Division by Zero in Common Meadows

Symposion and Festkolloquium in honor of Martin Wirsing

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March 5, 2015

Fiske Common meadow, Lexington, Massachusetts

Formally real meadows ($\mathbb{R}$ is a formally real field; Artin-Schreier)

Md-axioms:

$$
\begin{aligned}
(x+y)+z &= x+(y+z) & x \cdot y &= y \cdot x \\
x+y &= y+x & 1 \cdot x &= x \\
x+0 &= x & x \cdot (y+z) &= (x \cdot y)+(x \cdot z) \\
x+(-x) &= 0 & (x^{-1})^{-1} &= x \\
(x \cdot y) \cdot z &= x \cdot (y \cdot z) & x \cdot (x \cdot x^{-1}) &= x
\end{aligned}
$$

$\mathbb{R}_0$ is a "formally real meadow".

$$EFR = \left\{ \frac{1+x_0^2+x_1^2+\ldots+x_n^2}{1+x_0^2+x_1^2+\ldots+x_n^2} = 1 \;\middle|\; n \in \mathbb{N} \right\}$$

(thus: 1 plus a sum of squares is not 0)

Theorem (Completeness of $\mathbb{R}_0$):

$$Md + EFR \vdash s = t \quad \Longleftrightarrow \quad \mathbb{R}_0 \models s = t$$

Common meadows ($0^{-1} = \mathbf{a}$)

$$
\begin{aligned}
(x+y)+z &= x+(y+z) & 0\cdot(x\cdot x) &= 0\cdot x \\
x+y &= y+x & (x^{-1})^{-1} &= x+(0\cdot x^{-1}) \\
x+0 &= x & x\cdot x^{-1} &= 1+(0\cdot x^{-1}) \\
x+(-x) &= 0\cdot x & (x\cdot y)^{-1} &= (x^{-1})\cdot(y^{-1}) \\
-(-x) &= x & 1^{-1} &= 1 \\
(x\cdot y)\cdot z &= x\cdot(y\cdot z) & 0^{-1} &= \mathbf{a} \\
x\cdot y &= y\cdot x & x+\mathbf{a} &= \mathbf{a} \\
1\cdot x &= x & x\cdot\mathbf{a} &= \mathbf{a} \\
x\cdot(y+z) &= (x\cdot y)+(x\cdot z) & &
\end{aligned}
$$

Basis theorem for common cancellation meadows of char. 0:

$$Md_{\mathbf{a}} + (Char0) \vdash s = t$$

$$\iff \qquad Md_{\mathbf{a}} + \{x \neq 0 \;\&\; x \neq \mathbf{a} \to x\cdot x^{-1} = 1\} \vdash s = t$$

Fracpairs over $\mathbb{Z}$: expressions $\frac{p}{q}$ with $p, q \in \mathbb{Z}$ (so $q = 0$ is allowed) modulo the equivalence generated by

$$\frac{x \cdot z}{y \cdot (z \cdot z)} = \frac{x}{y \cdot z}$$

This equivalence appears to be a congruence with respect to the common meadow signature under natural definitions:

$$0 = \frac{0}{1}, \qquad 1 = \frac{1}{1}, \qquad \mathbf{a} = \frac{1}{0}, \qquad (\frac{p}{q}) + (\frac{r}{s}) = \frac{p \cdot s + r \cdot q}{q \cdot s}$$

$$(\frac{p}{q}) \cdot (\frac{r}{s}) = \frac{p \cdot r}{q \cdot s}, \qquad -(\frac{p}{q}) = \frac{-p}{q}, \qquad (\frac{p}{q})^{-1} = \frac{q \cdot q}{p \cdot q}$$

Theorem: The initial common meadow is isomorphic to the initial algebra of fracpairs over the integers $\mathbb{Z}$.

Theorem: The initial algebra of fracpairs over $\mathbb{Z}$ constitutes a homomorphic pre-image of the common meadow $\mathbb{Q}_{\mathbf{a}}$.