GLOBAL TORELLI FOR COMPLEX K3 SURFACES

Abstract. Overview of statement and proof of Global Torelli, following Verbitsky. With an emphasis on a strong form of the theorem, giving a complete description of the category of complex K3s (and iso's).

1. Motivation: classification of complex tori

2. Kähler classes and Weyl chambers

Theorem 1. *Let X be a complex K3. Let ω be a Kähler form on X. Then $[\omega] \in \mathrm{H}^2(X, \mathbf{R})$ satisfies*

1. $[\omega] \in \mathrm{H}^{1,1}(X)$
2. $[\omega]^2 > 0$
3. $[\omega] \cdot \delta \neq 0$ *for every* $\delta \in \mathrm{Pic}(X)$ *with* $\delta^2 = -2$.

Proof. First two we have seen already. For the last, note that for every curve C on X we have

$$[\omega] \cdot [C] = \int_C \omega > 0,$$

since ω is a volume form on C. Now if D is a divisor with $D^2 = -2$, then by RR+SD, we have

$$h^0(X, D) + h^0(X, -D) = h^1(X, D) + 2 + \frac{D^2}{2} \geq 1$$

so that either D or $-D$ is equivalent to an effective divisor, hence $[\omega] \cdot [D] \neq 0$. □

In other words, if $\Lambda := \mathrm{H}^2(X, \mathbf{Z})$, then the class $[\omega]$ lands in

$$C(\Lambda) := \{x \in \Lambda_{\mathbf{R}} \cap \Lambda^{1,1} \mid x^2 > 0, \text{ and } x \cdot \delta \neq 0 \text{ for all } \delta \in \Delta(\Lambda)\},$$

where

$$\Delta(\Lambda) := \{x \in \Lambda \cap \Lambda^{1,1} \mid x^2 = -2\}.$$

The elements of Δ are called *roots*. This set can be empty, finite or infinite. The connected components of $C(\Lambda)$ are called *chambers*, separated by the walls $x^{\perp}$ with $x \in \Delta(\Lambda)$.

Proposition 1. *Let X be a complex K3. If ω_0 and ω_1 are Kähler forms, then $[\omega_0]$ and $[\omega_1]$ land in the same chamber in $C(\mathrm{H}^2(X, \mathbf{Z}))$.*

Proof. $\lambda\omega_0 + \mu\omega_1$ with $\lambda, \mu > 0$ is also a Kähler form. □

Hence X determines a well-defined chamber $c(X)$ in $C(\mathrm{H}^2(X, \mathbf{Z}))$.

3. Classification of complex K3 surfaces: statement of the theorem

Theorem 2. *The functor* $X \mapsto (\mathrm{H}^2(X,\mathbf{Z}), c(X))$ *from complex K3s (and iso's) to the category of pairs* (Λ, c) *with*

- Λ *a weight* 2 *even unimodular Hodge lattice of rank* 22 *and signature* $(3,19)$
- c *a connected component of* $C(\Lambda)$

is an anti-equivalence of categories.

Goal for coming weeks is: to prove this theorem. No known proof on K3 by K3 basis. Must consider all K3s at same time, and use geometry of moduli spaces.

The proof is quite longe and intricate, combining different kinds of mathematics. The goal for today: give a global overview, sketch the argument, and divide the work.

4. The period map

4.1. **The period map.** Let $\mathcal{N}$ be the set of isoclasses of pairs (X, ϕ) with

(1) X a complex K3 surface
(2) $\phi\colon \mathrm{H}^2(X,\mathbf{Z}) \to \Lambda_{K3}$ an isometry

The set $\mathcal{N}$ has a natural structure of 20-dimensional complex manifold as follows. Let $(X,\phi) \in \mathcal{N}$. Choose a universal deformation $\mathcal{X} \to S$ (for some open polydisk S of dimension 20, see Lance), so that $\mathcal{X}_0 = X$. Since S is contractible, we have natural isomorphisms $\mathrm{H}^2(\mathcal{X}_s,\mathbf{Z}) \xrightarrow{\sim} \mathrm{H}^2(\mathcal{X}_0,\mathbf{Z})$ for all $s \in S$. Hence we get an injective map $S \hookrightarrow \mathcal{N}$. We define the topology and complex structure on $\mathcal{N}$ by glueing these S'es around varying (X,ϕ). To see that these are compatible (when an S and S' overlap), we use (Lance) that $\mathcal{X} \to S$ is also a universal deformation for its other fibers X_s.

Let $\mathcal{D} \subset \mathbf{P}(\Lambda_{K3,\mathbf{C}})$ be the period domain:

$$\mathcal{D} = \{x \in \mathbf{P}(\Lambda_{K3,\mathbf{C}}) \mid x^2 = 0 \text{ and } x\bar{x} > 0\}.$$

This is an open subset of the nonsingular projective algebraic variety $\{x^2 = 0\}$, hence $\mathcal{D}$ is a complex manifold (of dimension 20). The set $\mathcal{D}$ is in bijection with the set of weight 2 Hodge structures on Λ_{K3} of type $(1,20,1)$, compatible with inner product (via $x = \mathrm{H}^{2,0}$).

So we obtain a map

$$\mathcal{P}\colon \mathcal{N} \to \mathcal{D}$$

which maps (X,ϕ) to the x corresponding to the Hodge structure on Λ_{K3} induced by ϕ. We have already seen one fundamental theorem regarding $\mathcal{P}$ (Lance):

Theorem 3 (Local Torelli)**.** *The map* $\mathcal{P}\colon \mathcal{N} \to \mathcal{D}$ *is a local isomorphism.* □

Warning. The set $\mathcal{N}$ is not Hausdorff (as we will see next). This is a global property, something we cannot see using local Torelli. In particular, the map $\mathcal{P}$ cannot be an isomorphism. (But we'll see, it is not too far from being an isomorphism).

5. Fundamental ingredients in the proof of the classification

Let $\mathcal{N} \twoheadrightarrow \bar{\mathcal{N}}$ be the Hausdorffification of $\mathcal{N}$, the universal continuous map to a Hausdorff topological space. This can be obtained by taking the quotient for the equivalence relation generated by the following relation: $x \sim x'$ if there exist (x_i) converging to both x and x'.

Since $\mathcal{D}$ is Hausdorff, the map $\mathcal{P}$ factors over a map $\bar{\mathcal{P}}\colon \bar{\mathcal{N}} \to \mathcal{D}$.

Theorem 4. *The map $\bar{\mathcal{P}}\colon \bar{\mathcal{N}} \to \mathcal{D}$ is a covering map.*

Sketch of proof. Local Torelli and playing with twistor lines (using the Hyperkähler structure) plus a bit of topology (criterion for a local homeomorphism between Hausdorff spaces to be a covering map). See Huybrechts, sections 7.3 and 7.4. □

Theorem 5. *The space $\mathcal{D}$ is simply connected.*

Sketch of proof. First show

$$\mathcal{D} \cong \mathrm{O}(3,19)/(\mathrm{SO}(2) \times \mathrm{O}(1,19))$$

then deduce $\pi_1(\mathcal{D}, *)$ is trivial. See also Huybrechts, chapter 6. □

So we see that $\bar{\mathcal{N}}$ is actually a disjoint union of copies of $\mathcal{D}$. Next we will show there are exactly two copies.

Theorem 6. *$\mathcal{N}$ and $\bar{\mathcal{N}}$ each consist of two connected components, interchanged by $(X, \varphi) \mapsto (X, -\varphi)$.*

Sketch of proof. Deformation equivalence (Juultje) plus 'monodromy is big'. The latter is shown by explicitly constructing elements of the monodromy group. See Huybrechts 7.5.5. □

Theorem 7 ('Global Torelli')**.** *Let X and Y be K3 surfaces. Let $\varphi\colon \mathrm{H}^2(Y, \mathbf{Z}) \to \mathrm{H}^2(X, \mathbf{Z})$ be a Hodge isometry with $\varphi(\mathcal{K}_Y) \cap \mathcal{K}_X \neq \emptyset$, then there is an isomorphism $f\colon X \to Y$ with $\varphi = f^*$.*

Sketch of proof. Let $\mathcal{N}_0$, $\bar{\mathcal{N}}_0$ and $\mathcal{D}_0$ be the open subsets corresponding to 'Picard rank 0'. Then first show that $\mathcal{N}_0 \to \bar{\mathcal{N}}_0$ is an isomorphism (Huybrechts 7.2.2), and hence that $\mathcal{N}_0$ consists of two copies of $\mathcal{D}_0$ interchanged by $(X, \varphi) \mapsto (X, -\varphi)$. From this the theorem follows in Picard rank 0. Now in general, use a degeneration argument, to obtain a correspondence between X and Y, and analyse the geometry to show that the correspondence must be an isomorphism (Huybrechts, §7.5). □

Theorem 8. *If X, Y are complex K3 surfaces, and $f, g\colon X \to Y$ are isomorphisms with $f^* = g^*$ as maps $\mathrm{H}^2(Y, \mathbf{Z}) \to \mathrm{H}^2(X, \mathbf{Z})$, then $f = g$.*

Sketch of proof. Reduce to $X = Y$, need to show $f^* = \mathrm{id}$ implies $f = \mathrm{id}$. First: show auromorphism preserves Kähler form (and not just its class in H^2), so that f preserves the Riemannian metric. But the automorphism group of a compact Riemannian manifold is compact, so that $\mathrm{Aut}(X)$ is compact. But by $\mathrm{H}^0(X, T_X) = 0$ we also have that $\mathrm{Aut}(X)$ is discrete. Hence $\mathrm{Aut}(X)$ is finite and f of finite order. Now show that f is transversal, if $f \neq \mathrm{id}$, and use both topological and holomorphic Lefschetz to compute the number of fixed points, obtaining a contradiction. All this is in Huybrechts, chapter 15 'Automorphisms'. □

Theorem 9. *The Kähler cone of a K3 surface is a full chamber in $C(\mathrm{H}^2(X, \mathbf{R}))$.*

Sketch of proof. See Huybrechts, Chapter 8. Note that the argument *uses* the (proof of) Theorem 7. □

Theorem 10. *For a Hodge lattice Λ of K3 type, the group $\mathrm{O}(\Lambda)$ acts transitively on the set of chambers in $C(\Lambda)$*

Proof. This is essentially a statement about hyperbolic Coxeter groups. See Huybrechts, Chapter 8 □

6. Proof of the classification

Proof of the classification. To show that $X \mapsto (H^2(X, \mathbf{Z}), \mathcal{K}_X)$ is an equivalence of categories, we show that it is full, faithful, and essentially surjective.

Essential surjectivity. By Theorem 4 the map $\mathcal{N} \to \mathcal{P}$ is surjective. Together with the transitivity of Theorem 10, this shows that every pair (Λ, C) is realized.

Faithfulness. This is Theorem 8.

Fullness. Let X and Y be K3 surfaces. Let $\psi \colon \mathrm{H}^2(Y, \mathbf{Z}) \to \mathrm{H}^2(X, \mathbf{Z})$ be a Hodge isometry mapping the Kähler chamber of Y to the Kähler chamber of X. Then by Theorem 9 we have $\psi(\mathcal{K}_Y) = \psi(\mathcal{K}_X)$ and by Theorem 7 there is an isomorphism $f \colon X \to Y$ with $\varphi = f^*$. □