathcal{S}(J_b)$), and $\phi_{1,p}^{*,(j)}$ is a transfer of $\phi_{1,p}^{(j)}$, for all $j\in\mathbb{Z}$.

The desired test function $f_{1,p}^{\mathrm{Ig},\mathfrak{e}}$ is described by the process in [Shi10, §6], with $J_{b_1},G_1^{\mathfrak{e}},G_1$ in place of J_b,H,G therein, followed by averaging on $\mathfrak{X}_1=\mathfrak{X}_1^{\mathfrak{e}}$. We point out that [Shi10] is applicable as G_1 has simply connected derived subgroup. For our purpose, we summarize the construction as follows:

$$\begin{aligned} f_{1,p}^{\mathrm{Ig},\mathfrak{e},(j)} &:= \sum_{\omega\in\Omega_{\mathfrak{e}_1,\nu_1}} f_{1,p,\omega}^{\mathrm{Ig},\mathfrak{e},(j)},\qquad\text{where} \\ f_{1,p,\omega}^{\mathrm{Ig},\mathfrak{e},(j)} &:= c_\omega\cdot\mathscr{J}_{\nu_{1,\omega}}\big(\mathrm{LS}^{\mathfrak{e}_1,\omega}(\delta_{P_{\nu_1}}^{1/2}\cdot\phi_{1,p}^{*,(j)})\big)\in\mathcal{H}(G_1^{\mathfrak{e}}(\mathbb{Q}_p),\chi_{1,p}^{\mathfrak{e},-1}),\end{aligned}\tag{7.4.1}$$

Here $c_\omega\in\mathbb{C}$ are constants (possibly zero) independent of ϕ_p. Note that $\mathscr{J}_{\nu_{1,\omega}}$ and $\mathrm{LS}^{\mathfrak{e}_1,\omega}$ denote the maps in the setup with fixed central character as in §3.5. We observe the following about the right hand side of (7.4.1).

$$\delta_{P_{\nu_1}}^{1/2}\cdot\phi_{1,p}^{*,(j)}=\delta_{P_{\nu_1}}^{1/2}(\nu_1(p))\big(\delta_{P_{\nu_1}}^{1/2}\cdot\phi_{1,p}^*\big)^{(j)}=p^{\langle\rho,\nu\rangle}\big(\delta_{P_{\nu_1}}^{1/2}\cdot\phi_{1,p}^*\big)^{(j)}.\tag{7.4.2}$$

7.5. **The stable trace formula for Igusa varieties.** Keep the preceding setup and notation from earlier in this section and also from §2.9. We are ready to state the key stabilized formula to compute the cohomology of Igusa varieties.

Theorem 7.5.1. *Given $\phi^{\infty,p}\in\mathcal{H}(G(\mathbb{A}^{\infty,p}))$ and $\phi_p\in\mathcal{H}(J_b(\mathbb{Q}_p))$, there exists $j_0=j_0(\phi^{\infty,p},\phi_p)\in\mathbb{Z}_{\ge1}$ such that $\phi_p^{(j)}\in\mathcal{H}_{\mathrm{acc}}(J_b(\mathbb{Q}_p))$ and for every integer $j\ge j_0$ divisible by r, the following formula holds:*

$$\mathrm{Tr}\left(\phi^{\infty,p}\phi_p^{(j)}\,\big|\,\iota H_c(\mathrm{Ig}_b,\overline{\mathbb{Q}}_\ell)\right)=\mathrm{ST}^{G^*}_{\mathrm{ell},\chi_0}(f^{\mathrm{Ig},*,(j)})+\sum_{\mathfrak{e}\in\mathcal{E}^<_{\mathrm{ell}}(G)}\iota(G,G^{\mathfrak{e}})\mathrm{ST}^{G_1^{\mathfrak{e}}}_{\mathrm{ell},\chi_1^{\mathfrak{e}}}\left(f_1^{\mathrm{Ig},\mathfrak{e},(j)}\right).$$

Proof. The point is to stabilize the main result of [MC] (generalizing that of [Shi09]), which obtains the following expansion for $\mathrm{Tr}\left(\phi^{\infty,p}\phi_p^{(j)}\,\big|\,\iota H_c(\mathrm{Ig}_b,\overline{\mathbb{Q}}_\ell)\right)$:

$$\sum_{\gamma_0\in\Sigma_{\mathbb{R}\text{-ell}}(G)}\frac{c_2(\gamma_0)\mathrm{Tr}\,\xi(\gamma_0)}{\bar{\iota}_G(\gamma_0)}\sum_{\mathfrak{c}=(\gamma_0,a,[b_0])}c_1(\mathfrak{c})O_\gamma^{G(\mathbb{A}^{\infty,p})}(\phi^{\infty,p})O_\delta^{J_b(\mathbb{Q}_p)}(\phi_p^{(j)}),\tag{7.5.1}$$

where the inner sum is over the set of acceptable b-admissible Kottwitz parameters $\mathfrak{c}$, and the conjugacy class (γ,δ) in $G(\mathbb{A}^{\infty,p})\times J_b(\mathbb{Q}_p)$ is determined by $\mathfrak{c}$ as in *loc. cit.* We do not recall the notation further, as it suffices for our purpose to observe that the right hand side resembles the analogous formula for Shimura varieties [KSZ, Thm. 5.4.3] except that different terms appear at p. Indeed, the stabilization of (7.5.1) is very close to the stabilization for Shimura varieties in [KSZ, §11], with an appropriate change at p following [Shi10, §6], we content ourselves with pointing out the necessary changes.[18]

The initial steps in stabilization are identical to [KSZ, §11.1] (but simpler as we can take $\mathfrak{X} = A_{G,\infty}$; in this case $\tau_{\mathfrak{X}}(\cdot) = \tau(\cdot)$, the usual Tamagawa measure). Stabilization away from p and ∞ is based on the Langlands–Shelstad transfer as in [KSZ, §11.2.1]. The terms at ∞ are stabilized according to Shelstad's real endoscopy as detailed in [Kot90, §7], cf. [Shi10, Lem. 5.4] and [KSZ, 11.2.3]. As for stabilization at p, we use [Shi10, Lem. 6.5] (adapted to the fixed central character setup) instead of [KSZ, §11.2.1]. Even though G_{der} is assumed to be simply connected in [Shi10], we can reduce to that case via z-extensions when $\mathfrak{e} \neq \mathfrak{e}^*$, at the expense of introducing a central character datum; this is done in the same way as explained in §3.5 and §2.7. The resulting function $f_1^{\mathrm{Ig},\mathfrak{e},(j)} \in \mathcal{H}(G_1^{\mathfrak{e}}(\mathbb{Q}_p), \chi_{1,p}^{\mathfrak{e},-1})$ is exactly as described in (7.4.1), which is simply an adaptation of the process in [Shi10, §6.3] to a fixed central character setup. The final steps in stabilization are carried out as in [Shi10, §7], again by working with z-extensions when $\mathfrak{e} \neq \mathfrak{e}^*$. This completes the sketch of stabilization of (7.5.1). □

Remark 7.5.2. In the argument above, we do not need a precise local normalization of transfer factors and Haar measures over all places of $\mathbb{Q}$ when $\mathfrak{e} \neq \mathfrak{e}^*$, since it will only affect error terms by constant factors in the estimate. For instance, we need not know what normalization of transfer factors should be taken at p and ∞ (as well as away from p and ∞) to satisfy the product formula for transfer factors, cf. Remark 2.6.1. We intend to work out precise normalizations in a future paper, thus removing ambiguity in the coefficients c_ω in (7.4.1).

7.6. **Completion of the proof of Theorem 7.1.1.** The main term in the right hand side of Theorem 7.5.1 will turn out to be the following.

Proposition 7.6.1. *Fix $\phi^{\infty,p}\phi_p \in \mathcal{H}(G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p))$, from which $f^{*,(j)} \in \mathcal{H}(G^*(\mathbb{A}))$ is given as in §7.4 for every $j \in \mathbb{Z}_{\geq 1}$ such that $j \geq j_0 = j_0(\phi^{\infty,p}\phi_p)$. As $j \geq j_0$ varies over positive integers divisible by r, we have the estimate*

$$T^{G^*}_{\mathrm{disc},\chi_0}\big(f^{\mathrm{Ig},*,(j)}\big) = \sum_{\substack{\pi \\ \dim \pi = 1}} \mathrm{Tr}\ (\phi^{\infty,p}|\pi^{\infty,p}) \cdot \mathrm{Tr}\ \left(\phi_p^{(j)}|\pi_p \otimes \delta_{P_b}\right) + o\left(p^{j\langle 2\rho,\nu_b\rangle}\right),$$

where the sum runs over one-dimensional automorphic representations π of $G(\mathbb{A})$ such that π_p is unramified and $\pi_\infty|_{G(\mathbb{R})_+} = 1$.

Proof. We have

$$T^{G^*}_{\mathrm{disc},\chi_0}\left(f^{\mathrm{Ig},*,(j)}\right) = \sum_{\pi^*} m(\pi^*)\mathrm{Tr}\ \left(f^{\mathrm{Ig},*,p}|\pi^{*,p}\right) \mathrm{Tr}\ \left(f_p^{\mathrm{Ig},*,(j)}|\pi_p^*\right). \tag{7.6.1}$$

Let $JH(J_{P_b^{\mathrm{op}}}(\pi_p^*))$ denote the multi-set of irreducible subquotients of $J_{P_b^{\mathrm{op}}}(\pi_p^*)$ (up to isomorphism). The central character of $\tau \in JH(J_{P_b^{\mathrm{op}}}(\pi_p^*))$ is denoted ω_τ. We see from Lemma 3.1.3 (ii) that

$$\begin{aligned} \mathrm{Tr}\ \left(f_p^{\mathrm{Ig},*,(j)}|\pi_p^*\right) &= \mathrm{Tr}\ \left(\delta_{P_b}^{1/2}\phi_p^{*,(j)}|J_{P_b^{\mathrm{op}}}(\pi_p^*)\right) = \mathrm{Tr}\ \left(\phi_p^{*,(j)}|J_{P_b^{\mathrm{op}}}(\pi_p^*) \otimes \delta_{P_b}^{1/2}\right) \\ &= \sum_{\tau \in JH(J_{P_b^{\mathrm{op}}}(\pi_p^*))} \omega_\tau(j\nu_b(p))\delta_{P_b}^{1/2}(j\nu_b(p))\,\mathrm{Tr}\,(\phi_p^*|\tau). \end{aligned} \tag{7.6.2}$$

As we saw in §7.4, $j\nu_b(p) \in A_{M_b}^{--}$. By Corollary 2.5.2 (our running assumption that b is $\mathbb{Q}$-non-basic implies ($\mathbb{Q}$-nb(P_b^{op})) by Lemma 5.3.7), the largest growth of $\omega_\tau(j\nu_b(p))$ as a function in j is achieved

[18]The stabilization in [KSZ] is written for arbitrary Shimura varieties (assuming the Shimura variety analogue of (7.5.1) when (G, X) is not of abelian type). In [Shi10], the main assumptions are that (G, X) is of certain PEL type and that G_{der} is simply connected. The former is irrelevant for stabilization, and the latter is removed as we explain in the current proof.

exactly when $\dim \pi^* = 1$. In that case, we have $m(\pi^*) = 1$ and π^*_p is a unitary character. Via Lemma 2.5.3, π^* corresponds to a unique one-dimensional automorphic representation π of $G(\mathbb{A})$. We have $\pi^*_p \simeq \pi_p$ via $G^*(\mathbb{Q}_p) \simeq G(\mathbb{Q}_p)$. Thus

$$\mathrm{Tr}\,(\phi_p^{*,(j)}|J_{P_b^{\mathrm{op}}}(\pi^*_p)\otimes\delta_{P_b}^{1/2}) = \mathrm{Tr}\,(\phi_p^{*,(j)}|\pi_p\otimes\delta_{P_b}) = \mathrm{Tr}\,(\phi_p^{(j)}|\pi_p\otimes\delta_{P_b}) \\ = \delta_{P_b}(j\nu_b(p))\mathrm{Tr}\,(\phi_p|\pi_p\otimes\delta_{P_b}) = p^{j\langle 2\rho,\nu_b\rangle}\mathrm{Tr}\,(\phi_p|\pi_p\otimes\delta_{P_b}). \qquad (7.6.3)$$

We used Lemma 2.3.6 for the second equality above.

Let $f_{\mathbf{1}}$ denote the averaged Lefschetz function on $G(\mathbb{R})$ as in §2.4 with $\xi = \mathbf{1}$ and $\zeta = \mathbf{1}$. Write $e(G^{\infty,p}) := \prod_{v\neq\infty,p} e(G_v)$ for the product of Kottwitz signs. We can rewrite (7.6.1) as

$$T^{G^*}_{\mathrm{disc},\chi_0}\big(f^{\mathrm{Ig},*,(j)}\big) = \sum_{\substack{\pi^*\\ \dim\pi^*=1}} \mathrm{Tr}\,\big(f^{\mathrm{Ig},*}_\infty|\pi^*_\infty\big)\,\mathrm{Tr}\,\big(f^{\mathrm{Ig},*,\infty,p}|\pi^{*,\infty,p}\big)\mathrm{Tr}\,\big(f^{\mathrm{Ig},*,(j)}_p|\pi^*_p\big) + o\big(p^{j\langle 2\rho,\nu_b\rangle}\big)$$

$$= \sum_{\substack{\pi\\ \dim\pi=1}} e(G_\infty)\mathrm{Tr}\,(f_{\mathbf{1}}|\pi_\infty)e(G^{\infty,p})\mathrm{Tr}\,(\phi^{\infty,p}|\pi^{\infty,p})\mathrm{Tr}\,\big(\phi_p^{(j)}|\pi_p\otimes\delta_{P_b}\big) + o\big(p^{j\langle 2\rho,\nu_b\rangle}\big),$$

where the last equality was obtained from (7.6.2) at p, Lemma 2.4.2 at ∞, and Lemma 2.3.6 at the places away from p. Since $e(G_p) = 1$ as G is quasi-split over $\mathbb{Q}_p$, we have $e(G_\infty)e(G^{\infty,p}) = 1$ by the product formula for Kottwitz signs. To conclude, we invoke Lemma 2.4.3 to see that $\mathrm{Tr}\,(f_{\mathbf{1}}|\pi_\infty) = 1$ if $\pi_\infty|_{G(\mathbb{R})_+} = \mathbf{1}$ and $\mathrm{Tr}\,(f_{\mathbf{1}}|\pi_\infty) = 0$ otherwise. □

Finally we complete the proof of Theorem 7.1.1 employing the main estimates of §4.

Corollary 7.6.2. *Theorem 7.1.1 is true.*

Proof. Let $q \neq p$ be an auxiliary prime such that $G_{\mathbb{Q}_q}$ is split. Fix $\phi^{\infty,p,q}\phi_p \in \mathcal{H}(G(\mathbb{A}^{\infty,p,q})\times J_b(\mathbb{Q}_p))$. Write $\mathcal{A}_{\mathbf{1}}(G)$ for the set of one-dimensional automorphic representations π of $G(\mathbb{A})$ such that π_p is unramified and $\pi_\infty|_{G(\mathbb{R})_+} = 1$. There exists a constant $C_{\mathfrak{e}} = C_{\mathfrak{e}}(\phi^{\infty,p,q},\phi_p) > 0$ such that for each $\phi_q \in \mathcal{H}(G(\mathbb{Q}_q))_{C_{\mathfrak{e}}\text{-reg}}$, we have the following bound on endoscopic terms in the stabilization of Theorem 7.5.1 by applying the last bound in Corollary 4.2.3 to $k = j/r$, $\nu = r\nu_{1,\omega}$, $\phi_p^{(k)} = c_\omega(\delta^{1/2}_{P_{\nu_1}}\phi^*_{1,p})^{(k)}$, and $\chi = \chi_0$ for each $\omega \in \Omega_{\mathfrak{e}_1,\nu_1}$. Notice that $f^{\mathrm{Ig},\mathfrak{e},(j)}_{1,p,\omega}$ is $p^{\langle\rho^{\mathfrak{e}},\nu_{1,\omega}\rangle}$ times $f_p^{(k)}$ of that corollary, in light of (7.4.1) and (7.4.2).

$$\mathrm{ST}^{G_1^{\mathfrak{e}}}_{\mathrm{ell},\chi_1^{\mathfrak{e}}}\left((f_1^{\mathrm{Ig},\mathfrak{e},p})f^{\mathrm{Ig},\mathfrak{e},(j)}_{1,p,\omega}\right) = O\left(p^{k(\langle 2\rho^{\mathfrak{e}},r\nu_{1,\omega}\rangle + \langle\chi_1^{\mathfrak{e}},r\overline{\nu}_{1,\omega}\rangle)}\right) = O\left(p^{j(\langle 2\rho^{\mathfrak{e}},\nu_{1,\omega}\rangle + \langle\chi_1^{\mathfrak{e}},\overline{\nu}_{1,\omega}\rangle)}\right).$$

To turn this into a more manageable bound, we use (a) and (b) from the proof of Corollary 4.2.3 and the fact that $\langle\chi_{0,\infty},\overline{\nu}\rangle = 0$ since χ_0 (which plays the role of χ there) is trivial. Thereby we see that the right hand side is $o\left(p^{j\langle 2\rho,\nu_b\rangle}\right)$. Taking the sum over $\omega\in\Omega_{\mathfrak{e}_1,\nu_1}$, we obtain

$$\mathrm{ST}^{G_1^{\mathfrak{e}}}_{\mathrm{ell},\chi_1^{\mathfrak{e}}}\left(f_1^{\mathrm{Ig},\mathfrak{e},(j)}\right) = o\left(p^{j\langle 2\rho,\nu_b\rangle}\right), \qquad \mathfrak{e}\in\mathcal{E}^<_{\mathrm{ell}}(G). \qquad (7.6.4)$$

By Lemma 2.9.3, there are only finitely many $\mathfrak{e}$ contributing to the sum in Theorem 7.5.1 for a fixed choice of $\phi^{\infty,p,q}\phi_p$. Thus the coefficients $\iota(G,G^{\mathfrak{e}})$ are bounded by a uniform constant (depending on $\phi^{\infty,p,q}\phi_p$). We deduce the following by applying Theorem 7.5.1, (7.6.4), and Proposition 7.6.1 in the order: there exists a constant $C = C(\phi^{\infty,p},\phi_p) > 0$ (e.g., the maximum of $C_{\mathfrak{e}}$ over the set of finitely many $\mathfrak{e}$ which contribute) such that for every $\phi_q \in \mathcal{H}(G(\mathbb{Q}_q))_{C\text{-reg}}$, we have

$$\mathrm{Tr}\,\left(\phi^{\infty,p}\phi_p^{(j)}|\iota H_c(\mathrm{Ig}_b,\overline{\mathbb{Q}}_\ell)\right) = \mathrm{ST}^{G^*}_{\mathrm{ell},\chi}\big(f^{\mathrm{Ig},*,(j)}\big) + o\left(p^{j\langle 2\rho,\nu_b\rangle}\right) = T^{G^*}_{\mathrm{disc},\chi}\big(f^{\mathrm{Ig},*,(j)}\big) + o\left(p^{j\langle 2\rho,\nu_b\rangle}\right)$$

$$= \sum_{\pi\in\mathcal{A}_{\mathbf{1}}(G)} \mathrm{Tr}\,(\phi^{\infty,p}|\pi^{\infty,p})\cdot\mathrm{Tr}\,\left(\phi_p^{*,(j)}|\pi_p\otimes\delta_{P_b}\right) + o\left(p^{j\langle 2\rho,\nu_b\rangle}\right).$$

We have seen in (7.6.3) that $\mathrm{Tr}\left(\phi_p^{*,(j)}|\pi_p\otimes\delta_{P_b}\right)$ is either 0 or a nonzero multiple of $p^{j\langle 2\rho,\nu_b\rangle}$ as j varies over multiples of r. Since $\dim \mathrm{Ig}_b = \langle 2\rho,\nu_b\rangle$, it is implied by (7.2.1) and the Lang–Weil bound that the leading term should be of order $p^{j\langle 2\rho,\nu_b\rangle}$. [19] Therefore

$$\mathrm{Tr}\left(\phi^{\infty,p}\phi_p^{(j)}|\iota H_c^{\langle 4\rho,\nu_b\rangle}(\mathrm{Ig}_b,\overline{\mathbb{Q}}_\ell)\right)=\sum_{\pi\in\mathcal{A}_{\mathbf{1}}(G)}\mathrm{Tr}\left(\phi^{\infty,p}|\pi^{\infty,p}\right)\cdot\mathrm{Tr}\left(\phi_p^{*,(j)}|\pi_p\otimes\delta_{P_b}\right).$$

Let B_q be a Borel subgroup of $G_{\mathbb{Q}_q}$ over $\mathbb{Q}_q$ with a Levi component T_q. By Lemma 3.4.9, we have an isomorphism of $G(\mathbb{A}^{\infty,p,q})\times J_b(\mathbb{Q}_p)\times T_q(\mathbb{Q}_q)$-representations

$$J_{B_q}\left(H_c^{\langle 4\rho,\nu_b\rangle}(\mathrm{Ig}_b,\overline{\mathbb{Q}}_\ell)\right)\simeq\sum_{\pi\in\mathcal{A}_{\mathbf{1}}(G)}\pi^{\infty,p,q}\otimes(\pi_p\otimes\delta_{P_b})\otimes J_{B_q}(\pi_q).$$

(A priori the isomorphism exists up to semisimplification, but distinct one-dimensional representations have no extensions with each other.) Repeating the same argument for another prime $q'\notin\{p,q\}$ such that $G(\mathbb{Q}_{q'})$ is unramified, the above isomorphism exists with q' in place of q. Comparing the two consequences, we deduce Theorem 7.1.1, which asserts that

$$\iota H_c^{\langle 4\rho,\nu_b\rangle}(\mathrm{Ig}_b,\overline{\mathbb{Q}}_\ell)\simeq\sum_{\pi\in\mathcal{A}_{\mathbf{1}}(G)}\pi^{\infty,p}\otimes(\pi_p\otimes\delta_{P_b})\qquad\text{as } G(\mathbb{A}^{\infty,p})\times J_b(\mathbb{Q}_p)\text{-modules.}$$

□

8. Applications to geometry

This section is devoted to interesting geometric consequences of Theorem 6.1.3, continuing in the setup $(\mathrm{Unr}(G,p,K_p))$ for Hodge-type Shimura varieties.

8.1. **The discrete Hecke orbit conjecture.** To see that Theorem 6.1.3 implies Theorem 5.4.4, we check a group-theoretic lemma. Recall that we fixed a maximal torus and a Borel subgroup $T\subset B\subset G$ over $\mathbb{Z}_p$ and that M_b is a $\mathbb{Q}_p$-rational Levi subgroup containing T. The hyperspecial vertex for G determining K_p (contained in the apartment for T) gives a reductive model for M_b over $\mathbb{Z}_p$ such that $M_b(\mathbb{Z}_p)=M_b(\mathbb{Q}_p)\cap G(\mathbb{Z}_p)$. Earlier we defined $J_b^{\mathrm{int}}\subset J_b(\mathbb{Q}_p)$.

Lemma 8.1.1. *The image of J_b^{int} in $J_b(\mathbb{Q}_p)^{\mathrm{ab}}$ is carried onto the image of $M_b(\mathbb{Z}_p)$ in $M_b(\mathbb{Q}_p)^{\mathrm{ab}}$ under the canonical isomorphism $J_b(\mathbb{Q}_p)^{\mathrm{ab}}\xrightarrow{\sim}M_b(\mathbb{Q}_p)^{\mathrm{ab}}$.*

Proof. The validity of the lemma is invariant when b is changed to a σ-conjugate element $b'\in G(\breve{\mathbb{Q}}_p)$. Indeed, under the canonical identifications $J_{b'}(\mathbb{Q}_p)^{\mathrm{ab}}=J_b(\mathbb{Q}_p)^{\mathrm{ab}}$, the images of $J_{b'}^{\mathrm{int}}$ and J_b^{int} are equal. (In this proof, we need not worry about preserving conditions (br1)–(br3) as they play no role.) To avoid confusion, recall from §5.3 that we are writing M_b for M_{b° with a fixed b°. So M_b does not change when σ-conjugating b.

By [Kot85, Prop. 6.2] we may replace b with a σ-conjugate to assume that $b\in M_b(\breve{\mathbb{Q}}_p)$ and that b is basic in M_b. We further reduce to the case when $G_{\mathrm{der}}=G_{\mathrm{sc}}$. Indeed, take an unramified z-extension $1\to Z_1\to G_1\to G\to 1$ over $\mathbb{Q}_p$ and a lift $b_1\in G_1(\breve{\mathbb{Q}}_p)$ of b as in Lemma 5.3.8. Then $J_{b_1}^{\mathrm{int}}\to J_b^{\mathrm{int}}$ is onto. (Use the fact that $Z_1(\breve{\mathbb{Z}}_p)\to Z_1(\breve{\mathbb{Z}}_p)$ given by $z\mapsto\sigma(z)^{-1}z$ is onto.) Since $M_{b_1}(\mathbb{Z}_p)\to M_b(\mathbb{Z}_p)$ is onto as well (as $M_{b_1}\to M_b$ is a smooth morphism over $\mathbb{Z}_p$), we see that the proof for G is reduced to that for G_1.

Thus we assume that $b\in M_b(\breve{\mathbb{Q}}_p)$ is basic and that $G_{\mathrm{der}}=G_{\mathrm{sc}}$. So $M_{b,\mathrm{der}}=M_{b,\mathrm{sc}}$ as well. The natural map $M_b\to M_b^{\mathrm{ab}}$ is surjective on $\mathbb{Z}_p$-points, by smoothness over $\mathbb{Z}_p$ and the surjectivity on $\mathbb{F}_p$-points by Lang's theorem. It suffices to verify that the image of J_b^{int} in $J_b^{\mathrm{ab}}(\mathbb{Q}_p)=M_b^{\mathrm{ab}}(\mathbb{Q}_p)$ is equal to $M_b^{\mathrm{ab}}(\mathbb{Z}_p)$.

[19] In fact, the Lang–Weil bound proves that $\dim \mathrm{Ig}_b=\langle 2\rho,\nu_b\rangle$ even if we did not know it a priori. This gives an alternative proof of the dimension formula in Proposition 5.3.4.

Since M_b is unramified over $\mathbb{Q}_p$, it contains an unramified elliptic maximal torus T' over $\mathbb{Q}_p$. (See the last paragraph in [DeB06, §2.4].) Then T' extends to a torus over $\mathbb{Z}_p$, which we still denote by T'. By σ-conjugating b, we may assume that $b \in T'(\breve{\mathbb{Q}}_p)$ by [Kot85, Prop. 5.3]. By the definition of J_b and $J_b^{\rm int}$, we have $T'(\mathbb{Q}_p) \subset J_b(\mathbb{Q}_p)$ and $T'(\mathbb{Z}_p) \subset J_b^{\rm int}$. Moreover we have a commutative diagram

$$\begin{array}{ccccc} T'(\mathbb{Z}_p) & \hookrightarrow & J_b^{\rm int} & \hookrightarrow & J_b(\mathbb{Q}_p) \\ \downarrow & & \downarrow & & \downarrow \\ M_b(\mathbb{Z}_p) & \longrightarrow & M_b^{\rm ab}(\mathbb{Z}_p) & \hookrightarrow & M_b^{\rm ab}(\mathbb{Q}_p) = J_b^{\rm ab}(\mathbb{Q}_p). \end{array}$$

(The second vertical map is given by the composite $J_b^{\rm int} \subset M_b(\breve{\mathbb{Z}}_p) \to M_b^{\rm ab}(\breve{\mathbb{Z}}_p)$, whose image is clearly σ-invariant. With this, the commutativity is elementary to check.) The proof will be done once we show that the map $T'(\mathbb{Z}_p) \to M_b^{\rm ab}(\mathbb{Z}_p)$ is surjective in the diagram.

We have an exact sequence of unramified tori $1 \to T' \cap M_{b,\rm der} \to T' \to M_b^{\rm ab} \to 1$, which extends to an exact sequence of tori over $\mathbb{Z}_p$. Then $T'(\mathbb{Z}_p) \to M_b^{\rm ab}(\mathbb{Z}_p)$ is surjective, for the same reason that $M_b(\mathbb{Z}_p) \to M_b^{\rm ab}(\mathbb{Z}_p)$ was surjective above. This finishes the proof. □

Remark 8.1.2. The above proof also works for a $\mathbb{Z}_p$-subtorus T' of M_b that is not unramified and elliptic over $\mathbb{Q}_p$. What we need is that $b \in T'(\breve{\mathbb{Q}}_p)$, that $T' \to M_b^{\rm ab}$ is surjective, and that $T' \cap M_{b,\rm der}$ is connected over $\mathbb{F}_p$ (to apply Lang's theorem). Under this condition, the same argument shows that the images of $T'(\mathbb{Z}_p)$ and $J_b^{\rm int}$ are equal in $M_b^{\rm ab}(\mathbb{Q}_p) = J_b^{\rm ab}(\mathbb{Q}_p)$. An exemplary situation is when T' is a (maximally split) maximal torus of M_b over $\mathbb{Z}_p$ with b central in M_b.

Corollary 8.1.3. *Theorem 5.4.4 is true. The discrete Hecke orbit conjecture holds when b is $\mathbb{Q}$-non-basic.*

Proof. We start by noting that perfection does not change topological information of a scheme such as the set of connected components or étale cohomology.

By Part (1) of Lemma 6.1.1 and Theorem 6.1.3, we have isomorphisms of $G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$-modules

$$\iota H^0(C_b, \overline{\mathbb{Q}}_\ell) = \iota H^0(C_b^{\rm perf}, \overline{\mathbb{Q}}_\ell) = \iota H^0(\mathfrak{Ig}_b, \overline{\mathbb{Q}}_\ell)^{J_b^{\rm int}} \simeq \bigoplus_\pi \pi^{\infty,p} \otimes (\pi_p \otimes \delta_{P_b})^{J_b^{\rm int}}, \tag{8.1.1}$$

where the sum runs over the same set of π as in Theorem 6.1.3.

By Lemma 8.1.1, the left hand side is fixed (pointwise) under $J_b^{\rm int}$ if and only if the right hand side is fixed under $M_b(\mathbb{Z}_p)$. The latter condition holds if and only if π_p is fixed under $G(\mathbb{Z}_p)$. (The if direction is trivial. For the other implication, note that $G(\mathbb{Z}_p)$ is generated by $M_b(\mathbb{Z}_p)$ and unipotent subgroups of $G(\mathbb{Z}_p)$, but unipotent subgroups act trivially as they lie in $G_{\rm der}(\mathbb{Z}_p)$.) Thereby we deduce Theorem 5.4.4 from (8.1.1) and Lemma 5.1.1. Finally ($\mathrm{HO}_{\rm disc}$) and ($\mathrm{HO}'_{\rm disc}$) follow from Corollary 5.4.5. □

8.2. **Irreducibility of Igusa varieties.** In §1.3, we reviewed earlier results on irreducibility of Igusa towers over the μ-ordinary Newton strata of certain PEL-type Shimura varieties. Now we explain that our main theorem implies a generalization thereof to Hodge-type Shimura varieties and to non-μ-ordinary strata.

We continue in the same setup as in the preceding subsections; thus ($\mathrm{Unr}(G, p, K_p)$) is assumed, and b is $\mathbb{Q}$-non-basic. Define $J(\mathbb{Q}_p)' := \ker(J_b(\mathbb{Q}_p) \to G(\mathbb{Q}_p)^{\rm ab})$ using the composite map in the diagram (6.1.3). Recall that $\mathrm{pr} : \mathfrak{Ig}_b \to C_{b,\overline{\mathbb{F}}_p}^{\rm perf}$ is a pro-étale $J_b^{\rm int}$-torsor. Accepting Theorem 6.1.3 for now (to be proved in the next section), we show that Igusa varieties are 'as irreducible as possible'.

Theorem 8.2.1 (Irreducibility of Igusa varieties)**.** *The stabilizer of each connected component of $\mathfrak{Ig}_b$ under the $J_b(\mathbb{Q}_p)$-action is equal to $J_b(\mathbb{Q}_p)'$.*

Proof. Fix a component $I \subset \mathfrak{Ig}_b$ and write $\mathrm{Stab}(I)$ for the stabilizer of I in $J_b(\mathbb{Q}_p)$. Since the $J_b(\mathbb{Q}_p)$-action on every π_p in Theorem 6.1.3 factors through $J_b(\mathbb{Q}_p)/J_b(\mathbb{Q}_p)'$, we see that $\mathrm{Stab}(I) \supset J_b(\mathbb{Q}_p)'$. To prove the reverse inclusion, we show that every $\delta \in J_b(\mathbb{Q}_p)\backslash J_b(\mathbb{Q}_p)'$ acts nontrivially on $H^0(\mathfrak{Ig}_b, \overline{\mathbb{Q}}_\ell)$. Write $\delta^{\mathrm{ab}} \in G(\mathbb{Q}_p)^{\mathrm{ab}}$ for the image of δ. Then $\delta^{\mathrm{ab}} \neq 1$ by assumption. It suffices to show that there exists π as in the summation of Theorem 6.1.3 such that π_p is nontrivial on δ^{ab} when π_p is viewed as a character of $G(\mathbb{Q}_p)^{\mathrm{ab}}$ via Corollary 2.3.3. This follows from Lemma 2.5.4. □

Corollary 8.2.2. *Let S be a connected component of $C^{\mathrm{perf}}_{b,\overline{\mathbb{F}}_p}$. Then the set $\pi_0(\mathrm{pr}^{-1}(S)) \subset \pi_0(\mathfrak{Ig}_b)$ is a torsor under the group $J_b^{\mathrm{int}}/(J_b^{\mathrm{int}} \cap J_b(\mathbb{Q}_p)')$. Every component of $\mathrm{pr}^{-1}(S)$ is a pro-étale torsor under $J_b^{\mathrm{int}} \cap J_b(\mathbb{Q}_p)'$, and conversely, if $I \subset \mathfrak{Ig}_b$ is an open subscheme such that $I \to S$ is a pro-étale $J_b^{\mathrm{int}} \cap J_b(\mathbb{Q}_p)'$-torsor via* pr*, then I is irreducible.*

Proof. As pr is a J_b^{int}-torsor, J_b^{int} acts transitively on $\pi_0(\mathrm{pr}^{-1}(S))$. Theorem 8.2.1 implies that the action factors through a simply transitive action of $J_b^{\mathrm{int}}/J_b^{\mathrm{int}} \cap J_b(\mathbb{Q}_p)'$, proving the first assertion. The second assertion again follows from the same theorem and the fact that pr is a J_b^{int}-torsor. □

Since similar irreducibility results are stated in the literature over μ-ordinary Newton strata, it is worth verifying that a μ-ordinary Newton stratum is itself a central leaf; thus $\mathfrak{Ig}_b$ is a pro-étale torsor over the entire μ-ordinary Newton stratum (after taking perfection). This should be well known, but we have not found a convenient reference for the general statement.

For the remainder of this subsection, we compare with similar irreducibility results in the μ-ordinary case. Thus we specialize to the case when $[b] \in B(G, \mu_p^{-1})$ is **μ-ordinary**, meaning either of the following equivalent conditions [Wor, Rem. 5.7 (2)]:

- $[b] = [\mu_p^{-1}(p)]$ in $B(G)$ (which implies $[b] \in B(G, \mu_p^{-1})$).
- $[b]$ is the unique minimal element in $B(G, \mu_p^{-1})$ for the partial order $\preceq$ therein.

In this case, we may and will take $b = b^\circ = \mu_p^{-1}(p)$. Indeed, we can change b within its σ-conjugacy class thanks to Proposition 6.2.2. Put $r := [k(\mathfrak{p}) : \mathbb{F}_p]$. By the convention of §5.3, μ_p is defined over $\mathbb{Q}_{p^r}$. With this choice of r, μ_p, b, we have $\nu_b = \frac{1}{r}\sum_{i=0}^{r-1} \sigma^i \mu_p^{-1}$ (this follows from (4.3.1)–(4.3.3) of [Kot85] with $n = r$ and $c = 1$), which is defined over $\mathbb{Q}_p$, and conditions (br2) and (br3) are satisfied.

We define the μ-ordinary Newton stratum N_{b,K^p} as in [Wor], that is, by changing the definition of C_{b,K^p} (§5.3) to require the existence of an isomorphism only after inverting p. Then $C_{b,K^p} \subset N_{b,K^p}$ is closed by [Ham17, §2.3, Prop. 2]. It is worth verifying that $C_{b,K^p} = N_{b,K^p}$, so that $\mathfrak{Ig}_b$ is a pro-étale torsor over $N^{\mathrm{perf}}_{b,K^p}$ (not just $C^{\mathrm{perf}}_{b,K^p}$); this is the perfection of the usual setup in the literature.

Lemma 8.2.3. *In the μ-ordinary setup above, $C_{b,K^p} = N_{b,K^p}$.*

Proof. Proposition 5.3.4 (for the first equality) and the σ-invariance of ρ (since positive roots are coming from a Borel subgroup over $\mathbb{Q}_p$; this is used for the third equality) imply that

$$\dim C_{b,K^p} = \langle \rho, \nu_b \rangle = r^{-1} \sum_{i=0}^{r-1} \langle \rho, \sigma^i \mu_p \rangle = \langle \rho, \mu_p \rangle = \dim \mathscr{S}_{K_pK^p,k(\mathfrak{p})}.$$

Combined with ($\mathrm{HO}'_{\mathrm{disc}}$), this tells us that C_{b,K^p} is dense in $\mathscr{S}_{K_pK^p,k(\mathfrak{p})}$. It follows that C_{b,K^p} is dense and closed in N_{b,K^p}. Since N_{b,K^p} is reduced (as it is open in the smooth variety $\mathscr{S}_{K_pK^p,k(\mathfrak{p})}$), we conclude that $C_{b,K^p} = N_{b,K^p}$ as schemes. □

We explain Corollary 8.2.2 gives another proof for the irreducibility of Igusa towers in the (μ-)ordinary case, in the setting of unitary similitude PEL-type Shimura varieties as in [CEF⁺16, EM], cf. [Hid11, §2,§3]. Analogous arguments can be made in other settings such as the elliptic/Hilbert/Siegel modular cases.

Write $\{\mathrm{Ig}^{\mu\text{-ord}}_{m,K^p}\}_{mge1}$ for the Igusa tower $\{(\mathrm{Ig}_\mu)_{m,1}\}_{m\geq 1}$ over the μ-ordinary stratum N_{b,K^p} in [EM, §3.2] (relative to the same K^p) with finite étale transition maps. The scheme $\mathrm{Ig}^{\mu\text{-ord}}_{K^p} = \varprojlim_m \mathrm{Ig}^{\mu\text{-ord}}_{m,K^p}$ is a pro-étale J_b^{int}-torsor over N_{b,K^p}. Then $\mathfrak{Ig}_{b,K^p} \simeq (\mathrm{Ig}^{\mu\text{-ord}}_{K^p})^{\mathrm{perf}}$ compatibly with the actions of $G(\mathbb{A}^{\infty,p}) \times S_b$ (see Prop. 4.3.8 and the paragraph above Cor. 4.3.9 in [CS17]; see also [CS, Rem. 2.3.7]), and we have a $J_b(\mathbb{Q}_p)$-equivariant bijection

$$\pi_0(\mathfrak{Ig}_{b,K^p}) \simeq \pi_0((\mathrm{Ig}^{\mu\text{-ord}}_{K^p})^{\mathrm{perf}}) \simeq \pi_0(\mathrm{Ig}^{\mu\text{-ord}}_{K^p}).$$

Therefore each connected component of $\mathrm{Ig}^{\mu\text{-ord}}_{K^p}$ has stabilizer $J_b(\mathbb{Q}_p)'$ in $J_b(\mathbb{Q}_p)$.

The $\mathbb{Z}_p$-group J_μ in [EM, Rem. 2.9.3] has the property that $J_b^{\mathrm{int}} = J_\mu(\mathbb{Z}_p)$. Let $I \subset \mathrm{Ig}^{\mu\text{-ord}}_{K^p}$ denote the open subscheme Ig^{SU}_μ over a fixed component S of N_{b,K^p} as defined in [EM, §3.3] (more precisely, we mean the special fiber of Ig^{SU}_μ over $\overline{\mathbb{F}}_p$). Then I is a pro-étale $J_b^{\mathrm{int}} \cap J(\mathbb{Q}_p)'$-torsor over S by construction. (The determinant map of [EM, §3.3] goes from a J_b^{int}-torsor to a torsor under $G^{\mathrm{ab}}(\mathbb{Z}_p)$, so the fiber is a torsor under $\ker(J_b^{\mathrm{int}} \to G^{\mathrm{ab}}(\mathbb{Z}_p))$.)[20] Hence I is irreducible by the preceding paragraph, cf. the proof of Corollary 8.2.2.

If $[b]$ is moreover **ordinary**, that is if $[b]$ is μ-ordinary and ν_b is conjugate to μ_p^{-1}, then μ_p is defined over $\mathbb{Q}_p$ (since the conjugacy class of ν_b is always defined over $\mathbb{Q}_p$) and $r = [E_{\mathfrak{p}} = \mathbb{Q}_p] = 1$. By our choice $b = \mu_p(p)^{-1}$, we have $\nu_b = \mu_p^{-1}$ (not just conjugate) in this case. The following lemma is handy when comparing with results in the ordinary case such as [Hid09, Hid11]. Note that trivially $\varrho(G_{\mathrm{sc}}(\mathbb{Q}_p)) = G_{\mathrm{der}}(\mathbb{Q}_p)$ when $G_{\mathrm{der}} = G_{\mathrm{sc}}$.

Lemma 8.2.4. *If μ is ordinary, we have $J_b = M_b$, $J_b^{\mathrm{int}} = M_b(\mathbb{Z}_p)$, and $J_b(\mathbb{Q}_p)' = M_b(\mathbb{Q}_p) \cap \varrho(G_{\mathrm{sc}}(\mathbb{Q}_p))$.*

Proof. By definition, M_b is the centralizer of $\nu_b = \mu_p^{-1}$ in G. From the definition (5.3.1) with $b = \mu_p^{-1}(p)$, we see that M_b is a closed $\mathbb{Q}_p$-subgroup of J_b. On the other hand, M_b is an inner form of J_b, so we conclude $M_b = J_b$. Then $J_b^{\mathrm{int}} = J_b(\mathbb{Q}_p) \cap G(\breve{\mathbb{Z}}_p) = M_b(\mathbb{Q}_p) \cap G(\breve{\mathbb{Z}}_p) = M_b(\mathbb{Z}_p)$. The description of $J_b(\mathbb{Q}_p)'$ is obvious from $J_b = M_b$. □

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