Every two K3 surfaces are deformation equivalent

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reference: Le Potier; Géométrie des surfaces K3 modules et périodes, Astérisque No. 126, 1985; pages 79-89

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Every two K3 surfaces are deformation equivalent

Theorem

Every two K3 surfaces are deformation equivalent

Proof: Take a K3 surface X_0 with $\phi : H^2(X_0, \mathbb{Z}) \simeq \Lambda_{K3}$. Let U be an open connected neighbourhood of X_0 in the period domain.

Theorem

The subset of K3 surfaces of type g lies dense in $Im(\alpha) \subseteq \Omega$.

Definition

A K3 surface X is called of type g if the Picard group of X is generated by an element L with $(L, L) = 2g - 2$.

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Theorem

The subset of K3 surfaces of type g lies dense in Im$(\alpha) \subseteq \Omega$.

Theorem

Every K3 surface of type 3 is a smooth quartic in $\mathbb{P}^3$.

There exist a quartic surface X' in U.
Local Torelli theorem $\implies$ X_0 and X' are deformation equivalent.

Theorem

Every two smooth quartic surfaces are deformation equivalent.

The subset of K3 surfaces of type g lies dense in $\mathrm{Im}(\alpha)$

Theorem

The subset of K3 surfaces of type g lies dense in $Im(\alpha) \subseteq \Omega$

Proof: Let $U \neq \emptyset$ be a open subset of $\mathrm{Im}(\alpha) \subseteq \Omega$. Define for a subgroup $G \subseteq \Lambda_{K3}$ the following:

$$\Sigma(G) := \{x \in \mathbb{P}(\Lambda_{\mathbb{C}}) : (x,x) = 0 \text{ and } (x,g) = 0 \ \forall g \in G\}.$$

Proposition

Let $U \subseteq K_{20}$ be a non-empty open subset. Then there exists a primitive $\beta \in \Lambda_{K3}$ such that:

i) $(\beta, \beta) = 2g - 2$,

ii) $U \cap \{z \in \mathbb{P}(\Lambda_{\mathbb{C}}) : (\beta, z) = 0\} \neq \emptyset$

$$\implies \Sigma(\beta\mathbb{Z}) \cap U \neq \emptyset.$$

The subset of K3 surfaces of type g lies dense in $\mathrm{Im}(\alpha) \subseteq \Omega$

Define: $G_\beta = \{H \subseteq \Lambda_{K3} : \beta \in H \text{ and } H \neq \beta\mathbb{Z}\}$. We have:

- every $H \in G_\beta$ has rank ≥ 2
- G_β is countable

Then it follows:

$$(\Sigma(\beta\mathbb{Z}) \cap U) \not\subseteq \bigcup_{H \in G_\beta} \Sigma(H).$$

Now take an element $(z) \in (\Sigma(\beta\mathbb{Z}) \cap U) \setminus \bigcup_{H \in G_\beta} \Sigma(H)$. Then (z) is the period of some K3 surface X. We have:

$$\begin{aligned} \mathrm{Pic}(X) &\simeq H^{1,1}(X) \cap H^2(X, \mathbb{Z}) && \text{(Lefschetz)} \\ &\simeq \{\gamma \in \Lambda_{K3} : (\gamma, z) = 0\} \end{aligned}$$

The subset of K3 surfaces of type g lies dense in $\text{Im}(\alpha) \subseteq \Omega$

This gives us: $\beta \in \text{Pic}(X)$ and $(z) \in \Sigma(\text{Pic}(X))$.

$$\implies \text{Pic}(X) \notin G_\beta$$

Therefore $\text{Pic}(X) \simeq \beta\mathbb{Z}$, hence U contains a K3 surface X of type g. □

Every K3 surface of type 3 is quartic in $\mathbb{P}^3_{\mathbb{C}}$

Theorem

Every K3 surface of type 3 is isomorphic with a quartic in $\mathbb{P}^3_{\mathbb{C}}$.

Proof: Let X be a K3 surface of type 3, and L a generator of $\mathrm{Pic}(X)$ with $(L, L) = 4$. We can assume that $h^0(X, L) \neq 0$. We have:

$$SD: \quad h^2(X, L) \simeq h^0(X, L^\vee) = 0$$

$$RR: \quad \chi(L) = h^0(X, L) - h^1(X, L) = \frac{1}{2}(L, L) + 2 = 4$$

Lemma

The linebundle L is globally generated.

Every K3 surface of type 3 is quartic in $\mathbb{P}^3_{\mathbb{C}}$

Corollary

The linebundle L is ample, a generic curve $Y \in |L|$ is smooth, and $h^1(X, L) = 0$.

So the linebundle gives a finite morphism

$$X \xrightarrow{\phi_L} X' \to \mathbb{P}^3_{\mathbb{C}} = \mathbb{P}(H^0(X, L)^\vee) \text{ with } \phi_L^*(\mathcal{O}(1)) \simeq L.$$

We have:

$$\deg(X') \cdot \deg(\phi_L) = (L, L) = 4$$

$$\begin{array}{ccc} X & \xrightarrow{\phi_L} & X' \subseteq \mathbb{P}^3_{\mathbb{C}} \\ \cup & & \cup \\ Y & \xrightarrow{\phi_L|_Y = \phi_{\omega_Y}} & Y' \subseteq \mathbb{P}^2_{\mathbb{C}} \end{array}$$

Every K3 surface of type 3 is quartic in $\mathbb{P}^3_{\mathbb{C}}$

1. Y *is non-hyperelliptic*: $\deg(\phi_L) = 1$
   $\implies \deg(X') = 4$.

2. Y *is hyperelliptic*: $\deg(\phi_L) = 2$
   $\implies X'$ is a irreducible quadric surface in $\mathbb{P}^3$.

   (a) *X' is smooth*: $X' \simeq \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathrm{Pic}(X') \simeq \mathbb{Z} \times \mathbb{Z}$.
   But then $\mathcal{O}(1)|_{X'} \simeq \mathcal{O}(0,1) \otimes \mathcal{O}(1,0)$,
   $\implies \phi_L^*(\mathcal{O}(1)) \simeq L^{\otimes m}$: **contradiction**.

   (b) *X' is singular*: X' is a cone in s, $\mathrm{Pic}(X' \setminus \{s\}) \simeq L(\gamma)\mathbb{Z}$.
   But then $\mathcal{O}(1)|_{X' \setminus \{s\}} \simeq L(2\gamma)$
   $\implies \phi_L^*(\mathcal{O}(1)) \simeq L^{\otimes 2m}$: **contradiction**.

Hence, X is isomorphic to a quartic surface in $\mathbb{P}^3_{\mathbb{C}}$. □

Every 2 smooth quartics in $\mathbb{P}^3$ are deformation equivalent

Theorem

Every 2 smooth quartic surfaces in $\mathbb{P}^3$ are deformation equivalent.

Proof: A quartic surface $X \subseteq \mathbb{P}^3_{\mathbb{C}}$ is given by an equation:

$$a_0 X^4 + \ldots + a_3 W^4 + a_4 X^2 Y^2 + \ldots.. + a_{34} XYZW.$$

So we get a morphism:

$$\begin{array}{ccc} \mathbb{P}^{34}_{\mathbb{C}} & \supset & U \\ \downarrow & & \downarrow \\ \{\text{quartics}\} & \supset & \{\text{smooth quartics}\} \end{array}$$

U is an open and connected subset of $\mathbb{P}^{34}_{\mathbb{C}}$, hence all quartic surfaces are deformation equivalent. $\square$

Proposition

Let $U \subseteq K_{20}$ be a non-empty open subset. Then there exists a primitive $\beta \in \Lambda_{K3}$ such that:

i) $(\beta, \beta) = 2g - 2$,

ii) $U \cap \{z \in \mathbb{P}(\Lambda_{\mathbb{C}}) : (\beta, z) = 0\} \neq \emptyset$

Lemma

The linebundle L is generated by global sections.

Corollary

The linebundle L is ample.

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We have $h^1(X, L) = 0$.

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i) $(\beta, \beta) = 2g - 2$,

ii) $U \cap \{z \in \mathbb{P}(\Lambda_{\mathbb{C}}) : (\beta, z) = 0\} \neq \emptyset$

Proof: Define $\Sigma := \{((\alpha), (z) \in \mathbb{P}(\Lambda_{\mathbb{C}}) \times K_{20} : (\alpha, z) = 0\}$.

$$\begin{array}{ccccc} & & \Sigma & \xrightarrow{p_2} & K_{20} \supseteq U \\ & & \downarrow{\scriptstyle p_1} & & \\ \mathbb{P}(\Lambda_{\mathbb{R}}) & \xrightarrow{i} & \mathbb{P}(\Lambda_{\mathbb{C}}) & & \end{array}$$

$V := p_1(p_2^{-1}(U))$ is open. One can show: $i^{-1}(V) \cap Q \neq \emptyset$.

Lemma

Let $Q = \{x \in \mathbb{P}(\Lambda_{\mathbb{R}}) : (x, x) = 0\}$ and V an open subset of $\mathbb{P}(\Lambda_{\mathbb{R}})$ with $V \cap Q \neq \emptyset$. Then V contains an element (β) with $\beta \in \Lambda_{K3}$, such that β is primitive and $(\beta, \beta) = 2g - 2$.

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$\exists$ a primitive $\beta \in \mathbb{P}(\Lambda_{K3})$ with $(\beta, \beta) = 2g - 2$. We also have: $i(\beta) \in V \implies \exists u \in U$ with $(\beta, u) = 0$. $\square$

Lemma

Let $Q = \{x \in \mathbb{P}(\Lambda_{\mathbb{R}}) : (x,x) = 0\}$ and V an open subset of $\mathbb{P}(\Lambda_{\mathbb{R}})$ with $V \cap Q \neq \emptyset$. Then V contains an element (β) with $\beta \in \Lambda_{K3}$, such that β is primitive and $(\beta,\beta) = 2g-2$.

Proof:

Let $B = \{\beta \in \Lambda_{K3} : (\beta,\beta) = 2g-2 \text{ and } \beta \text{ is primitive}\}$,
(B) the image of B in $\mathbb{P}(\Lambda_{\mathbb{R}})$, and
$F = \{\text{limit points of } (B)\} = \overline{(B)} \setminus (B)^0$.

Claim: $F \subseteq Q$.

Suppose $(\beta_i) \to \beta$ in $\mathbb{P}(\Lambda_{\mathbb{R}})$ with $\beta_i \in \Lambda_{K3}$, $\beta_i^2 = 2g-2$, and $\beta^2 \neq 0$. We can choose β such that $\beta^2 = \beta_i^2$.

$$(\beta_i) \to \beta \iff \lambda_i\beta_i \to \beta \text{ in } \Lambda_{\mathbb{R}}$$

Since $\lambda_i^2\beta_i^2 \to \beta^2$ we can assume $\lambda_i = 1$. But Λ_{K3} is discrete, so the sequence β_i must become constant: **contradiction**

In fact we have $F = Q$:

Lemma

The image of the subset $\{x \in \Lambda_{K3} : (x, x) = 0\}$ *lies dense in* Q*. Furthermore, the orbits of* $O(\Lambda_{K3})$ *lie dense in* Q*.*

So if $V \subseteq \mathbb{P}(\Lambda_{\mathbb{R}})$ open and $V \cap Q \neq \emptyset$, then V contains a limitpoint of (B).

$\implies V$ contains a $\beta_i \in (B)$. □

Lemma

The linebundle L is generated by global sections.

Proof: Let s be a non-trivial global section of L. The section s defines a reduced and irreducible curve $Y \subseteq X$. We have:

$$\mathrm{Bs}|L| = \bigcap_{s' \in H^0(X,L)} \mathcal{Z}(s') \subseteq Y$$

The following sequence is exact:

$$0 \to \mathbb{C} \to H^0(X, L) \to H^0(Y, L|_Y) \to H^1(X, \mathcal{O}_X) = 0 \to ..$$

Lemma

Let Y be a reduced irreducible curve of genus $g \geq 1$ on a compact complex surface X. Then the linebundle $\omega_Y = (\omega_X \otimes \mathcal{O}(Y))|_Y$ is globally generated.

□

Corollary

The linebundle L is ample.

Proof: "L is a ample linebundle $\iff \phi_L : X \longrightarrow \mathbb{P}(H^0(X, L)^\vee)$ is a finite morphism"

Suppose ϕ_L isn't a finite morphism.
$\implies \exists$ curve $C \subseteq X$ on which ϕ_L is constant.
$\implies (C, C) < 0$.

But L generates $\mathrm{Pic}(X)$ and $(L, L) = 4$: **contradiciton**. □

Corollary

We have $h^1(X, L) = 0$.

Proof: Let Y be a smooth curve on X. RR and SD gives us:

$$\begin{aligned} h^1(Y, \omega_Y) &= h^0(Y, \omega_Y) - \deg(\omega_Y) - (1 - g) \\ &= h^1(Y, \mathcal{O}_Y) - 2g + 2 - 1 + g \\ &= 1 \end{aligned}$$

Consider the short exact sequence
$0 \to \mathcal{O}_X \to L \to L|_Y \simeq \omega_Y \to 0$. This gives the exact sequence:

$$0 \to H^1(X, L) \to \mathbb{C} \to \mathbb{C} \to 0 = H^2(X, L) \simeq H^0(X, L)^\vee$$

Hence we have $H^1(X, L) \simeq 0$. □