Kähler (& hyper-Kähler) manifolds

Arithmetic & Algebraic Geometry Seminar, KdVI

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October 23, 2015

Contents

Introduction

These notes are based on two talks given at the Arithmetic & Algebraic Geometry Seminar of the Korteweg-de Vriesinstituut for mathematics of the Universiteit van Amsterdam. They are intended to give a short introduction to the theory of Kähler manifolds, with a slight focus of applicability to the subject of K3 surfaces. However, they also include other interesting results not related to K3 surfaces, most noteably at the end.

Most of these notes (all except the section on Hyper-Kähler manifolds) are based on a course on complex manifolds and an essay written by myself, respectively lectured and supervised by Julius Ross, DPMMS, Cambridge.

I will use the summation convention for repeated indices, where convenient.

1 — Basic definitions

Recall that a complex manifold has an underlying real (smooth) manifold, with an almost complex structure. Any almost complex structure which arises in this way is said to be integrable. By the Newlander-Nirenberg theorem, an almost complex structure is integrable if and only if its Nijenhuis tensor vanishes.

The complexified tangent bundle splits as $T_{\mathbb{C}}X := TX \otimes_{\mathbb{R}} \mathbb{C} = TX^{(1,0)} \oplus TX^{(0,1)}$, its holomorphic and antiholomorphic parts, and similarly $T^*_{\mathbb{C}}X = T^*X \otimes_{\mathbb{R}} \mathbb{C} = (T^*X)^{(1,0)} \oplus (T^*X)^{(0,1)}$.

We write $\underline{\mathbb{C}}$ for the trivial holomorphic line bundle, with sheaf of holomorphic sections $\mathscr{O}_X$ on X, $\underline{\mathbb{R}}$ for the trivial real line bundle, with sheaf of smooth sections O_X on X, and

$$\begin{aligned} T^*X^{(p,q)} &:= \bigwedge^p (T^*X)^{(1,0)} \otimes_{\underline{\mathbb{R}}} \bigwedge^q (T^*X)^{(0,1)} \\ \mathscr{A}^k(U) &:= \Gamma\Big(U, \bigwedge^k T^*_{\mathbb{C}}X\Big) \\ \mathscr{A}^{(p,q)}(U) &:= \Gamma\Big(U, T^*X^{(p,q)}\Big) \\ \Omega^p(U) &:= \{\alpha \in \mathscr{A}^p(U) \mid \bar{\partial}\alpha = 0\} \end{aligned}$$

Definition 1.1. Let X be a complex manifold with induced almost complex structure I.

- A Riemannian metric g on X is *Hermitian* with respect to I if $g_x(I_x v, I_x w) = g_x(v, w)$ for all $x \in X$ and all $v, w \in T_x X$;
- In this case, define the *fundamental form* $\omega \in \mathscr{A}^2(X)$ by $\omega(v, w) = g(Iv, w)$;
- We extend g to $T_{\mathbb{C}}X$ by $g(\lambda v, \mu w) = \lambda\bar{\mu}\, g(v, w)$. Then $\omega \in \mathscr{A}^{1,1}(X)$.

Remark. Any two of a triple (g, I, ω) determine the third.

Definition 1.2. A Hermitian metric is *Kähler* if $d\omega = 0$. We then call ω the *Kähler form*, or sometimes just the *Kähler metric*.

The *Kähler class* of a Kähler metric g is the class $[\omega] \in H^{1,1}(X)$.

Remark. Any Kähler form is a symplectic form on the underlying real manifold, by definition. This implies e.g. that on a compact Kähler manifold X, $[\omega^p] \neq 0 \in H^{p,p}(X)$ for $0 \le p \le \dim_{\mathbb{C}} X$.

Example 1.3. Complex space $\mathbb{C}^n$ with standard orthogonal coordinaties $\{x^j, y^j\}_{j=1}^n$ (where $z^j = x^j + iy^j$) has fundamental form $\omega = \frac{i}{2}\sum_j dz^j d\bar{z}^j$, which is clearly Kähler.

Example 1.4. If $\dim_{\mathbb{R}} X = 2$, any two-form is trivially closed, so any Hermitian metric is Kähler. Hence, all Riemann surfaces are Kähler.

Lemma 1.5. *If $f : X \hookrightarrow Y$ is a holomorphic embedding, and Y has a Kähler metric g with Kähler form ω, then $f^*\omega$ is a Kähler form on X.*

Proof. As Df is injective, f^*g is non-degenerate, so it is a metric. Furthermore, $df^*\omega = f^*d\omega = 0$, so ω is a Kähler form. □

Definition 1.6. The set of Kähler classes in $H^{1,1}(X)$ is called the *Kähler cone* of X.

This is indeed a cone, as a positive linear combination of metrics is a metric and both $\ker d$ and $\operatorname{im} d$ are linear subspaces of $\mathscr{A}^{1,1}(X)$.

Remark. The Kähler cone will not be used in these notes, but it is important in the proof of the Global Torelli theorem.

2 — Metrics and connections on vector bundles

Just as in real geometry, metrics and all related notions can be defined on arbitrary vector bundles, not just on the tangent bundle. We will give these definitions, and develop the notions parallel to each other.

For a vector bundle $E \to X$, its sheaf of smooth sections is denoted $O(E)$. If E is holomorphic, its sheaf of holomorphic sections is denoted $\mathscr{O}(E)$.

Definition 2.1. Let $\pi : E \to X$ be a holomorphic vector bundle.

- A *Hermitian metric* h on E is a smooth family of Hermitian inner products on E_x, i.e. it is a (global) smooth section of $E^* \otimes_{\underline{\mathbb{R}}} \bar{E}^*$ such that for local smooth sections s, s' of E, $h(s, s') = \overline{h(s', s)}$;

- Writing $\mathscr{A}^{p,q}(E) = \mathscr{A}^{p,q} \otimes_{\mathscr{O}_X} \mathscr{O}(E)$, we can define an operator $\bar{\partial}_E$ by defining for holomorphic sections s of E and (p,q)-forms α,

$$\bar{\partial}_E : \mathscr{A}^{p,q}(E) \to \mathscr{A}^{p,q+1}(E) : \alpha \otimes s \mapsto (\bar{\partial}\alpha) \otimes s \tag{1}$$

Remark. A Hermitian metric on X induces one on the holomorphic vector bundle $TX^{(1,0)}$ via sesquilinear extension to $T_{\mathbb{C}}X$, and this procedure is invertible.

Definition 2.2. Given a Hermitian vector bundle $(E,h) \to X$, its *Chern connection* is the unique holomorphic connection ∇_h on E such that $\nabla_h h = 0$, i.e.

$$\nabla_h = \nabla'_h + \bar{\partial}_E, \qquad \nabla'_h : \mathscr{A}^0(E) \to \mathscr{A}^{1,0}(E)$$
$$dh(v,w) = h(\nabla_h v, w) + h(v, \nabla_h w)$$

The proof of the unique existence will be given later.

Remark. A metric g on X is Kähler if and only if its Levi-Civita connection equals its Chern connection.

Definition 2.3. For a Hermitian vector bundle $(E,h) \to X$, define its *curvature* to be $\nabla_h^2 : \mathscr{A}^0(E) \to \mathscr{A}^2(E)$.
If $E = T^{(1,0)}X$, we call this the *Riemannian curvature* and denote it by R_h.

Lemma 2.4. *Let $(E,h) \to X$ be a Hermitian vector bundle. Then*

(1) The Chern connection exists uniquely;
(2) The curvature is a skew-Hermitian element of $\mathscr{A}^{1,1}(\mathrm{End}(E))(X)$.

Proof. (1) Taking a local holomorphic frame $\{e^j\}$ of E and using that the Chern connection is holomorphic, we have $\nabla e_j = \Theta_j^k e_k$ for some matrix of $(1,0)$-forms Θ, so locally $\nabla = d + \Theta$. Writing $h_{jk} = h(e_j, e_k)$, we get

$$dh_{jk} = dh(e_j, e_k) = h(\Theta_j^l e_l, e_k) + h(e_j, \Theta_k^l e_l) = \Theta_j^l h_{lk} + \bar{\Theta}_k^l h_{jl} \tag{2}$$

and hence $\partial h_{jk} = \Theta_j^l h_{lk}$ and $\bar{\partial} h_{jk} = \bar{\Theta}_k l h_{jl}$. So we need $\Theta = \partial h \cdot h^{-1}$, proving uniqueness.
Taking this as a local defintion, it patches, proving existence.

(2) We can assume that at a point $h_{jk} = \delta_{jk}$ and $dh = 0$, so by (2), $\Theta^* = -\Theta$.
We have

$$\begin{aligned}\nabla^2 s &= (d+\Theta)^2 s = d^2 s + d(\Theta s) + \Theta ds + \Theta \wedge \Theta s \\ &= (d\Theta)s - \Theta ds + \Theta ds + \Theta \wedge \Theta s = (d\Theta + \Theta \wedge \Theta)s\end{aligned}$$

So locally $\nabla^2 = d\Theta + \Theta \wedge \Theta$, showing that

$$(\nabla^2)^* = (d\Theta + \Theta \wedge \Theta)^* = d\Theta^* - \Theta^* \wedge \Theta^* = -d\Theta - \Theta \wedge \Theta = -\nabla^2$$

Now, because $\nabla = \nabla' + \bar{\partial}_E$, the $(0,2)$-component of ∇^2 is zero. By skew-hermitianness, so is its $(2,0)$-component.

□

Definition 2.5. The *Ricci form* of a Hermitian metric g on X is $\mathrm{Ric}(\omega) := \frac{i}{2\pi} \operatorname{Tr} R_g$.
The *curvature form* of a Hermitian line bundle $(L,h) \to X$ is $F(h) := \frac{i}{2\pi}\nabla_h^2$.

Remark. The normalisation in the above definition is not completely standard, but it will have the advantage to be integral in cohomology, as will be shown in the next few lemma's.

Lemma 2.6. *The following equations hold locally*

$$\mathrm{Ric}(\omega) = \frac{1}{2\pi i}\partial\bar{\partial} \log \det g \tag{3}$$
$$F(h) = \frac{1}{2\pi i}\partial\bar{\partial} \log h_s \tag{4}$$

where $h_s = h(s,s)$, for a local holomorphic section s of a line bundle $(L,h) \to X$.
Hence, $\mathrm{Ric}(\omega) = -F(\det g^{-1})$, where $\det g^{-1}$ is the natural induced metric on the canonical line bundle.

Proof. Using the proof of lemma 2.4 with holomorphic frame s, we get locally

$$\begin{aligned} F(h) &= \frac{i}{2\pi}\nabla_h^2 = \frac{i}{2\pi}(d\Theta + \Theta \wedge \Theta) \\ &= \frac{i}{2\pi}(d(\partial h_s \cdot h_s^{-1}) + (\partial h_s \cdot h_s^{-1}) \wedge (\partial h_s \cdot h_s^{-1})) \\ &= \frac{i}{2\pi}(d\partial \log h_s + 0) = \frac{i}{2\pi}\bar{\partial}\partial \log h_s \\ &= \frac{1}{2\pi i}\partial\bar{\partial}\log h_s \end{aligned}$$

The proof for the Ricci form is similar, using $\operatorname{Tr}\log = \log\det$. □

2.1 — Curvature as Chern classes

Lemma 2.7. *(1) For a Hermitian line bundle $(L, h) \to X$, $F(h)$ is closed and its cohomology class only depends on L;*
(2) For a Kähler manifold (X, g), $\mathrm{Ric}(\omega)$ is closed and its cohomology class only depends on X

Proof. Closedness is clear from lemma 2.6.

Suppose h' is a different Hermitian metric on L. Then $\frac{h'_s}{h_s}$ is a non-vanishing function, so it can be represented by e^f, where f is globally well-defined up to a constant. Hence,

$$F(h') - F(h) = \frac{1}{2\pi i}\partial\bar{\partial} f$$

which is an exact form.

The statement about the Ricci curvature follows from the last part of lemma 2.6. □

Lemma 2.8. *Curvature forms represent Chern classes, i.e. $c_1(X) = [\mathrm{Ric}(\omega)]$ and $c_1(L) = [F(h)]$ for any Kähler class ω on X or Hermitian metric h on a line bundle L.*

Proof. Again, we only need to prove the line bundle case. As the definition of the Chern class involves the isomorphism $\mathrm{Pic}(X) \cong \check{H}^1(X, \mathscr{O}^*)$ and the connecting homomorphism in Čech cohomology of the fundamental short exact sequence

$$0 \longrightarrow \underline{\mathbb{Z}}_X \xrightarrow{2\pi i} \mathscr{O}_X \xrightarrow{\exp} \mathscr{O}_X^* \longrightarrow 0$$

while on the other hand, the curvature forms define de Rham classes, we need to chase the connecting homomorphism, using the de Rham theorem to move from Čech to de Rham cohomology at some point.

Take a cover $\{U_i\}$ of X over which L can be trivialised by holomorphic sections s_i, with holomorphic transition functions $\psi_{ij} = s_i s_j^{-1}$. The first step of the connecting homomorphism is to lift this along the exponential map, obtaining $\log \psi_{ij}$.

On $U_i \cap U_j$ we get

$$\log h_{s_i} - \log h_{s_j} = \log \frac{h(\psi_{ij}s_j, \psi_{ij}s_j)}{h(s_j, s_j)} = \log \psi_{ij} + \log \bar{\psi}_{ij}$$

so we get that $\delta(\log h_{s_i})_i = (\log \psi_{ij} + \log \bar{\psi}_{ij})_{ij}$ as Čech cycles. Hence, the de Rham isomorphism sends $[\log \psi_{ij} + \log \bar{\psi}_{ij}]$ to $[d \log h_s]$. Taking the holomorphic part, $[\log \psi_{ij}]$ corresponds to $[\bar{\partial} \log h_s]$ (as holomorphic forms are those on which $\bar{\partial}$ acts as zero).

The second part of the connecting homomorphism (now in de Rham cohomology) consists of taking the differential, getting $d\bar{\partial}\log h_s = \partial\bar{\partial}\log h_s$. Finally, this lifts along $2\pi i$, so we get the class $\left[\frac{1}{2\pi i}\partial\bar{\partial}\log h_s\right] = [F(h)]$.

This is summarised in the following diagram:

$$\begin{array}{ccccccccccc} \mathrm{Pic}\, X & \xrightarrow{\sim} & \check{H}^1(X, \mathscr{O}^*) & \xleftarrow{\exp} & \check{C}^1(X, \mathscr{O}) & \to & \mathscr{A}^1(X) & \xrightarrow{d} & \mathscr{A}^2(X) & \xleftarrow{2\pi i} & H^2(X, \mathbb{Z}) \\ L & \longmapsto & [\psi_{ij}]_{ij} & \longmapsto & \log \psi_{ij} & \mapsto & \bar{\partial}\log h_s & \mapsto & \partial\bar{\partial}\log h_s & \mapsto & \left[\frac{1}{2\pi i}\partial\bar{\partial}\log h_s\right] = [F(h)] \end{array}$$

□

3 — Positive line bundles

Example 3.1 (The Fubini-Study metric)**.** Consider projective space, $\mathbb{CP}^n$ with line bundle $\mathcal{O}(-1)$, which can be viewed as an extension of the standard projection $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ to the blow-up of $\mathbb{C}^{n+1}$ at the origin. Endowing this line bundle with the Euclidean metric $\|\cdot\|$ on the total space, which is clearly Hermitian, we get a curvature form. We define the *Fubini-Study metric* to be the negative of this curvature.

Over the open $U = \{(1 : z_1 : \ldots : z_n)\}$ we get a section $s : (1 : z_1 : \ldots : z_n) \mapsto (1, z_1, \ldots, z_n)$, so we can calculate the metric:

$$\begin{aligned}\omega_{\mathrm{FS}} &= -F(\|\cdot\|) = \frac{i}{2\pi}\partial\bar\partial \log \|s\|^2 \\ &= \frac{i}{2\pi}\partial\bar\partial \log(1 + z_j \bar z^j) = \frac{i}{2\pi}\frac{1}{(1+z_j\bar z^j)^2}\Big((1 + z_k \bar z^k) dz_l d\bar z^l - \bar z^k z_l dz_k d\bar z^l\Big)\end{aligned}$$

Hence $\omega_{\mathrm{FS}}(1 : 0 : \ldots : 0) = \frac{i}{2\pi} dz_j d\bar z^j$, which is non-degenerate. The Fubini-Study metric is invariant under a transitive group action on $\mathbb{CP}^n$, so it is everywhere non-degenerate, hence is an actual metric.

Corollary 3.2. *(1) Every projective manifold has a Kähler structure;*
(2) If a line bundle $L \to X$ is very ample, it defines a Kähler form ω by the previous point, and $[\omega] = c_1(L)$.

Proof. (1) This follows directly from lemma 1.5.
(2) Because part of the definition of a very ample line bundle is that $L = f^*\mathcal{O}(1)$, we have

$$[\omega] = [f^*\omega_{\mathrm{FS}}] = f^*[-F_{\|\cdot\|}] = -f^* c_1(\mathcal{O}(-1)) = c_1(f^*\mathcal{O}(1)) = c_1(L)$$

□

Definition 3.3. A $(1,1)$-class in cohomology is called *positive* if it is a Kähler class.

A holomorphic line bundle $L \to X$ is *positive* if there exists a Hermitian metric h on L such that $[F(h)]$ is positive.

This is a very useful notion, as the following two theorems show.

Theorem 3.4 (Kodaira vanishing)**.** *Let $L \to X$ be a positive line bundle over a compact Kähler manifold X. Then $H^q(X, \Omega^p \otimes L) = 0$ if $p + q > \dim_{\mathbb{C}} X$.*

Theorem 3.5 (Kodaira embedding)**.** *Let $L \to X$ be a line bundle over a compact complex manifold. Then L is ample if and only if it is positive.*

These proofs of these theorems are very involved, so we do not give them here.

4 — Hyper-Kähler manifolds

Definition 4.1. A *hyper-Kähler manifold* is a tuple (X, g, I, J) such that both (X, g, I) and (X, g, J) are Kähler and $IJ + JI = 0$.

A hyper-Kähler manifold has an action of the quaternions on each of its tangent spaces by defining $K = IJ$ and then setting $(a + bi + cj + dk)v = av + bIv + cJv + dKv$. If $a = 0$ and $b^2 + c^2 + d^2 = 1$, the action of the quaternion again gives a Kähler structure for (X, g), so we get an $S^2 \cong \mathbb{CP}^1$ of Kähler structures.

Definition 4.2. Given a hyper-Kähler manifold (X, g, I, J) define its *twistor space* to be the almost complex manifold with underlying Riemannian manifold $(X, g) \times (\mathbb{CP}^1, g_{\mathrm{FS}})$ and almost complex structure at the point $(x, (a, b, c)) \in X \times S^2 \subset X \times \mathbb{R}^3$ given by $(aI + bJ + cK, I_{\mathbb{CP}^1})$. This is integrable by the Newlander-Nirenberg theorem, and hence a Kähler manifold.

Remark. A hyper-Kähler manifold can be recovered from its twistor space, and twistor spaces can be characterised without referring to their associated hyper-Kähler manifolds, see e.g. [Saf11].

Proposition 4.3. *Let (X, g, I) be a Kähler manifold of real dimension 4 with trivial canonical bundle. Then X admits a hyper-Kähler structure.*

Proof. [BHPvdV] Choose a flat section τ of K_X and define a $J_1 \in \mathrm{End}(T_pX)$ by $g(J_1X,Y) = \mathrm{Re}\,\tau(X,Y)$. Then

$$\begin{aligned} g(IJ_1X,Y) &= -g(J_1X,IY) = -\mathrm{Re}\,\tau(X,IY) = -\mathrm{Re}\,\tau(IX,Y) \qquad \text{as } \tau \text{ is of type } (0,2) \\ &= -g(J_1IX,Y) \end{aligned}$$

So $IJ_1 = -J_1I$. Also, by antisymmetry of τ, J_1 is skew-adjoint, so J_1^2 is self-adjoint with non-positive eigenvalues. As $J_1 \neq 0$, J_1^2 has an eigenvalue $-\lambda^{-2}$ with eigenspace V. Define $J = \lambda J_1$, so $J^2|_V = -\mathrm{id}_V$.

Then $[I,J^2] = 0$, so I preserves V, and I,J,IJ turn into a V a quaternionic vector space. Therefore, its real dimension is at least 4, meaning that $V = T_pX$.

Defining J_1 and J globally this way, we get

$$\begin{aligned} 0 &= d\Big(g(J_1X,Y) - \mathrm{Re}\,d\tau(X,Y)\Big) \\ &= g(\nabla(J_1X),Y) + g(J_1X,\nabla Y) - \mathrm{Re}\,\Big(\tau(\nabla X,Y) + \tau(X,\nabla Y)\Big) \\ &= g((\nabla J_1)X,Y) + g(J_1\nabla X,Y) - \mathrm{Re}\,\tau(\nabla X,Y) \\ &= g((\nabla J_1)X,Y) \end{aligned}$$

Hence J_1 is flat, implying that λ is constant, so J is flat. Also $g(JX,JY) = -g(X,J^2Y) = g(X,Y)$, so (X,g,J) is Kähler. □

Remark. This shows in particular that any Kähler K3 surface (hence any K3 surface) is hyper-Kähler.

5 — The Kähler identities

A Kähler manifold has many operators on forms:

$$\begin{aligned} *&: \mathscr{A}^k \to \mathscr{A}^{n-k} & d&: \mathscr{A}^k \to \mathscr{A}^{k+1} \\ d^* = -*d*&: \mathscr{A}^k \to \mathscr{A}^{k-1} & \Delta_d = dd^* + d^*d&: \mathscr{A}^k \to \mathscr{A}^k \end{aligned}$$

$$\begin{aligned} \partial&: \mathscr{A}^{p,q} \to \mathscr{A}^{p+1,q} & \bar\partial&: \mathscr{A}^{p,q} \to \mathscr{A}^{p,q+1} \\ \partial^* = -*\partial*&: \mathscr{A}^{p,q} \to \mathscr{A}^{p-1,q} & \bar\partial^* = -*\bar\partial*&: \mathscr{A}^{p,q} \to \mathscr{A}^{p,q-1} \\ \Delta_\partial = \partial\partial^* + \partial^*\partial&: \mathscr{A}^{p,q} \to \mathscr{A}^{p,q} & \Delta_{\bar\partial} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial&: \mathscr{A}^{p,q} \to \mathscr{A}^{p,q} \end{aligned}$$

$$\begin{aligned} L&: \mathscr{A}^{p,q} \to \mathscr{A}^{p+1,q+1} : \alpha \mapsto \alpha \wedge \omega & h&: \mathscr{A}^k \to \mathscr{A}^k : \alpha \mapsto (n-k)\alpha \\ \Lambda&: \mathscr{A}^{p,q} \to \mathscr{A}^{p-1,q-1} : \alpha \mapsto *^{-1} \circ L \circ *\alpha \end{aligned}$$

These operators obey certain relations:

Theorem 5.1 (Kähler identities)**.** *On a Kähler manifold, the following identities hold:*

$$\begin{aligned} [\bar\partial^*, L] &= i\partial & [\bar\partial, \Lambda] &= -i\partial^* \\ [\partial^*, L] &= i\bar\partial & [\partial, \Lambda] &= -i\bar\partial^* \end{aligned}$$

The proof of this theorem is an uninsightful local computation, so we do not give it here.

Corollary 5.2. *On a Kähler manifold, we have,*

(1) $\Delta_d = 2\Delta_\partial = 2\Delta_{\bar\partial}$*;*
(2) h*,* L*, and* Λ *all commute with* Δ*;*
(3) $[\Lambda, L] = h$*,* $[h, L] = -2L$*, and* $[h, \Lambda] = 2\Lambda$*.*

This corollary is purely a formal consequence of theorem 5.1, calculating first that $\bar\partial^*\partial + \partial\bar\partial^* = 0$.

So we get a representation of $\mathfrak{sl}(2)$ on $\mathscr{A}^{*,*}$ using h, L, and Λ, which descends to cohomology via Hodge decomposition. Also, the Hodge decompositions for d, ∂, and $\bar\partial$ are equal.

Corollary 5.3 (Hard Lefschetz)**.** *The map* $L^k : H^{n-k}(X,\mathbb{C}) \to H^{n+k}(X,\mathbb{C})$ *is an isomorphism.*

Proof. This follows from standard representation theory of $\mathfrak{sl}(2)$. □

We start with a definition.

Definition 6.1. A Kähler metric ω is called *Kähler-Einstein* if for some $\lambda \in \mathbb{R}$,

$$\mathrm{Ric}(\omega) = \lambda\omega \tag{5}$$

By rescaling the metric, we can and will always restrict ourselves to one of three cases: $\mathrm{Ric}(\omega) = \omega$, $\mathrm{Ric}(\omega) = 0$ or $\mathrm{Ric}(\omega) = -\omega$. In these cases the first Chern class is, by definition, positive, zero, or negative respectively.

Remark. The 'Einstein' part of the name 'Kähler-Einstein' originates from the Einstein equations for gravity. A Kähler-Einstein metric is a Kähler metric which satisfies the Einstein equations in the absense of matter. The parameter λ is called the *cosmological constant* in this context. In current cosmology, it is interpreted as dark energy.

The next theorem is a fundamental result about the first Chern class of Kähler manifolds:

Theorem 6.2 (Calabi-Yau theorem[Yau77])**.** *Given a compact Kähler manifold (X,ω), there exists for any real (1,1)-form $\alpha \in c_1(X)$ a unique Kähler metric η, cohomologous to ω, such that* $\mathrm{Ric}(\eta) = \alpha$.

For a proof, see e.g. Tian [Tia00].

Lemma 6.3 ($\partial\bar{\partial}$-lemma)**.** *For any two real cohomologous (1,1)-forms τ and η on a compact Kähler manifold X, there exists a smooth function $f : X \to \mathbb{R}$ such that*

$$\tau = \eta + i\partial\bar{\partial}f \tag{6}$$

Proof. [Szé14] As τ and η are cohomologous, there exists a real 1-form α such that $\tau = \eta + d\alpha$. We can decompose α in its $(1,0)$ part α' and its $(0,1)$ part α'', and as α is real, $\alpha'' = \overline{\alpha'}$. As τ and η are $(1,1)$-forms, we get that $\partial\alpha' = \bar{\partial}\alpha'' = 0$ and

$$\tau = \eta + \bar{\partial}\alpha' + \partial\alpha'' = \eta + \bar{\partial}\alpha' + \overline{\bar{\partial}\alpha'}$$

Choosing a Kähler metric g on X, with associated Kähler form ω, volume form $\mathrm{dvol} := \omega^{\dim_{\mathbb{C}} X}$, and Hodge star $*$, we get

$$\begin{aligned}\int_X (\partial^*\alpha')\mathrm{dvol} &= -\int_X (*\partial * \alpha')\cdot \mathrm{dvol} = \int_X (\partial * \alpha')\cdot(*\mathrm{dvol})\\ &= \int_X \partial * \alpha' = 0\end{aligned}$$

So by Hodge theory (and the Poisson equation), there exists a function f' such that $\partial^*\alpha' = \Delta_\partial f' = \partial^*\partial f'$.

As $\partial\alpha' = 0$, we get

$$\partial(\alpha' - \partial f') = \partial^*(\alpha' - \partial f') = 0$$

Therefore, $\alpha' - \partial f'$ is ∂-harmonic, and hence also $\bar{\partial}$-harmonic, as g is Kähler. This gives $\bar{\partial}\alpha' = \bar{\partial}\partial f'$. Putting everything together,

$$\begin{aligned}\tau - \eta &= \bar{\partial}\alpha' + \overline{\bar{\partial}\alpha'} = \bar{\partial}\partial f' + \overline{\bar{\partial}\partial f'}\\ &= -\partial\bar{\partial}f' + \partial\bar{\partial}\bar{f}' = -2i\partial\bar{\partial}\,\mathrm{Im}(f')\end{aligned}$$

Choosing $f = -2\,\mathrm{Im}(f')$ gives the result. □

Remark. Because of this lemma, Kähler metrics (in the right class) always obey $\mathrm{Ric}(\omega) = \lambda\omega + i\partial\bar{\partial}f$ for some smooth real f. The main strategy in finding Kähler-Einstein metrics is to vary ω and see how f varies.

A main problem in this field has been to find out when a compact Kähler manifold admits a Kähler-Einstein metric. Trivially, the first Chern class should be definite, i.e. $c_1(X) = \lambda[\omega]$ for some Kähler form ω. The answer then turns out to depend on the sign of this λ:

- For the case $c_1(X) < 0$, the solution to the problem is given by the following theorem:

Theorem 6.4 (Aubin-Yau)**.** *[Aub76, Yau78] Suppose X is a compact Kähler manifold such that $c_1(X) < 0$. Then there exists a unique Kähler-Einstein metric ω on X. This metric must necessarily lie in the cohomology class $-c_1(X)$.*

This case is hence completely solved.

0 For the case $c_1(X) = 0$, there is no a priori confinement of Kähler-Einstein metrics to certain cohomology classes. It turns out there is no such restriction at all in this case.

Theorem 6.5 (Yau)**.** *Let (X, ω) be a compact Kähler manifold with trivial first Chern class. Then there exists a unique Kähler metric η, with $[\omega] = [\eta]$, such that $\mathrm{Ric}(\eta) = 0$.*

This is an easy corollary of the Calabi-Yau theorem, obtained by setting $\alpha = 0$.

So this case is also solved.

+ In the case $c_1(X) > 0$, things are more complicated. It turns out not all Kähler manifolds with positive first Chern class admit a Kähler-Einstein metric. According to the Yau-Tian-Donaldson conjecture, the obstruction is algebro-geometric in nature. This conjecture is now proven, leading to the following theorem:

Theorem 6.6 (Berman-Chen-Donaldson-Sun)**.** *[Ber15, CDS12, CDS15a, CDS15b, CDS15c] A Fano variety admits a Kähler-Einstein metric if and only if it is K-stable.*

6.1 — Explanation of the Berman-Chen-Donaldson-Sun theorem

In order to understand this theorem, we need to introduce several definitions. This part is adapted from [Ber15].

Definition 6.7. A *Fano variety* X is a normal projective complex variety whose anti-canonical line bundle $-K_X$ on the regular locus X_{reg} extends to an ample $\mathbb{Q}$-line bundle on X, i.e. some power extends to an ample line bundle over X.

Definition 6.8. A Fano variety X has *Kawamata log terminal (klt) singularities* if there exist a metric h and a smooth metric h' on $-K_X$ such that $F(h)$ is a Kähler form, $\log h_s - \log h'_s$ is a locally bounded function (for some holomorphic section s), and the volume form of h has finite mass.

In this setting, the definition of a Kähler-Einstein needs to be adapted slightly as well:

Definition 6.9. Let X be a Fano variety. A Kähler metric ω on X is said to be *Kähler-Einstein* if $\mathrm{Ric}(\omega) = \omega$ and $\int_{X_{\mathrm{reg}}} \omega^n = c_1(-K_X)^n =: V$.

Definition 6.10. A normal variety X is *$\mathbb{Q}$-Gorenstein* if nK_X is a Cartier divisor for some positive integer n.

Definition 6.11. A *test configuration* for a Fano variety X consists of a flat family $\pi : \mathcal{X} \to \mathbb{A}^1 := \mathbb{A}^1_{\mathbb{C}}$ with a relatively ample $\mathbb{Q}$-line bundle $\mathcal{L} \to \mathcal{X}$ (so some power of $\mathcal{L}$ is a line bundle), endowed with a $\mathbb{G}_m(\mathbb{C})$-action ρ covering the standard action on $\mathbb{A}^1$ such that

- $X = \mathcal{X}_1 := \pi^{-1}(1)$ and $(\mathcal{X}_1, \mathcal{L}_1) \cong (X, -K_X)$;
- The total space $\mathcal{X}$ and the central fibre $\mathcal{X}_0 := \pi^{-1}(0)$ are normal $\mathbb{Q}$-Gorenstein varieties with klt singularities. The latter is integral.

The condition that $\mathcal{L} \to \mathcal{X}$ be relatively ample means that, for all $\tau \in \mathbb{A}^1$, $\mathcal{L}_\tau \to \mathcal{X}_\tau$ is ample. That $\mathcal{X}$ and $\mathcal{X}_0$ are $\mathbb{Q}$-Gorenstein means that $K_{\mathcal{X}/\mathbb{A}^1} = K_{\mathcal{X}} - \pi^* K_{\mathbb{A}^1}$ is trivial near all singularities, and hence can be extended over them.

Lemma 6.12. *If $(\mathcal{X}, \mathcal{L})$ is a test configuration for the Fano variety X, then $\mathcal{L}$ is isomorphic to $-K_{\mathcal{X}/\mathbb{A}^1}$.*

Proof. As $\mathcal{L}_1 \cong -K_X$, and $(\mathcal{X}_\tau, \mathcal{L}_\tau) \cong (\mathcal{X}_1, \mathcal{L}_1)$ for all $\tau \in \mathbb{C}^*$ via the action of ρ, the $\mathbb{Q}$-Cartier divisors $\mathcal{L}$ and $-K_{\mathcal{X}}$ are linearly equivalent on $\mathcal{X}^* := \mathcal{X} \setminus \mathcal{X}_0$. Hence $n(\mathcal{L} + K_{\mathcal{X}})$ is linearly equivalent to a Weil divisor supported in $\mathcal{X}_0$ for some n. But as the central fibre is integral, $n(\mathcal{L} + K_{\mathcal{X}})$ is equivalent to a multiple of $\mathcal{X}_0$. As $\mathcal{X}_0$ is cut out by $\pi^*\tau$, where τ is the coordinate on $\mathbb{A}^1$, and this coordinate, inducing the action on $\mathcal{L}$, vanishes with order one there, $\mathcal{X}_0$ is a Cartier divisor. By adjunction, $\mathcal{L}|_{\mathcal{X}_0} \sim -K_{\mathcal{X}_0}$, so $\mathcal{L}$ is indeed isomorphic to $-K_{\mathcal{X}} + \pi^* K_{\mathbb{A}^1} = -K_{\mathcal{X}/\mathbb{A}^1}$. ◻

Test configurations have an associated numerical invariant, called the Donaldson-Futaki invariant. To define it, consider for all $k \geq 1$ the vector space $H^0(\mathcal{X}_0, -kK_0)$, having a $\mathbb{G}_m(\mathbb{C})$-action induced by that of K, whose dimension will be called d_k. The total weight of the action will be denoted w_k. By Hilbert function theory, it can be shown that d_k and w_k, for large k, become polynomials of degree n and $n+1$, respectively, see [CDS12]. Hence, the following definition makes sense.

Definition 6.13. The *Donaldson-Futaki invariant* $\mathrm{DF}(\mathcal{X})$ of a test configuration $\mathcal{X}$ is given by the following expansion:

$$\frac{w_k}{kd_k} = c_0 + \frac{1}{2}\mathrm{DF}(\mathcal{X})k^{-1} + O(k^{-2}) \tag{7}$$

Definition 6.14. A Fano variety X is called *K-stable* if $\mathrm{DF}(\mathcal{X}) \leq 0$ for all test configurations $\mathcal{X}$ and $\mathrm{DF}(\mathcal{X}) = 0$ only if $\mathcal{X}$ is isomorphic $X \times \mathbb{C}$.

Remark. We have now defined the notions in theorem 6.6. The proof, however, goes way beyond the scope of these notes. It uses functional analysis and advanced algebraic geometry, showing the strength of complex geometry, especially when considering Kähler manifolds.

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