Every two K3 surfaces are deformation equivalent

Juultje Kok

reference: Le Potier; Géométrie des surfaces K3 modules et périodes, Astérisque No. 126, 1985; pages 79-89

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Every two K3 surfaces are deformation equivalent

Theorem

Every two K3 surfaces are deformation equivalent

Proof: Take a K3 surface $X_0$ with $\phi : H^2(X_0, \mathbb{Z}) \sim \Lambda_{K3}$. Let $U$ be an open connected neighbourhood of $X_0$ in the period domain.

Theorem

The subset of K3 surfaces of type $g$ lies dense in $\text{Im}(\alpha) \subseteq \Omega$.

Definition

A K3 surface $X$ is called of type $g$ if the Picard group of $X$ is generated by an element $L$ with $(L, L) = 2g - 2$. 

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**Theorem**

*Every two K3 surfaces are deformation equivalent*

**Proof:** Take a K3 surface $X_0$ with $\phi : H^2(X_0, \mathbb{Z}) \simeq \Lambda_{K3}$. Let $U$ be an open connected neighbourhood of $X_0$ in the period domain.

**Theorem**

*The subset of K3 surfaces of type $g$ lies dense in $\text{Im}(\alpha) \subseteq \Omega$.*

**Theorem**

*Every K3 surface of type 3 is a smooth quartic in $\mathbb{P}^3$."

There exist a quartic surface $X'$ in $U$.

Local Torelli theorem $\iff X_0$ and $X'$ are deformation equivalent.

**Theorem**

*Every two smooth quartic surfaces are deformation equivalent.*
The subset of K3 surfaces of type $g$ lies dense in $\text{Im}(\alpha) \subseteq \Omega$

**Proof:** Let $U \neq \emptyset$ be a open subset of $\text{Im}(\alpha) \subseteq \Omega$. Define for a subgroup $G \subseteq \Lambda_{K3}$ the following:

$$\Sigma(G) := \{ x \in \mathbb{P}(\Lambda_C) : (x, x) = 0 \text{ and } (x, g) = 0 \ \forall g \in G \}.$$ 

**Proposition**

Let $U \subseteq K_{20}$ be a non-empty open subset. Then there exists a primitive $\beta \in \Lambda_{K3}$ such that:

i) $(\beta, \beta) = 2g - 2,$

ii) $U \cap \{ z \in \mathbb{P}(\Lambda_C) : (\beta, z) = 0 \} \neq \emptyset$

$$\implies \Sigma(\beta \mathbb{Z}) \cap U \neq \emptyset.$$
The subset of K3 surfaces of type $g$ lies dense in $\text{Im}(\alpha) \subseteq \Omega$

Define: $G_\beta = \{ H \subseteq \Lambda_{K3} : \beta \in H \text{ and } H \neq \beta \mathbb{Z} \}$. We have:

- every $H \in G_\beta$ has rank $\geq 2$
- $G_\beta$ is countable

Then it follows:

$$(\Sigma(\beta \mathbb{Z}) \cap U) \notin \bigcup_{H \in G_\beta} \Sigma(H).$$

Now take an element $(z) \in (\Sigma(\beta \mathbb{Z}) \cap U) \setminus \bigcup_{H \in G_\beta} \Sigma(H)$. Then $(z)$ is the period of some K3 surface $X$. We have:

$$\text{Pic}(X) \simeq H^{1,1}(X) \cap H^2(X, \mathbb{Z}) \quad \text{(Lefschetz)}$$
$$\simeq \{ \gamma \in \Lambda_{K3} : (\gamma, z) = 0 \}$$

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The subset of K3 surfaces of type $g$ lies dense in $\text{Im}(\alpha) \subseteq \Omega$

This gives us: $\beta \in \text{Pic}(X)$ and $(z) \in \Sigma(\text{Pic}(X))$.

$$\implies \text{Pic}(X) \not\subseteq G_\beta$$

Therefore $\text{Pic}(X) \simeq \beta \mathbb{Z}$, hence $U$ contains a K3 surface $X$ of type $g$. 

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Every two K3 surfaces are deformation equivalent
Every K3 surface of type 3 is isomorphic with a quartic in $\mathbb{P}^3_\mathbb{C}$.

**Proof:** Let $X$ be a K3 surface of type 3, and $L$ a generator of $\text{Pic}(X)$ with $(L, L) = 4$. We can assume that $h^0(X, L) \neq 0$. We have:

$$SD : \quad h^2(X, L) \simeq h^0(X, L^\vee) = 0$$

$$RR : \quad \chi(L) = h^0(X, L) - h^1(X, L) = \frac{1}{2}(L, L) + 2 = 4$$

**Lemma**

The linebundle $L$ is globally generated.
Every K3 surface of type 3 is quartic in $\mathbb{P}^3_{\mathbb{C}}$

**Corollary**

The linebundle $L$ is ample, a generic curve $Y \in |L|$ is smooth, and $h^1(X, L) = 0$.

So the linebundle gives a finite morphism

$$X \xrightarrow{\phi_L} X' \rightarrow \mathbb{P}^3_{\mathbb{C}} = \mathbb{P}(H^0(X, L)^\vee) \text{ with } \phi^*_L(O(1)) \simeq L.$$  

We have:

$$\deg(X') \cdot \deg(\phi_L) = (L, L) = 4$$
Every K3 surface of type 3 is quartic in $\mathbb{P}^3_C$

1. **$Y$ is non-hyperelliptic**: $\deg(\phi_L) = 1$
   \[\implies \deg(X') = 4.\]

2. **$Y$ is hyperelliptic**: $\deg(\phi_L) = 2$
   \[\implies X'$ is a irreducible quadric surface in $\mathbb{P}^3$.\]

   (a) **$X'$ is smooth**: $X' \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $\text{Pic}(X') \cong \mathbb{Z} \times \mathbb{Z}$.
      But then $O(1)|_{X'} \cong O(0, 1) \otimes O(1, 0)$,
      \[\implies \phi^*_L(O(1)) \cong L \otimes^m: \text{contradiction}.\]

   (b) **$X'$ is singular**: $X'$ is a cone in $s$, $\text{Pic}(X' \setminus \{s\}) \cong L(\gamma)\mathbb{Z}$.
      But then $O(1)|_{X' \setminus \{s\}} \cong L(2\gamma)$
      \[\implies \phi^*_L(O(1)) \cong L \otimes^{2m}: \text{contradiction}.\]

Hence, $X$ is isomorphic to a quartic surface in $\mathbb{P}^3_C$. □

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Every two K3 surfaces are deformation equivalent
Every 2 smooth quartics in $\mathbb{P}^3$ are deformation equivalent.

**Theorem**

Every 2 smooth quartic surfaces in $\mathbb{P}^3$ are deformation equivalent.

**Proof:** A quartic surface $X \subseteq \mathbb{P}^3_{\mathbb{C}}$ is given by an equation:

$$a_0X^4 + \ldots + a_3W^4 + a_4X^2Y^2 + \ldots + a_{34}XYZW.$$

So we get a morphism: $\mathbb{P}^3_{\mathbb{C}} \supset U$

$$\downarrow \quad \downarrow$$

$$\{\text{quartics}\} \supset \{\text{smooth quartics}\}$$

$U$ is an open and connected subset of $\mathbb{P}^3_{\mathbb{C}}$, hence all quartic surfaces are deformation equivalent.
**Proposition**

Let $U \subseteq K_{20}$ be a non-empty open subset. Then there exists a primitive $\beta \in \Lambda_{K3}$ such that:

i) $(\beta, \beta) = 2g - 2$,

ii) $U \cap \{z \in \mathbb{P}(\Lambda_{\mathbb{C}}) : (\beta, z) = 0\} \neq \emptyset$

**Lemma**

The linebundle $L$ is generated by global sections.

**Corollary**

The linebundle $L$ is ample.

**Corollary**

We have $h^1(X, L) = 0$. 

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Proposition

Let $U \subseteq K_{20}$ be a non-empty open subset. Then there exists a primitive $\beta \in \Lambda_{K3}$ such that:

i) $(\beta, \beta) = 2g - 2$,

ii) $U \cap \{z \in \mathbb{P}(\Lambda_{\mathbb{C}}) : (\beta, z) = 0\} \neq \emptyset$

Proof: Define $\Sigma := \{(\alpha, z) \in \mathbb{P}(\Lambda_{\mathbb{C}}) \times K_{20} : (\alpha, z) = 0\}$.

\[
\begin{array}{ccc}
\Sigma & \xrightarrow{p_2} & K_{20} \supseteq U \\
\downarrow p_1 & & \\
\mathbb{P}(\Lambda_{\mathbb{R}}) & \xrightarrow{i} & \mathbb{P}(\Lambda_{\mathbb{C}})
\end{array}
\]

$V := p_1(p_2^{-1}(U))$ is open. One can show: $i^{-1}(V) \cap Q \neq \emptyset$.

Lemma

Let $Q = \{x \in \mathbb{P}(\Lambda_{\mathbb{R}}) : (x, x) = 0\}$ and $V$ an open subset of $\mathbb{P}(\Lambda_{\mathbb{R}})$ with $V \cap Q \neq \emptyset$. Then $V$ contains an element $(\beta)$ with $\beta \in \Lambda_{K3}$, such that $\beta$ is primitive and $(\beta, \beta) = 2g - 2$. 
Proposition

Let $U \subseteq K_{20}$ be a non-empty open subset. Then there exists a primitive $\beta \in \Lambda_{K3}$ such that:

i) $(\beta, \beta) = 2g - 2$,

ii) $U \cap \{z \in \mathbb{P}(\Lambda_{\mathbb{C}}) : (\beta, z) = 0\} \neq \emptyset$

Proof: Define $\Sigma := \{((\alpha), (z)) \in \mathbb{P}(\Lambda_{\mathbb{C}}) \times K_{20} : (\alpha, z) = 0\}$.

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\begin{array}{ccc}
\Sigma & \xrightarrow{p_2} & K_{20} \supseteq U \\
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\]

$V := p_1(p_2^{-1}(U))$ is open. One can show: $i^{-1}(V) \cap Q \neq \emptyset$.

$\exists$ a primitive $\beta \in \mathbb{P}(\Lambda_{K3})$ with $(\beta, \beta) = 2g - 2$. We also have:

$i(\beta) \in V \implies \exists u \in U$ with $(\beta, u) = 0$. 

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Lemma

Let \( Q = \{ x \in \mathbb{P}(\Lambda_{\mathbb{R}}) : (x, x) = 0 \} \) and \( V \) an open subset of \( \mathbb{P}(\Lambda_{\mathbb{R}}) \) with \( V \cap Q \neq \emptyset \). Then \( V \) contains an element \((\beta)\) with \( \beta \in \Lambda_{K3} \), such that \( \beta \) is primitive and \((\beta, \beta) = 2g - 2\).

Proof:

Let \( B = \{ \beta \in \Lambda_{K3} : (\beta, \beta) = 2g - 2 \text{ and } \beta \text{ is primitive} \} \),

\((B)\) the image of \( B \) in \( \mathbb{P}(\Lambda_{\mathbb{R}}) \), and

\[ F = \{ \text{limit points of } (B) \} = (\overline{B}) \setminus (B)^0. \]

Claim: \( F \subseteq Q \).

Suppose \((\beta_i) \to \beta \) in \( \mathbb{P}(\Lambda_{\mathbb{R}}) \) with \( \beta_i \in \Lambda_{K3}, \beta_i^2 = 2g - 2, \) and \( \beta^2 \neq 0 \). We can choose \( \beta \) such that \( \beta^2 = \beta_i^2 \).

\[ (\beta_i) \to \beta \iff \lambda_i \beta_i \to \beta \text{ in } \Lambda_{\mathbb{R}} \]

Since \( \lambda_i^2 \beta_i^2 \to \beta^2 \) we can assume \( \lambda_i = 1 \). But \( \Lambda_{K3} \) is discrete, so the sequence \( \beta_i \) must become constant: \textbf{contradiction}
In fact we have $F = Q$:

**Lemma**

The image of the subset \( \{ x \in \Lambda_{K3} : (x, x) = 0 \} \) lies dense in $Q$. Furthermore, the orbits of $O(\Lambda_{K3})$ lie dense in $Q$.

So if $V \subseteq \mathbb{P}(\Lambda_{\mathbb{R}})$ open and $V \cap Q \neq \emptyset$, then $V$ contains a limitpoint of $(B)$.

\[ \implies V \text{ contains a } \beta_i \in (B). \]
Lemma

The linebundle $L$ is generated by global sections.

Proof: Let $s$ be a non-trivial global section of $L$. The section $s$ defines a reduced and irreducible curve $Y \subseteq X$. We have:

$$ Bs|L| = \bigcap_{s' \in H^0(X,L)} \mathcal{Z}(s') \subseteq Y $$

The following sequence is exact:

$$ 0 \rightarrow \mathbb{C} \rightarrow H^0(X,L) \rightarrow H^0(Y,L|_Y) \rightarrow H^1(X,\mathcal{O}_X) = 0 \rightarrow .. $$

Lemma

Let $Y$ be a reduced irreducible curve of genus $g \geq 1$ on a compact complex surface $X$. Then the linebundle $\omega_Y = (\omega_X \otimes \mathcal{O}(Y))|_Y$ is globally generated.
Corollary

The linebundle $L$ is ample.

Proof: "$L$ is a ample linebundle $\iff \phi_L : X \to \mathbb{P}(H^0(X, L)^\vee)$ is a finite morphism"

Suppose $\phi_L$ isn’t a finite morphism.

$\implies \exists$ curve $C \subseteq X$ on which $\phi_L$ is constant.

$\implies (C, C) < 0.$

But $L$ generates $\text{Pic}(X)$ and $(L, L) = 4$: contradiciton.  

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Corollary

We have $h^1(X, L) = 0$.

Proof: Let $Y$ be a smooth curve on $X$. RR and SD gives us:

$$h^1(Y, \omega_Y) = h^0(Y, \omega_Y) - \deg(\omega_Y) - (1 - g)$$
$$= h^1(Y, \mathcal{O}_Y) - 2g + 2 - 1 + g$$
$$= 1$$

Consider the short exact sequence

$$0 \rightarrow \mathcal{O}_X \rightarrow L \rightarrow L|_Y \simeq \omega_Y \rightarrow 0.$$ This gives the exact sequence:

$$0 \rightarrow H^1(X, L) \rightarrow \mathbb{C} \rightarrow \mathbb{C} \rightarrow 0 = H^2(X, L) \simeq H^0(X, L)^\vee$$

Hence we have $H^1(X, L) \simeq 0$. 

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