Fracterm Calculus for Partial Meadows

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Abstract. Partial algebras and partial data types are discussed with the use of signatures that allow partial functions, and a three-valued short-circuit (sequential) first order logic with a Tarski semantics. The propositional part of this logic is also known as McCarthy calculus and has been studied extensively.

Axioms for the fracterm calculus of partial meadows are given. The case is made that in this way a rather natural formalisation of fields with division operator is obtained. It is noticed that the logic thus obtained cannot express that division by zero must be undefined.

An interpretation of the three-valued sequential logic into ⊥-enlargements of partial algebras is given, for which it is concluded that the consequence relation of the former logic is semi-computable, and that the ⊥-enlargement of a partial meadow is a common meadow.

Keywords: fracterm calculus · partial meadow · common meadow · abstract data type

1 Introduction

Following [10], we use *meadow* for fields equipped with an inverse or a division function. A *partial* meadow is a field equipped with a partial division function, here written $\frac{x}{y}$ in fractional notation. A partial meadow is a partial algebra because division is partial as $\frac{x}{0}$ is undefined. A *fracterm* is an expression of the form $\frac{p}{q}$, where p and q are terms over the signature under consideration.

Fracterm calculus for partial meadows (FTCpm) fully accepts $\frac{1}{0}$ as a fracterm, in spite of it having no value. At the same time a three valued logic is adopted. For instance in FTCpm it is not the case that $\frac{1}{0} = \frac{1}{0}$ and neither is it the case that $\frac{1}{0} \neq \frac{1}{0}$. Both assertions are considered syntactically valid, however. We will use *partial equality*, written $=_{\mathsf{p}}$ for the equality sign for partial data types. If either t or r has no value then $t =_{\mathsf{p}} r$ has no classical truth value, i.e. true or false. The assertion $\phi(x)$ with

$$\phi(x) \equiv x \neq_{\mathsf{p}} 0 \rightarrow \frac{x}{x} =_{\mathsf{p}} 1$$

This paper is dedicated to Sjouke Mauw, a full version including all longer proofs appeared as [7].

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is taken for a fundamental fact about partial meadows which holds for all x. Contemplating the substitution $x = 0$ suggests that the most plausible reading of the implication in $\phi(x)$ is a short-circuit implication (also referred to as sequential implication, or as the implication of McCarthy's logic), a state of affairs which we wish to make explicit in the symbolic notation. Following [2] we will denote short-circuit implication with the connective $\circ\!\!\rightarrow$, thereby changing ϕ to ϕ':

$$\phi'(x) \equiv x \neq_{\mathsf{p}} 0 \circ\!\!\rightarrow \frac{x}{x} =_{\mathsf{p}} 1.$$

We notice that evaluation of $\phi'(0)$ will not involve an attempt to evaluate the fracterm $\frac{0}{0}$, because $0 \neq_{\mathsf{p}} 0$ evaluates to false, thereby rendering the implication at hand true without further inspection of its consequent. Moreover, once division is modelled as a partial function, assigning a Boolean truth value to $\frac{0}{0} =_{\mathsf{p}} \frac{0}{0}$ becomes artificial and the use of a non-classical logic with three or more truth values becomes plausible if not unavoidable. One may object that $\frac{0}{0} =_{\mathsf{p}} \frac{0}{0}$ is an assertion that "asks" for division by 0 which for that reason must be excluded. The reasons for understanding logical connectives in a short-circuited manner (i.e. McCarthy logic) are these: the most plausible alternatives are versions of strong Kleene logic which allow $\mathsf{T} \vee x = x \vee \mathsf{T} = \mathsf{T}$ for all truth values x including non-Boolean ones and T a constant for truth. The use of such logics is plausible in computer science, for instance in [19] the equality relation $=_{\mathsf{p}}$ (in [19] simply denoted with $=$) is combined with a strong Kleene logic. The preference of [19] for that logic is based on the symmetry of conjunction and disjunction, symmetries which are lost in the short-circuit case. By consequence one finds that $\psi(x) \equiv \frac{x}{0} = 2 \vee x = 3$ would be true for $x = 3$. Evaluating $\psi(3)$, however, seems to call for an attempt to evaluate $\frac{3}{0}$ which is impossible for elementary arithmetic. For these reasons we deviate from [19] by adopting short-circuit logic as preferable in the case of rational numbers with a partial division function. Some general background on short-circuit logic is given in Sect. 6.

We are unaware of any existing work which investigates the details of the above view of partial meadows, or stated differently of formalising arithmetics involving a partial division function.

1.1 Survey of the Paper

The contents of this paper corresponds to [7], only some explanation of *Prover9* and *Mace4* results has been omitted, as well as the proofs of Propositions 1 and 2, and Theorems 2 and 3(see [7, Prop.2.2.1 & 3.3.1, Thm.3.3.3 & 5.2.1]).

We consider the following results of this paper to be new:

(i) The claim that short-circuit logic is the most natural logic for use in the context of partial meadows, plus a listing of reasons, together constituting a rationale, for that claim (see below).

(ii) A complete axiomatisation of the fracterm calculus of partial meadows. Here we notice that two variants of completeness must be distinguished:
   - Axiomatisation completeness: complete axiomatisation of a class of (partial) algebras using a logic with given semantics.

– Logical completeness: completeness of a proof system for a logic with given semantics.

Axiomatisation completeness is the focus of our paper. Axiomatisation completeness (with fracterm calculus for partial meadows as the intended application) can be understood and appreciated without any regard to logical completeness. We intend this paper to be fully self-contained, both mathematically and in terms of motivation, and to be independent from any proof theoretic considerations.

(iii) A result on fracterm flattening in the context of the fracterm calculus of partial meadows.

(iv) An interpretation of the short-circuit logic for partial algebras into the first order logic of algebras enlarged with an element $\bot$ which serves as an absorptive element. The interpretation demonstrates that the consequence relation for short-circuit logic on partial algebras is semi-computable. Moreover, said interpretation indirectly provides a proof system, or rather a formal proof method for short-circuit logic over partial algebras, thereby demonstrating that a proof theory can be developed in principle. As a proof of concept, it is shown that the $\bot$-enlargement of a partial meadow is a *common* meadow, see e.g. [4,5,12] for common meadows.

1.2 On the Rationale of Fracterm Calculus for Partial Meadows

Fracterm calculus for partial meadows aims at formalising elementary arithmetic including division. Modelling division as a partial function is motivated by the idea that this is the most conventional viewpoint on division. All views where division has been made total are, however useful these views may be in the context of certain specific objectives, somehow unconventional.

Even if formalisation of elementary arithmetic can be simplified by having division total (which may be achieved in different ways), then it is still the case that using a partial function for modelling division provides a very relevant alternative which merits systematic investigation. At this stage it is unclear whether or not, and if so to which extent, totalisation of division is helpful for the formalisation and understanding of elementary arithmetic. Once the notion that division is modelled as a partial function, undefined when the denominator equals zero, has been adopted, it is plausible to require (i.e. as a design constraint for texts) that texts are to be written in such a manner and ordering that a reader, when reading in the order of presentation, will never be asked or invited to contemplate what happens when dividing by 0.

Dedication. We wrote this paper in honour of Sjouke Mauw. Sjouke was a very enthusiastic and effective member of the process algebra group in the late 1980s. His contribution ranged from theory (see, e.g., [21]) to the design and, in joint work with Gert Veltink and Bob Diertens the implementation, of the specification language PSF where process algebra and ASF-style algebraic specifications of abstract data types were combined into a language and the PSF Toolkit that were used for many years in educational practice, see, e.g., [17,22]. PSF uses total functions for data types, here we depart from that path and look at partial functions for abstract data types with a tailor-made logic for these.

Table 1. The set EqCL of axioms of CℓSCL with T, F and the connective $\circ\!\!\longrightarrow$

$$\mathsf{F} = \neg\mathsf{T} \tag{e1}$$
$$\phi \mathbin{\overset{\circ}{\vee}} \psi = \neg(\neg\phi \mathbin{\underset{\circ}{\wedge}} \neg\psi) \tag{e2}$$
$$\mathsf{T} \mathbin{\underset{\circ}{\wedge}} \phi = \phi \tag{e3}$$
$$\phi \mathbin{\underset{\circ}{\wedge}} (\phi \mathbin{\overset{\circ}{\vee}} \psi) = \phi \tag{e4}$$
$$(\phi \mathbin{\overset{\circ}{\vee}} \psi) \mathbin{\underset{\circ}{\wedge}} \xi = (\neg\phi \mathbin{\underset{\circ}{\wedge}} (\psi \mathbin{\underset{\circ}{\wedge}} \xi)) \mathbin{\overset{\circ}{\vee}} (\phi \mathbin{\underset{\circ}{\wedge}} \xi) \tag{e5}$$
$$(\phi \mathbin{\underset{\circ}{\wedge}} \psi) \mathbin{\overset{\circ}{\vee}} (\psi \mathbin{\underset{\circ}{\wedge}} \phi) = (\psi \mathbin{\underset{\circ}{\wedge}} \phi) \mathbin{\overset{\circ}{\vee}} (\phi \mathbin{\underset{\circ}{\wedge}} \psi) \tag{e6}$$
$$\phi \circ\!\!\longrightarrow \psi = \neg\phi \mathbin{\overset{\circ}{\vee}} \psi \tag{e7}$$

2 Partial Algebras with 3-Valued Sequential Logic

In Sect. 2.1 we define a three-valued sequential first order logic for partial algebras, and in Sect. 2.2 we discuss its semantics.

2.1 Sequential First Order Logic for Partial Algebras

We start by recalling an equational logic that defines the sequential, short-circuited connectives $\underset{\circ}{\wedge}$ and $\overset{\circ}{\vee}$, called *short-circuit* (or *sequential*) disjunction and conjunction. In [6] we introduced so-called CℓSCL, Conditional Short-Circuit Logic, together with a relatively simple semantics based on evaluation trees for the sequential evaluation of atoms (propositional variables). In this paper, atoms will be partial equalities.

CℓSCL is equivalent with three-valued Conditional Logic, as introduced by Guzmán and Squier in [18], but is distinguished by the use of the specific notation for the short-circuit connectives mentioned above. In [6] we provided several complete, independent equational axiomatisations of CℓSCL, based on whether or not to include the constants T, F and U for the three truth values true, false and $\mathsf{undefined}$, respectively.

In Table 1, we define the set EqCL of axioms of CℓSCL with constants T and F and and with the addition of the connective $\circ\!\!\longrightarrow$ that defines short-circuit implication. The axioms of EqCL imply double negation elimination, i.e.

$$\neg\neg\phi = \phi, \tag{DNE}$$

and therewith a sequential version of the duality principle: in equations without occurrences of $\circ\!\!\longrightarrow$, the connectives $\underset{\circ}{\wedge}$ and $\overset{\circ}{\vee}$ and occurrences of T and F can be swapped. Axiom (e4) is a sequential version of the absorption law. Axioms (e5) and (e6) imply some other properties of these connectives and (e7) defines $\circ\!\!\rightarrow$. Next to (DNE), other useful consequences of EqCL are the following:

(a) $\phi \mathbin{_{\circ}\!\!\wedge} \mathsf{T} = \phi$ and $\mathsf{T} \mathbin{^{\circ}\!\!\vee} \phi = \mathsf{T}$,
(b) $(\phi \mathbin{_{\circ}\!\!\wedge} \psi) \mathbin{_{\circ}\!\!\wedge} \xi = \phi \mathbin{_{\circ}\!\!\wedge} (\psi \mathbin{_{\circ}\!\!\wedge} \xi)$, so the connective $\mathbin{_{\circ}\!\!\wedge}$ is associative,
(c) $\phi \mathbin{_{\circ}\!\!\wedge} (\psi \mathbin{_{\circ}\!\!\wedge} \phi) = \phi \mathbin{_{\circ}\!\!\wedge} \psi$, so, with $\psi = \mathsf{T}$, $\mathbin{_{\circ}\!\!\wedge}$ is idempotent, i.e. $\phi \mathbin{_{\circ}\!\!\wedge} \phi = \phi$,
(d) $\phi \mathbin{_{\circ}\!\!\wedge} (\psi \mathbin{^{\circ}\!\!\vee} \xi) = (\phi \mathbin{_{\circ}\!\!\wedge} \psi) \mathbin{^{\circ}\!\!\vee} (\phi \mathbin{_{\circ}\!\!\wedge} \xi)$, so $\mathbin{_{\circ}\!\!\wedge}$ is left-distributive,
(e) $\phi \mathbin{_{\circ}\!\!\wedge} \neg\phi = \neg\phi \mathbin{_{\circ}\!\!\wedge} \phi$,
(f) $(\phi \mathbin{_{\circ}\!\!\wedge} \psi) \circ\!\!\rightarrow \xi = \phi \circ\!\!\rightarrow (\psi \circ\!\!\rightarrow \xi)$.

Of course, the duals of (a)–(e), say (a)′–(e)′, are also consequences of EqCL. Consequences (DNE) and (a)–(f) follow quickly with help of the theorem prover *Prover9* [23]. Simple short proofs of (DNE) and (a)–(e) from (only) (e1)–(e5) are included in [9, App.A.3 & A.4]. At the end of Sect. 2.2 we discuss an example that refutes the commutativity of $\mathbin{^{\circ}\!\!\vee}$ and give a completeness result for EqCL.

Based on EqCL, we define a sequential first order logic for partial algebras, and in Sect. 3 we wil refine this logic to the specific case of partial meadows. Let Σ be a signature that contains one or more partial functions and a constant symbol c. Consider a partial Σ-algebra A with non-empty domain $|A|$. There is an equality relation called *partial equality* and written $=_{\mathsf{p}}$ which behaves on all domains as usual, i.e. as a normal equality relation. This means that for $a, b \in |A|$, $a =_{\mathsf{p}} b$ if, and only if, $a = b$, where $=$ is the normal equality relation on $|A|$. For clarification, we will illustrate further axioms and rules of inference with informal use of the semantics defined in Sect. 2.2.

With $\mathsf{L}_{\mathsf{sfol}}(\Sigma)$ we denote the collection of formulae inductively defined as follows, assuming a set V_{var} of variables:

- $\mathsf{T}, \mathsf{F} \in \mathsf{L}_{\mathsf{sfol}}(\Sigma)$,
- for Σ-terms t and r the *atomic* formulae $(t =_{\mathsf{p}} r) \in \mathsf{L}_{\mathsf{sfol}}(\Sigma)$,
- if $\phi \in \mathsf{L}_{\mathsf{sfol}}(\Sigma)$ then $\neg\phi \in \mathsf{L}_{\mathsf{sfol}}(\Sigma)$,
- if $\phi, \psi \in \mathsf{L}_{\mathsf{sfol}}(\Sigma)$, then $\phi \mathbin{^{\circ}\!\!\vee} \psi$, $\phi \mathbin{_{\circ}\!\!\wedge} \psi$, and $\phi \circ\!\!\rightarrow \psi$ are in $\mathsf{L}_{\mathsf{sfol}}(\Sigma)$,
- if $\phi \in \mathsf{L}_{\mathsf{sfol}}(\Sigma)$ and $x \in V_{var}$, then $\exists_{\mathsf{p}} x.\phi$ and $\forall_{\mathsf{p}} x.\phi$ are in $\mathsf{L}_{\mathsf{sfol}}(\Sigma)$.

So, in $\mathsf{L}_{\mathsf{sfol}}(\Sigma)$, there is explicit quantification over variables that occur in atomic formulae. For example, we will see that a partial meadow satisfies

$$(\forall_{\mathsf{p}} x. x \neq_{\mathsf{p}} 0 \circ\!\!\rightarrow \frac{x}{x} =_{\mathsf{p}} 1) = \mathsf{T}.$$

We extend EqCL by adding the following axiom:

$$\exists_{\mathsf{p}} x.\phi = \neg\forall_{\mathsf{p}} x.\neg\phi. \tag{e8}$$

Next, we define $\mathsf{UL}_{\mathsf{sfol}}(\Sigma)$, the universal quantifier-free fragment of $\mathsf{L}_{\mathsf{sfol}}(\Sigma)$, and $\mathsf{Eq}(\mathsf{L}_{\mathsf{sfol}}(\Sigma))$, the collection of equations over $\mathsf{L}_{\mathsf{sfol}}(\Sigma)$ and the largest language we consider:

- If $\phi \in \mathsf{L}_{\mathsf{sfol}}(\Sigma)$ is quantifier-free, then $\phi \in \mathsf{UL}_{\mathsf{sfol}}(\Sigma)$,

– If $\phi, \psi \in \mathsf{L}_{\mathsf{sfol}}(\Sigma)$ then $(\phi = \psi) \in \mathsf{Eq}(\mathsf{L}_{\mathsf{sfol}}(\Sigma))$.

Hence, in both $\mathsf{UL}_{\mathsf{sfol}}(\Sigma)$ and $\mathsf{Eq}(\mathsf{L}_{\mathsf{sfol}}(\Sigma))$, variables can occur free. We give an example of a valid $\mathsf{Eq}(\mathsf{L}_{\mathsf{sfol}}(\Sigma))$-formula: a partial meadow A satisfies both $(x \neq_{\mathsf{p}} 0 \mathbin{\circ\!\!\rightarrow} \frac{x}{x} =_{\mathsf{p}} 1) = \mathsf{T}$ and $(\forall_{\mathsf{p}} x. x \neq_{\mathsf{p}} 0 \mathbin{\circ\!\!\rightarrow} \frac{x}{x} =_{\mathsf{p}} 1) = \mathsf{T}$, which is characterised by the equivalence

$$A \models (x \neq_{\mathsf{p}} 0 \mathbin{\circ\!\!\rightarrow} \tfrac{x}{x} =_{\mathsf{p}} 1) = \mathsf{T} \iff A \models (\forall_{\mathsf{p}} x.\, x \neq_{\mathsf{p}} 0 \mathbin{\circ\!\!\rightarrow} \tfrac{x}{x} =_{\mathsf{p}} 1) = \mathsf{T}.$$

More generally, for a partial Σ-algebra A, the following equivalences hold true for $\phi \in \mathsf{UL}_{\mathsf{sfol}}(\Sigma)$:

$$A \models \phi = \mathsf{T} \iff A \models (\forall_{\mathsf{p}} x.\phi) = \mathsf{T} \iff A \models (\neg\exists_{\mathsf{p}} x.\neg\phi) = \mathsf{T}, \qquad \text{(qF)}$$

where the second equivalence is a consequence of axiom (e8). The related equivalence $A \models (\exists_{\mathsf{p}} x.\phi) = \mathsf{T} \iff A \models (\neg\forall_{\mathsf{p}} x.\neg\phi) = \mathsf{T}$ is more complex, see Sect. 2.2.

We are especially interested in $\mathsf{Eq}(\mathsf{L}_{\mathsf{sfol}}(\Sigma))$-formulae of the form

$$\phi = \mathsf{T},$$

and give below axioms and rules for deriving such equations, accompanied by examples for partial meadows (which will be formally defined in Sect. 3). When proving results, and in some cases when writing axiom systems or designing operations on the syntax, we assume for simplification that all formulas can be written with a subset of the sequential connectives, for example, with connectives $\neg$ and $\mathbin{{}^{\circ}\!\vee}$ and with the universal quantifier $\forall_{\mathsf{p}}$ only (by axiom (e8)). Alternatively one may assume that connectives $\mathbin{{}_{\circ}\!\wedge}$ and $\mathbin{\circ\!\!\rightarrow}$ are used as well as both quantifiers $\exists_{\mathsf{p}}$ and $\forall_{\mathsf{p}}$, on top of equations and denial inequations, that is negated equations written in the form $t \neq_{\mathsf{p}} r$.

Axioms for the Relation $=_{\mathsf{p}}$. This relation is not a congruence relation because it is not reflexive on expressions, for example, in a partial meadow it is not the case that $\frac{1}{0} =_{\mathsf{p}} \frac{1}{0}$ and neither is it the case that $\frac{1}{0} \neq_{\mathsf{p}} \frac{1}{0}$. Equations of the form $t =_{\mathsf{p}} t$ are used to express that t is defined and occur in the 'weak substitution property' (p6) and (p7) (cf. [3]). We write $\mathop{{}_{\circ}\!\bigwedge}_{i=1}^{1} \phi_i = \phi_1$ and for $k > 0$, $\mathop{{}_{\circ}\!\bigwedge}_{i=1}^{k+1} \phi_i = \phi_1 \mathbin{{}_{\circ}\!\wedge} (\mathop{{}_{\circ}\!\bigwedge}_{i=2}^{k+1} \phi_i)$.

Axioms (and axiom schemes) for the relation $=_{\mathsf{p}}$ are the following, for any constant $c \in \Sigma$ and $x \in V_{var}$ and (open) Σ-terms t_i:

(p1) $(c =_{\mathsf{p}} c) = \mathsf{T}$, (reflexivity for constants)
(p2) $(x =_{\mathsf{p}} x) = \mathsf{T}$, (reflexivity for variables)
(p3) $((\mathop{{}_{\circ}\!\bigwedge}_{i=1}^{2} t_i =_{\mathsf{p}} t_i) \mathbin{\circ\!\!\rightarrow} (t_1 =_{\mathsf{p}} t_2 \mathbin{\circ\!\!\rightarrow} t_2 =_{\mathsf{p}} t_1)) = \mathsf{T}$, (symmetry)
(p4) $((\mathop{{}_{\circ}\!\bigwedge}_{i=1}^{3} t_i =_{\mathsf{p}} t_i) \mathbin{\circ\!\!\rightarrow} ((t_1 =_{\mathsf{p}} t_2 \mathbin{{}_{\circ}\!\wedge} t_2 =_{\mathsf{p}} t_3) \mathbin{\circ\!\!\rightarrow} t_1 =_{\mathsf{p}} t_3)) = \mathsf{T}$. (transitivity)

For each k-ary $f \in \Sigma$ that is total and all (open) Σ-terms $t_1, ..., t_k$:
(p5) $((\mathop{{}_{\circ}\!\bigwedge}_{i=1}^{k} t_i =_{\mathsf{p}} t_i) \mathbin{\circ\!\!\rightarrow} f(t_1, ..., t_k) =_{\mathsf{p}} f(t_1, ..., t_k)) = \mathsf{T}$. (definedness)

For all k-ary $f \in \Sigma$ and all (open) Σ-terms $t_1, ..., t_k,\ r_1, ..., r_k$, the *weak substitution property*:
(p6) $(f(t_1, ..., t_k) =_{\mathsf{p}} f(t_1, ..., t_k) \mathbin{\circ\!\!\!\rightarrow} (\ {}_{\circ}\!\!\bigwedge_{i=1}^{k} t_i =_{\mathsf{p}} t_i)) = \mathsf{T}$,
(p7) $(((f(t_1, ..., t_k) =_{\mathsf{p}} f(t_1, ..., t_k) \mathbin{{}_{\circ}\!\!\wedge} (\ {}_{\circ}\!\!\bigwedge_{i=1}^{k} t_i =_{\mathsf{p}} r_i)))\ \mathbin{\circ\!\!\!\rightarrow}$
$f(t_1, ..., t_k) =_{\mathsf{p}} f(r_1, ..., r_k)) = \mathsf{T}$.

For example, if in a partial meadow $\frac{t}{t}$ is defined then so is t by (p6), and if $\frac{t}{t}$ is defined and $t =_{\mathsf{p}} r$, then $\frac{t}{t} =_{\mathsf{p}} \frac{r}{r}$ by (p7).

More Axioms for $\mathsf{Eq}(\mathsf{L}_{\mathsf{sfol}}(\Sigma))$. For t a Σ-term, $x \in V_{var}$, $\phi \in \mathsf{L}_{\mathsf{sfol}}(\Sigma)$, and constant $c \in \Sigma$:
(a1) $((t =_{\mathsf{p}} t \mathbin{{}_{\circ}\!\!\wedge} \forall_{\mathsf{p}} x.\phi) \mathbin{\circ\!\!\!\rightarrow} \phi[t/x]) = \mathsf{T}$ if no variables in t occur bounded in ϕ,
(a2) $(\forall_{\mathsf{p}} x.\ x =_{\mathsf{p}} c \mathbin{{}^{\circ}\!\!\vee} x \neq_{\mathsf{p}} c) = \mathsf{T}$.

For example, with respect to a partial meadow, axiom (a1) excludes the substitution $x \mapsto \frac{y}{0}$ (because that would introduce undefinedness). A more extensive example using (a2) is the derivation of (Ex.1) on page 8. Note that by consequence (f) of EqCL, (a1) can also be written as

$$(t =_{\mathsf{p}} t \mathbin{\circ\!\!\!\rightarrow} (\forall_{\mathsf{p}} x.\phi \mathbin{\circ\!\!\!\rightarrow} \phi[t/x])) = \mathsf{T}.$$

Below we define two replacement rules (i1) and (i2) with help of the notion of a 'context': if a formula ϕ in $\mathsf{L}_{\mathsf{sfol}}(\Sigma)$ has a subformula ψ, then ϕ is a *context* for ψ, notation $\phi \equiv C[\boldsymbol{\psi}]$, where ψ can be written in boldface to indicate a single occurrence. For example, if $\phi \equiv (\psi \mathbin{{}_{\circ}\!\!\wedge} \xi) \mathbin{{}^{\circ}\!\!\vee} (\neg\psi)$, then

$$\phi \equiv C_1[\boldsymbol{\psi}] \equiv (\boldsymbol{\psi} \mathbin{{}_{\circ}\!\!\wedge} \xi) \mathbin{{}^{\circ}\!\!\vee} (\neg\psi) \quad \text{and} \quad \phi \equiv C_2[\boldsymbol{\psi}] \equiv (\psi \mathbin{{}_{\circ}\!\!\wedge} \xi) \mathbin{{}^{\circ}\!\!\vee} (\neg\boldsymbol{\psi}).$$

Rules of Inference for $\mathsf{Eq}(\mathsf{L}_{\mathsf{sfol}}(\Sigma))$. We write $\vdash \phi = \psi$ for the derivability of $\phi = \psi$ from the above axioms, in particular including those of EqCL (Table 1) extended with axiom (e8):
(i1) If $\vdash C[\boldsymbol{t =_{\mathsf{p}} r}] = \mathsf{T}$ and $\vdash (r =_{\mathsf{p}} s) = \mathsf{T}$ then $\vdash C[\boldsymbol{t =_{\mathsf{p}} s}] = \mathsf{T}$, (replacement rule 1)
(i2) If $\vdash C[\boldsymbol{\phi}] = \mathsf{T}$ and $\vdash \phi = \psi$ then $\vdash C[\boldsymbol{\psi}] = \mathsf{T}$, (replacement rule 2)
(i3) If $\vdash \phi = \mathsf{T}$ and $\vdash (\phi \mathbin{\circ\!\!\!\rightarrow} \psi) = \mathsf{T}$ then $\vdash \psi = \mathsf{T}$, (modus ponens for $\circ\!\!\!\rightarrow$)
(i4) If $\vdash \phi = \mathsf{T}$ and $\vdash \psi = \mathsf{T}$, then $\vdash (\phi \mathbin{{}_{\circ}\!\!\wedge} \psi) = \mathsf{T}$, (${}_{\circ}\!\!\wedge$-introduction)
(i5) If $\vdash (\phi \mathbin{\circ\!\!\!\rightarrow} \psi) = \mathsf{T}$ and $\vdash (\psi \mathbin{{}^{\circ}\!\!\vee} \neg\psi) = \mathsf{T}$ and $\vdash (\xi \mathbin{{}^{\circ}\!\!\vee} \neg\xi) = \mathsf{T}$,
then $\vdash ((\phi \mathbin{{}^{\circ}\!\!\vee} \xi) \mathbin{\circ\!\!\!\rightarrow} (\psi \mathbin{{}^{\circ}\!\!\vee} \xi)) = \mathsf{T}$. (${}^{\circ}\!\!\vee$-introduction)

Inference rule (i1) is a replacement rule that captures some properties of $=_{\mathsf{p}}$. For example, a partial meadow has the properties that $\vdash (1 =_{\mathsf{p}} 1 + 0) = \mathsf{T}$ and $\vdash (0 \neq_{\mathsf{p}} 1) = \mathsf{T}$. So, with $C[0 =_{\mathsf{p}} 1] = \neg(0 =_{\mathsf{p}} 1)$ it follows by (i1) that $\vdash (0 \neq_{\mathsf{p}} 1 + 0) = \mathsf{T}$.

As for (i2), we first note that this rule expresses the common congruence rule for EqCL-expressions. Furthermore, ϕ is a subformula of itself ($\phi \equiv C[\phi]$), so if

$\vdash \phi = \psi$ and $\vdash \phi = \mathsf{T}$, then $\psi \equiv C[\psi]$ and $\psi = \mathsf{T}$ is derived by (i2). Below, we give an example that uses (i2).

For an example of (i3), instantiate ϕ with $\forall_{\mathsf{p}} x.\ x =_{\mathsf{p}} c \mathbin{\overset{\circ}{\vee}} x \neq_{\mathsf{p}} c$, i.e. the left-hand side of (a2): $\vdash (x =_{\mathsf{p}} x \circ\!\!\rightarrow ((\forall_{\mathsf{p}} x. x =_{\mathsf{p}} c \mathbin{\overset{\circ}{\vee}} x \neq_{\mathsf{p}} c) \circ\!\!\rightarrow (x =_{\mathsf{p}} c \mathbin{\overset{\circ}{\vee}} x \neq_{\mathsf{p}} c))) = \mathsf{T}$. By (p1), $\vdash (x =_{\mathsf{p}} x) = \mathsf{T}$, so by (i3),

$$\vdash ((\forall_{\mathsf{p}} x.\ x =_{\mathsf{p}} c \mathbin{\overset{\circ}{\vee}} x \neq_{\mathsf{p}} c) \circ\!\!\rightarrow (x =_{\mathsf{p}} c \mathbin{\overset{\circ}{\vee}} x \neq_{\mathsf{p}} c)) = \mathsf{T}.$$

With (a2) and (i3), the leftmost consequence of (Ex.1) below is obtained, and hence, with consequence (e)′ (i.e. $\vdash \phi_1 \mathbin{\overset{\circ}{\vee}} \neg\phi_1 = \neg\phi_1 \mathbin{\overset{\circ}{\vee}} \phi_)$, say $\vdash \phi = \psi$) and (i2) with $\phi \equiv C[\phi]$, also the second consequence of (Ex.1), i.e. $C[\psi] = \mathsf{T}$:

$$\vdash (x =_{\mathsf{p}} c \mathbin{\overset{\circ}{\vee}} x \neq_{\mathsf{p}} c) = \mathsf{T}, \quad \text{and thus} \quad \vdash (x \neq_{\mathsf{p}} c \mathbin{\overset{\circ}{\vee}} x =_{\mathsf{p}} c) = \mathsf{T}. \qquad \text{(Ex.1)}$$

As for rule (i5), we note that the conditions $\vdash (\psi \mathbin{\overset{\circ}{\vee}} \neg\psi) = \mathsf{T}$ and $\vdash (\xi \mathbin{\overset{\circ}{\vee}} \neg\xi) = \mathsf{T}$ exclude undefinedness of ψ and ξ; for an application of (i5), see the proof of Theorem 2 in [7, Thm.3.3.3]. Inference rules (i2)–(i5) are derivable from EqCL, this follows easily with *Prover9* [23].

Lemma 1. *The relation* $\{(t, r) \mid t, r$ *Σ-terms such that* $\vdash (t =_{\mathsf{p}} t) = \mathsf{T}$, $\vdash (r =_{\mathsf{p}} r) = \mathsf{T}$ *and* $\vdash (t =_{\mathsf{p}} r) = \mathsf{T}\}$ *is a congruence.*

Proof. The combination of (p5) and (p7) implies with (i4) that for each total k-ary function f and for all defined Σ-terms $t_1, ..., t_k$ and $r_1, ..., r_k$ that satisfy $\vdash (t_i =_{\mathsf{p}} r_i) = \mathsf{T}$, $(f(t_1, ..., t_k) =_{\mathsf{p}} f(r_1, ..., r_k)) = \mathsf{T}$. So, with (p1)–(p4) we are done. □

2.2 Inductively Defined Tarski Semantics of $\mathsf{Eq}(\mathsf{L}_{\mathsf{sfol}}(\Sigma))$

Assume A is a partial Σ-algebra and σ is a valuation that assigns values from $|A|$ to the variables in V_{var}. We write $\sigma[a/x]$ with $a \in |A|$ for the valuation that assigns a to x and is otherwise defined as σ. A valuation σ extends to Σ-terms provided their constituents are defined: if $f(\sigma(t_1), ..., \sigma(t_k))$ is defined, then $\sigma(f(t_1, ..., t_k)) = f(\sigma(t_1), ..., \sigma(t_k))$.

We first consider formulae of the form $\phi = \mathsf{T}$ (and their universal quantifications) and define $A, \sigma \models \phi = \mathsf{T}$ if, and only if, $A, \sigma \models \phi$. A key role is played by the negation of partial equality $A, \sigma \models t \neq_{\mathsf{p}} r$, also called denial inequality. For denial inequality we will have three notations:

$$A, \sigma \models \neg(t =_{\mathsf{p}} r), A, \sigma \models t \neq_{\mathsf{p}} r, \text{ and } A, \sigma\ ! \models t =_{\mathsf{p}} r.$$

We notice that when working with total algebras, $=_{\mathsf{p}}$ coincides with $=$ and denial inequality coincides with dissatisfaction: $A, \sigma \models t \neq_{\mathsf{p}} r \iff A, \sigma \not\models t =_{\mathsf{p}} r$.

For $\phi \in \mathsf{L}_{\mathsf{sfol}}(\Sigma)$, satisfaction $A, \sigma \models \phi$ complemented with denial satisfaction $A, \sigma\ ! \models \phi$ is inductively defined as follows:

(1) $A, \sigma \models \mathsf{T}$,

(2) $A, \sigma \mathrel{!{\models}} \mathsf{F}$,
(3) $A, \sigma \models t =_{\mathsf{p}} r$ if, and only if, both evaluation results $\sigma(t)$ and $\sigma(r)$ are defined (exist) and are equal: $\sigma(t) = \sigma(r)$,
(4) $A, \sigma \mathrel{!{\models}} t =_{\mathsf{p}} r$ if, and only if, both evaluation results $\sigma(t)$ and $\sigma(r)$ are defined (exist) and are different elements of $|A|$: $\sigma(t) \neq \sigma(r)$,
(5) $A, \sigma \models \neg\phi$ if $A, \sigma \mathrel{!{\models}} \phi$,
(6) $A, \sigma \mathrel{!{\models}} \neg\phi$ if $A, \sigma \models \phi$,
(7) $A, \sigma \models \phi_1 \mathbin{{}^{\circ}\!\!\vee} \phi_2$ if either (i) $A, \sigma \models \phi_1$ or (ii) $A, \sigma \mathrel{!{\models}} \phi_1$ and $A, \sigma \models \phi_2$,
(8) $A, \sigma \mathrel{!{\models}} \phi_1 \mathbin{{}^{\circ}\!\!\vee} \phi_2$ if $A, \sigma \mathrel{!{\models}} \phi_1$ and $A, \sigma \mathrel{!{\models}} \phi_2$,
(9) $A, \sigma \models \forall_{\mathsf{p}} x.\phi$ if for all $a \in |A|$, it is the case that $A, \sigma[a/x] \models \phi$,
(10) $A, \sigma \mathrel{!{\models}} \forall_{\mathsf{p}} x.\phi$ if for some $a \in |A|$, it is the case that $A, \sigma[a/x] \mathrel{!{\models}} \phi$ and for all $b \in |A|$, it is the case that $A, \sigma[b/x] \models \phi \mathbin{{}^{\circ}\!\!\vee} \neg\phi$ (i.e. ϕ is not undefined).

It follows that $A, \sigma \models \phi \mathbin{\circ\!\!\rightarrow} \psi$ if either (i) $A, \sigma \mathrel{!{\models}} \phi$ or (ii) $A, \sigma \models \phi$ and $A, \sigma \models \psi$. By duality and inference rule (i2), it follows that $A, \sigma \models \phi \mathbin{{}_{\circ}\!\!\wedge} \psi$ if $A, \sigma \models \phi$ and $A, \sigma \models \psi$. This does not contradict the fact that $\mathbin{{}_{\circ}\!\!\wedge}$ and $\mathbin{{}^{\circ}\!\!\vee}$ are *not* commutative, as is demonstrated at the end of this section. Furthermore, it follows that

$$\begin{aligned} &A, \sigma \models \exists_{\mathsf{p}} x.\phi \text{ if for some } a \in |A|, \text{ it is the case that } A, \sigma[a/x] \models \phi \\ &\quad \text{and for all } b \in |A|, \text{ it is the case that } A, \sigma[b/x] \models \phi \mathbin{{}^{\circ}\!\!\vee} \neg\phi. \\ &A, \sigma \mathrel{!{\models}} \exists_{\mathsf{p}} x.\phi \text{ if for all } a \in |A|, \text{ it is the case that } A, \sigma[a/x] \mathrel{!{\models}} \phi. \end{aligned} \tag{qE}$$

Note that in a quantification $\forall_{\mathsf{p}} x.\phi$ or $\exists_{\mathsf{p}} x.\phi$ the free variable x ranges over $|A|$ and not over Σ-terms. So a term t may only be substituted for the variable x when the definedness of t is valid.

As usual, with

$$A \models \phi$$

it is denoted that for all valuations σ it is the case that $A, \sigma \models \phi$. With $A \mathrel{!{\models}} \phi$ it is denoted that for all valuations σ, $A, \sigma \mathrel{!{\models}} \phi$. In [7, Prop.2.2.1], we prove:

Proposition 1. *The axioms and inference rules for* $\mathsf{Eq}(\mathsf{L}_{\mathsf{sfol}}(\Sigma))$ *given in Sect. 2.1 are valid in all partial* Σ*-algebras.*

A partial equality $t =_{\mathsf{p}} r$ is undefined if at least one of t and r is undefined. In the following we generalise ‘undefinedness’ to $\mathsf{L}_{\mathsf{sfol}}(\Sigma)$-formulae and $\mathsf{Eq}(\mathsf{L}_{\mathsf{sfol}}(\Sigma))$-formulae using an adaptation of the notation for satisfiability. For a partial Σ-algebra A and a valuation σ, we write

$$A, \sigma \uparrow\models \phi$$

to express that ϕ is undefined in A according to σ, and we define this notion inductively:

(1) $A, \sigma \uparrow\models t =_{\mathsf{p}} r$ if, and only if, at least one of both evaluation results $\sigma(t)$ and $\sigma(r)$ is undefined,
(2) $A, \sigma \uparrow\models \neg\phi$ if $A, \sigma \uparrow\models \phi$,

(3) $A, \sigma \uparrow\models \phi_1 \curlyvee \phi_2$ if either (i) $A, \sigma \uparrow\models \phi_1$ or (ii) $A, \sigma\ !\models \phi_1$ and $A, \sigma \uparrow\models \phi_2$,
(4) $A, \sigma \uparrow\models \forall_\mathsf{p} x.\phi$ if for some $a \in |A|$, it is the case that $A, \sigma[a/x] \uparrow\models \phi$.

Some consequences that follow from clause (2) and axiom (e8) (i.e. $\forall_\mathsf{p} x.\phi = \neg\exists_\mathsf{p} x.\neg\phi$):

$$\begin{aligned} &A, \sigma \uparrow\models \phi_1 \curlywedge \phi_2 && \text{iff either (i) } A, \sigma \uparrow\models \phi_1 \text{ or (ii)} A, \sigma \models \phi_1 \text{ and } A, \sigma \uparrow\models \phi_2, \\ &A, \sigma \uparrow\models \forall_\mathsf{p} x.\neg\phi && \text{iff } A, \sigma \uparrow\models \forall_\mathsf{p} x.\phi, \\ &A, \sigma \uparrow\models \exists_\mathsf{p} x.\phi && \text{iff } A, \sigma \uparrow\models \forall_\mathsf{p} x.\phi. \end{aligned}$$

It easily follows that for any formula $\phi \in \mathsf{UL}_\mathsf{sfol}(\Sigma)$ (thus ϕ quantifier-free) and any σ, either $A, \sigma \models \phi$, or $A, \sigma\ !\models \phi$, or $A, \sigma \uparrow\models \phi$.

We write $A \uparrow\models \phi$ if for all valuations σ, $A, \sigma \uparrow\models \phi$. Some examples where we take A to satisfy $0, 1 \in |A|$, $A \models 0 \neq_\mathsf{p} 1$, $A \uparrow\models \frac{1}{0} =_\mathsf{p} t$ for any term t and $A \models \frac{1}{1} =_\mathsf{p} 1$ (properties of a partial meadow):

(i) $A \uparrow\models \frac{1}{0} =_\mathsf{p} 1$, so $A \uparrow\models \frac{1}{0} =_\mathsf{p} 1 \curlyvee 0 \neq_\mathsf{p} 1$, while $A \models 0 \neq_\mathsf{p} 1 \curlyvee \frac{1}{0} =_\mathsf{p} 1$, so $\curlyvee$ is not commutative.
(ii) $A \uparrow\models \forall_\mathsf{p} x.\frac{x}{x} =_\mathsf{p} 1$ because $A \uparrow\models \frac{0}{0} =_\mathsf{p} 1$, and thus also $A \uparrow\models \exists_\mathsf{p} x.\frac{x}{x} =_\mathsf{p} 1$.

More generally, if $A, \sigma \uparrow\models \forall_\mathsf{p} x.\phi$ then neither $A, \sigma \models \forall_\mathsf{p} x.\phi$ nor $A, \sigma \models \forall_\mathsf{p} x.\neg\phi$. This follows by choosing $a \in |A|$ that witnesses $A, \sigma[a/x] \uparrow\models \phi$ (and hence also $A, \sigma[a/x] \uparrow\models \neg\phi$).

For $\mathsf{Eq}(\mathsf{L}_\mathsf{sfol}(\Sigma))$-formulae and a partial Σ-algebra A we define $A, \sigma \models \phi = \psi$ by the following three clauses:

(1) either $A, \sigma \models \phi$ and $A, \sigma \models \psi$,
(2) or $A, \sigma\ !\models \phi$ and $A, \sigma\ !\models \psi$,
(3) or $A, \sigma \uparrow\models \phi$ and $A, \sigma \uparrow\models \psi$.

We write $A \models \phi = \psi$ if $A, \sigma \models \phi = \psi$ for all σ. For example, if A is a partial meadow (like in (i)–(ii) above), we find $A \models (\frac{1}{0} =_\mathsf{p} 1) = (\frac{1}{0} =_\mathsf{p} 0)$. For another example, see Proposition 3.

The following theorem concerns the axiomatisation EqCL in Table 1 and is a minor generalisation of the result cited in [6, Thm.5.2.(ii)] because short-circuit implication $\circ\!\!\rightarrow$ has been added as a definable connective. It immediately follows that this addition preserves that result.

Theorem (Cf. Thm.5.2 in [6]). *Conditional logic with* T *and* F *distinguished is completely axiomatised by the seven axioms (e1)–(e7) of* EqCL *in Table 1. Moreover, these axioms are independent.*

3 Fracterm Calculus for Partial Meadows

In Sect. 3.1 we define 'fracterm calculus for partial meadows', FTCpm in short, and prove a completeness result for FTCpm. In Sect. 3.2, we introduce a convention for concise notation for FTCpm for the sake of readability of axioms and proofs. In Sect. 3.3 we derive some properties of partial meadows and prove some results, among which "conditional flattening", and pay attention to some $\mathsf{Eq}(\mathsf{L}_\mathsf{sfol}(\Sigma))$-identities.

3.1 FTCpm: a Specification

The signature of partial meadows with divisive notation Σ_m^{pd} is obtained by extending the signature of unital rings with a two place division operator (denoted $\frac{x}{y}$), where it is indicated in the signature description that division is a partial function. The sort of numbers involved is named Number.

Table 2. Specification of the signature Σ_m^{pd} of fracterm calculus of partial meadows and a set $\mathsf{AxFTCpm}$ of axioms in the format of $\mathsf{Eq}(\mathsf{L}_{\mathsf{sfol}}(\Sigma_m^{pd}))$

$$
\begin{aligned}
\mathsf{signature} &: \Sigma_{wcr,\perp} = \{ \\
\mathsf{sort} &: \mathsf{Number} \\
\mathsf{constants} &: 0, 1, \perp : \mathsf{Number} \\
\mathsf{total\ functions} &: \_ + \_ \,, \_ \cdot \_ : \mathsf{Number} \times \mathsf{Number} \to \mathsf{Number}; \\
&\quad - \_ : \mathsf{Number} \to \mathsf{Number} \\
\mathsf{equality\ relation} &: \_ = \_ \subseteq \mathsf{Number} \times \mathsf{Number}\} \\
\mathsf{variables} &: x, y, z : \mathsf{Number}
\end{aligned}
$$

$$((x + y) + z =_{\mathsf{p}} x + (y + z)) = \mathsf{T} \tag{pm1a}$$

$$(x + 0 =_{\mathsf{p}} x) = \mathsf{T} \tag{pm2a}$$

$$(x + (-x) =_{\mathsf{p}} 0) = \mathsf{T} \tag{pm3a}$$

$$(x \cdot (y \cdot z) =_{\mathsf{p}} (x \cdot y) \cdot z) = \mathsf{T} \tag{pm4a}$$

$$(x \cdot y =_{\mathsf{p}} y \cdot x) = \mathsf{T} \tag{pm5a}$$

$$(1 \cdot x =_{\mathsf{p}} x) = \mathsf{T} \tag{pm6a}$$

$$(x \cdot (y + z) =_{\mathsf{p}} (x \cdot y) + (x \cdot z)) = \mathsf{T} \tag{pm7a}$$

$$(y \neq_{\mathsf{p}} 0 \multimap \frac{x}{y} =_{\mathsf{p}} x \cdot \frac{1}{y}) = \mathsf{T} \tag{pm8a}$$

$$(x \neq_{\mathsf{p}} 0 \multimap \frac{x}{x} =_{\mathsf{p}} 1) = \mathsf{T} \tag{pm9a}$$

$$(0 \neq_{\mathsf{p}} 1) = \mathsf{T} \tag{pm10a}$$

$$((x \neq_{\mathsf{p}} 0 \wedge y \neq_{\mathsf{p}} 0) \multimap x \cdot y \neq_{\mathsf{p}} 0) = \mathsf{T} \tag{pm11a}$$

Definition 1. *A **partial meadow** is a structure F^{pd} with signature Σ_m^{pd} that is obtained by expanding a field F with a partial division operator (with the usual definition, i.e. $\frac{a}{b} = c$ if $b \neq 0$ and $b \cdot c = a$).*

In Σ_m^{pd} constants are supposed to have a value and functions, except division, are supposed to be total.

Starting from our definition in Sect. 2.1 of a sequential first order logic, we understand 'fracterm calculus for partial meadows', FTCpm in short, as the collection of $\mathsf{UL}_{\mathsf{sfol}}(\Sigma_m^{pd})$-formulae ϕ (thus ϕ quantifier-free), such that ϕ is valid in all partial meadows, i.e. for each partial meadow F^{pd},

$$F^{pd} \models \phi$$

(that is, for all valuations σ, $F^{pd}, \sigma \models \phi$).

A specification of Σ_m^{pd} with a set $\mathsf{AxFTCpm}$ of axioms for FTCpm is given in Table 2. Observe that $(\frac{1}{1} =_{\mathsf{p}} 1) = \mathsf{T}$ follows immediately from axiom (pm9a), which in turn with axiom (pm8a) implies $(\frac{x}{1} =_{\mathsf{p}} x) = \mathsf{T}$. We note that axioms (pm10a) and (pm11a) capture common properties of a field: $0 \neq 1$ and absence of zero divisors.

Theorem 1 (Soundness and completeness of $\mathsf{AxFTCpm}$).

(i) (Soundness) The axioms of $\mathsf{AxFTCpm}$ are sound for the class of partial meadows, and

(ii) (Completeness) If a universal formula $\phi \in \mathsf{UL}_{\mathsf{sfol}}(\Sigma_m^{pd})$ is true in all partial meadows, then $\mathsf{AxFTCpm} \models \phi$.

Table 3. The set $\mathsf{AxFTCpm}$ of axioms according to the first equivalence of (qF) (see page 6)

$$\mathsf{import} : \Sigma_m^{pd} \text{ (Table 2)}$$
$$\mathsf{variables} : x, y, z \colon \mathsf{Number}$$

$$(\forall_{\mathsf{p}} x.\forall_{\mathsf{p}} y.\forall_{\mathsf{p}} z.\ (x+y)+z =_{\mathsf{p}} x+(y+z)) = \mathsf{T} \tag{pm1b}$$
$$(\forall_{\mathsf{p}} x.\ x+0 =_{\mathsf{p}} x) = \mathsf{T} \tag{pm2b}$$
$$(\forall_{\mathsf{p}} x.\ x+(-x) =_{\mathsf{p}} 0) = \mathsf{T} \tag{pm3b}$$
$$(\forall_{\mathsf{p}} x.\forall_{\mathsf{p}} y.\forall_{\mathsf{p}} z.\ x\cdot(y\cdot z) =_{\mathsf{p}} (x\cdot y)\cdot z) = \mathsf{T} \tag{pm4b}$$
$$(\forall_{\mathsf{p}} x.\forall_{\mathsf{p}} y.\ x\cdot y =_{\mathsf{p}} y\cdot x) = \mathsf{T} \tag{pm5b}$$
$$(\forall_{\mathsf{p}} x.\ 1\cdot x =_{\mathsf{p}} x) = \mathsf{T} \tag{pm6b}$$
$$(\forall_{\mathsf{p}} x.\forall_{\mathsf{p}} y.\forall_{\mathsf{p}} z.\ x\cdot(y+z) =_{\mathsf{p}} (x\cdot y)+(x\cdot z)) = \mathsf{T} \tag{pm7b}$$
$$\forall_{\mathsf{p}} x.\forall_{\mathsf{p}} y.(y \neq_{\mathsf{p}} 0 \mathrel{\circ\!\!\longrightarrow} \frac{x}{y} =_{\mathsf{p}} x\cdot\frac{1}{y}) = \mathsf{T} \tag{pm8b}$$
$$(\forall_{\mathsf{p}} x.\ x \neq_{\mathsf{p}} 0 \mathrel{\circ\!\!\longrightarrow} \frac{x}{x} =_{\mathsf{p}} 1) = \mathsf{T} \tag{pm9b}$$
$$(0 \neq_{\mathsf{p}} 1) = \mathsf{T} \tag{pm10b}$$
$$(\forall_{\mathsf{p}} x.\forall_{\mathsf{p}} y.(x \neq_{\mathsf{p}} 0 \mathrel{\circ\!\!\wedge} y \neq_{\mathsf{p}} 0) \mathrel{\circ\!\!\longrightarrow} x\cdot y \neq_{\mathsf{p}} 0) = \mathsf{T} \tag{pm11b}$$

Proof Soundness is obvious by inspection of the axioms of AxFTCpm. For completeness assume that ϕ is valid in all partial meadows. Now consider a model B of AxFTCpm. B is an expansion with a division function of a ring and in fact of a field. B may differ from a partial meadow because the division function may be defined for some pairs of arguments (a, b) with $b = 0$. Now let B' be obtained from B by replacing the division function of B by the standard partial division function. B' is a partial meadow and therefore $B' \models \phi$. It can be shown by induction on the structure of open formulae ψ that for all valuations σ it is the case that $B', \sigma \models \psi$ implies $B \models \psi$. We find that each model of AxFTCpm satisfies ϕ, as required. □

3.2 FTCpm: Concise Notations

To enhance readability, we introduce the following convention.

Convention 1. (1) *We adopt the convention of writing ϕ instead of $\phi = \mathsf{T}$.*
(2) $\forall_\mathsf{p}$*-elimination: for expressions of the form $\forall_p x.\phi$ (with $\phi \in \mathsf{L}_{\mathsf{sfol}}(\Sigma_m^{pd})$) we adopt the convention of writing ϕ.*

If not explicitly mentioned, ϕ here either stands for a syntactic formula (an element of $\mathsf{L}_{\mathsf{sfol}}(\Sigma_m^{pd})$), or for $\vdash \phi$, or for $\models \phi$, and this should then always be clear from the context.

Table 4. Representation of the axioms of AxFTCpm according to Convention 1.(1)

$$\mathsf{import} : \Sigma_m^{pd} \text{ (Table 2)}$$
$$\mathsf{variables} : x, y, z \colon \mathsf{Number}$$

$$\forall_\mathsf{p} x.\forall_\mathsf{p} y.\forall_\mathsf{p} z.\ (x + y) + z =_\mathsf{p} x + (y + z) \tag{pm1c}$$
$$\forall_\mathsf{p} x.\ x + 0 =_\mathsf{p} x \tag{pm2c}$$
$$\forall_\mathsf{p} x.\ x + (-x) =_\mathsf{p} 0 \tag{pm3c}$$
$$\forall_\mathsf{p} x.\forall_\mathsf{p} y.\ x \cdot (y \cdot z) =_\mathsf{p} (x \cdot y) \cdot z \tag{pm4c}$$
$$\forall_\mathsf{p} x.\forall_\mathsf{p} y.\ x \cdot y =_\mathsf{p} y \cdot x \tag{pm5c}$$
$$\forall_\mathsf{p} x.\ 1 \cdot x =_\mathsf{p} x \tag{pm6c}$$
$$\forall_\mathsf{p} x.\forall_\mathsf{p} y.\ x \cdot (y + z) =_\mathsf{p} (x \cdot y) + (x \cdot z) \tag{pm7c}$$
$$\forall_\mathsf{p} x.\forall_\mathsf{p} y.\ y \neq_\mathsf{p} 0 \circ\!\!\longrightarrow \frac{x}{y} =_\mathsf{p} x \cdot \frac{1}{y} \tag{pm8c}$$
$$\forall_\mathsf{p} x.\ x \neq_\mathsf{p} 0 \circ\!\!\longrightarrow \frac{x}{x} =_\mathsf{p} 1 \tag{pm9c}$$
$$0 \neq_\mathsf{p} 1 \tag{pm10c}$$
$$\forall_\mathsf{p} x.\forall_\mathsf{p} y.\ (x \neq_\mathsf{p} 0 \mathbin{\overset{\circ}{\wedge}} y \neq_\mathsf{p} 0) \circ\!\!\longrightarrow x \cdot y \neq_\mathsf{p} 0 \tag{pm11c}$$

Table 5. Representation of the set $\mathsf{AxFTCpm}$ of axioms according to Convention 1

$$\begin{aligned}\mathsf{import}\ &:\ \Sigma_m^{pd}\ \text{(Table 2)}\\ \mathsf{variables}\ &:\ x, y, z\colon \mathsf{Number}\end{aligned}$$

$$(x+y)+z =_{\mathsf{p}} x+(y+z) \tag{pm1}$$

$$x+0 =_{\mathsf{p}} x \tag{pm2}$$

$$x+(-x) =_{\mathsf{p}} 0 \tag{pm3}$$

$$x\cdot(y\cdot z) =_{\mathsf{p}} (x\cdot y)\cdot z \tag{pm4}$$

$$x\cdot y =_{\mathsf{p}} y\cdot x \tag{pm5}$$

$$1\cdot x =_{\mathsf{p}} x \tag{pm6}$$

$$x\cdot(y+z) =_{\mathsf{p}} (x\cdot y)+(x\cdot z) \tag{pm7}$$

$$y \neq_{\mathsf{p}} 0 \mathbin{\circ\!\!\rightarrow} \frac{x}{y} =_{\mathsf{p}} x\cdot\frac{1}{y} \tag{pm8}$$

$$x \neq_{\mathsf{p}} 0 \mathbin{\circ\!\!\rightarrow} \frac{x}{x} =_{\mathsf{p}} 1 \tag{pm9}$$

$$0 \neq_{\mathsf{p}} 1 \tag{pm10}$$

$$(x \neq_{\mathsf{p}} 0 \mathbin{\circ\!\!\wedge} y \neq_{\mathsf{p}} 0) \mathbin{\circ\!\!\rightarrow} x\cdot y \neq_{\mathsf{p}} 0 \tag{pm11}$$

In Tables 3, 4 and 5, we display the axioms of $\mathsf{AxFTCpm}$ in different, equivalent formats. Table 2 represents the $\forall_{\mathsf{p}}$-elimination of the axioms of Table 3, and Table 5 shows the $\forall_{\mathsf{p}}$-elimination of the axioms of Table 4. Of course, Convention 1 is justified by the Tarski-semantics discussed in Sect. 2.2.

As an example, we give for axiom (pm9a) of Table 2 a diagram that illustrates this convention, using the first equivalence of (qF), i.e. $F^{pd} \models \phi = \mathsf{T} \iff F^{pd} \models (\forall_{\mathsf{p}} x.\phi) = \mathsf{T}$ (see page 6), in which the prefix "$F^{pd} \models$" is omitted:

$$\begin{array}{llcll}
\text{(Tbl.2)} & (x \neq_{\mathsf{p}} 0 \mathbin{\circ\!\!\rightarrow} \frac{x}{x} =_{\mathsf{p}} 1) = \mathsf{T} & \iff & (\forall_{\mathsf{p}} x. x \neq_{\mathsf{p}} 0 \mathbin{\circ\!\!\rightarrow} \frac{x}{x} =_{\mathsf{p}} 1) = \mathsf{T} & \text{(Tbl.3)}\\
 & \downarrow & & \downarrow & \\
\text{(Tbl.5)} & x \neq_{\mathsf{p}} 0 \mathbin{\circ\!\!\rightarrow} \frac{x}{x} =_{\mathsf{p}} 1 & \twoheadleftarrow & \forall_{\mathsf{p}} x. x \neq_{\mathsf{p}} 0 \mathbin{\circ\!\!\rightarrow} \frac{x}{x} =_{\mathsf{p}} 1 & \text{(Tbl.4)}
\end{array}$$

Convention 1 and these examples relate to both the axioms in Sect. 2.1 and those for partial meadows. In what follows, we will work mostly according to this convention and the axioms in Table 5, but where practical, we will use more explicit representation. A typical example is the following consequence:

If t is defined, thus $t =_{\mathsf{p}} t$, and $\forall_{\mathsf{p}} x.\phi(x)$ is a valid assertion, then so is $\phi(t)$.

This 'instantiation consequence' is justified by axiom (a1), i.e. $((t =_{\mathsf{p}} t \mathbin{\circ\!\!\wedge} \forall_{\mathsf{p}} x.\phi) \mathbin{\circ\!\!\rightarrow} \phi[t/x]) = \mathsf{T}$. An example of this consequence applied to axiom (pm11) and the (derivable) identity $-1 =_{\mathsf{p}} -1$ is $(-1 \neq_{\mathsf{p}} 0 \mathbin{\circ\!\!\wedge} y \neq_{\mathsf{p}} 0) \mathbin{\circ\!\!\rightarrow} -1\cdot y \neq_{\mathsf{p}} 0$.

We end this section with a final remark on the axioms for partial meadows. Observe that the condition $(x \neq_{\mathsf{p}} 0 \mathbin{\&} y \neq_{\mathsf{p}} 0)$ of axiom (pm11a) in Table 2 and in its related versions in Tables 3–5 can be replaced by $(y \neq_{\mathsf{p}} 0 \mathbin{\&} x \neq_{\mathsf{p}} 0)$.

3.3 Some Consequences of FTCpm and $\mathsf{Eq}(\mathsf{L}_{\mathsf{sfol}}(\Sigma_m^{pd}))$

Table 6 lists some familiar consequences (in the sense of $\models$) of the axioms of AxFTCpm in the format of Table 5. Note that the order of the three conjuncts in Assertion (A6) is not relevant.

Table 6. Assertions of fracterm calculus of partial meadows (with $x, y, u, v \in V_{var}$)

$$x + y =_{\mathsf{p}} y + x \tag{A1}$$

$$0 \cdot x =_{\mathsf{p}} 0 \tag{A2}$$

$$x \neq_{\mathsf{p}} 0 \hookrightarrow -x \neq_{\mathsf{p}} 0 \tag{A3}$$

$$y \neq_{\mathsf{p}} 0 \hookrightarrow -\frac{x}{y} =_{\mathsf{p}} \frac{-x}{y} \tag{A4}$$

$$(y \neq_{\mathsf{p}} 0 \mathbin{\&} v \neq_{\mathsf{p}} 0) \hookrightarrow \frac{x}{y} \cdot \frac{u}{v} =_{\mathsf{p}} \frac{x \cdot u}{y \cdot v} \tag{A5}$$

$$(y \neq_{\mathsf{p}} 0 \mathbin{\&} u \neq_{\mathsf{p}} 0 \mathbin{\&} v \neq_{\mathsf{p}} 0) \hookrightarrow \frac{(\frac{x}{y})}{(\frac{u}{v})} =_{\mathsf{p}} \frac{x \cdot v}{y \cdot u} \tag{A6}$$

$$(y \neq_{\mathsf{p}} 0 \mathbin{\&} v \neq_{\mathsf{p}} 0) \hookrightarrow \frac{x}{y} + \frac{u}{v} =_{\mathsf{p}} \frac{(x \cdot v) + (y \cdot u)}{y \cdot v} \tag{A7}$$

Proposition 2. *The assertions (A1)–(A7) in Table 6 follow from* AxFTCpm*.*

Proof. See [7, Prop.3.3.1]. □

Definition 2. *A **flat fracterm** is an expression of the form $\frac{p}{q}$ that contains precisely one occurrence (i.e. the top level occurrence) of the division operator, thus p and q are **division free** terms.*

Theorem 2 (Conditional fracterm flattening for partial meadows). *For each term t there is a division free term s and a flat fracterm r such that*

(i) $s \neq_{\mathsf{p}} 0 \hookrightarrow t =_{\mathsf{p}} r$ holds in all partial meadows, and
(ii) $s =_{\mathsf{p}} 0 \mathbin{\&} t =_{\mathsf{p}} t$ does not hold in any partial meadow under any valuation.

In [7, Thm.3.3.3], this theorem is proven by induction on the structure of t (which may be an open term). When using the same notation as in the theorem, except that we write $\frac{p}{q}$ for r (thus, p and q division free), the following equation is not valid unless t is always defined: $(t =_{\mathsf{p}} \frac{p}{q}) = \mathsf{T}$. Also, $(t =_{\mathsf{p}} \frac{p \cdot s}{q \cdot s}) = \mathsf{T}$ fails.

To see this complication, notice that for instance $(\frac{1}{x} =_{\mathsf{p}} \frac{1}{x}) = \mathsf{T}$ fails, while $(x \neq_{\mathsf{p}} 0 \circ\!\!\rightarrow \frac{1}{x} =_{\mathsf{p}} \frac{1}{x}) = \mathsf{T}$ holds. Hence, we also find the following $\mathsf{Eq}(\mathsf{L}_{\mathsf{sfol}}(\Sigma_m^{pd}))$ identities:

$$(s \neq_{\mathsf{p}} 0 \circ\!\!\rightarrow x =_{\mathsf{p}} t) = (s \neq_{\mathsf{p}} 0 \circ\!\!\rightarrow x =_{\mathsf{p}} \frac{p}{q}),$$
$$(s =_{\mathsf{p}} 0 \circ\!\!\rightarrow x =_{\mathsf{p}} t) = (s =_{\mathsf{p}} 0 \circ\!\!\rightarrow 0 =_{\mathsf{p}} \frac{1}{0}).$$

In $\mathsf{Eq}(\mathsf{L}_{\mathsf{sfol}}(\Sigma_m^{pd}))$, conditional fracterm flattening can be obtained in a more direct manner.

Proposition 3. *For each term t there is a division free term s and a flat fracterm $\frac{p}{q}$ such that the $\mathsf{Eq}(\mathsf{L}_{\mathsf{sfol}}(\Sigma_m^{pd}))$-identity*

$$(x =_{\mathsf{p}} t) = (x =_{\mathsf{p}} \tfrac{p \cdot s}{q \cdot s})$$

is valid in all partial meadows.

Proof. Given t, let s and $r = \frac{p}{q}$ be as found in Theorem 2. Given a partial meadow F^{pd} and a valuation σ, two cases are distinguished:

$$F^{pd}, \sigma \models s \neq_{\mathsf{p}} 0 \quad \text{and} \quad F^{pd}, \sigma \models s =_{\mathsf{p}} 0.$$

In the first case, $F^{pd}, \sigma \models t =_{\mathsf{p}} \frac{p}{q}$ and $F^{pd}, \sigma \models \frac{s}{s} =_{\mathsf{p}} 1$ so that $F^{pd}, \sigma \models t =_{\mathsf{p}} \frac{p \cdot s}{q \cdot s}$, in the second case, both t and $\frac{p \cdot s}{q \cdot s}$ are undefined in F^{pd} under valuation σ. □

In $\mathsf{Eq}(\mathsf{L}_{\mathsf{sfol}}(\Sigma_m^{pd}))$, the equation $(\frac{1}{0} =_{\mathsf{p}} \frac{1}{0}) = (\frac{1}{0} \neq_{\mathsf{p}} \frac{1}{0})$ expresses that division is not total. However, this cannot be expressed in $\mathsf{L}_{\mathsf{sfol}}(\Sigma_m^{pd})$:

Proposition 4. *It is impossible to express in* FTCpm*, thus by formulae in* $\mathsf{L}_{\mathsf{sfol}}(\Sigma_m^{pd})$*, that division is not total.*

Proof. Suppose that $F^{pd} \models \phi$ for some partial meadow F^{pd}, then we may totalise division in accordance with the Suppes-Ono convention $\frac{x}{0} = 0$, thereby obtaining $\mathsf{Tot}_0(F^{pd})$. Now we claim that $\mathsf{Tot}_0(F^{pd}) \models \phi$. With simultaneous induction on the structure of ϕ one easily proves that: (i) if $F^{pd} \models \phi$ then $\mathsf{Tot}_0(F^{pd}) \models \phi$ and (ii) if $F^{pd} \models \neg\phi$ then $\mathsf{Tot}_0(F^{pd}) \models \neg\phi$. It follows that no formula ϕ can distinguish between F^{pd} and $\mathsf{Tot}_0(F^{pd})$ by being true for F^{pd} and not for $\mathsf{Tot}_0(F^{pd})$. □

Two assertions in $\mathsf{L}_{\mathsf{sfol}}(\Sigma_m^{pd})$ that involve $\exists_{\mathsf{p}}$ and hold in all partial meadows are these:

$$x \neq_{\mathsf{p}} 0 \circ\!\!\rightarrow \exists_{\mathsf{p}} y.(x \cdot y =_{\mathsf{p}} 1), \tag{1}$$
$$x \neq_{\mathsf{p}} 0 \circ\!\!\rightarrow \exists_{\mathsf{p}} y.(y \neq_{\mathsf{p}} 0 \mathbin{\circ\!\!\wedge} x =_{\mathsf{p}} \frac{1}{y}). \tag{2}$$

Note that (1) is related to axiom (pm11) and that in (2), the conjunct $y \neq_{\mathsf{p}} 0$ cannot be omitted.

In the case of $F^{pd} = \mathbb{Q}^{pd}$, the partial meadow of rationals, we find a computable partial algebra. A specification of the abstract partial data type of rationals is given in Table 7, where the notation x^2 in (3) abbreviates $x \cdot x$.

Proposition 5. *The axioms in Table 7 are satisfied in $\mathbb{Q}^{pd}$ and prove each closed equation and inequation that is true in $\mathbb{Q}^{pd}$.*

Table 7. AxFTCpm/4sq: A specification of the partial meadow of rationals

import : AxFTCpm (Table 5)
variables : x, y, z, u: Number

$$1 + ((x^2 + y^2) + (z^2 + u^2)) \neq_{\mathsf{p}} 0 \qquad (3)$$

4 ⊥-Enlargements and Consequence Relations

In Sects. 4.1 and 4.2 we define a notion of 'enlargement' in order to connect partial algebras and their logic to ordinary first order logic with an absorptive element ⊥ that models partiality. In Sect. 4.3 we provide alternative notions of computability for partial algebras, though limited to the case of minimal partial algebras. We will focus on minimal algebras only and then with much simpler definitions.

4.1 ⊥-Enlargement and Its Converse

Let the *absorptive element* ⊥ be a new constant symbol, intended to represent "no proper value" (i.e. $t \neq \bot$ corresponds to t being defined) and let the set of first order conditional formulas $T_\bot$ contain the assertions that express that functions produce ⊥ on any series of arguments involving ⊥. For instance, for a three place function f, $T_\bot$ contains

$$f(\bot, y, z) = \bot, \quad f(x, \bot, z) = \bot, \quad f(x, y, \bot) = \bot.$$

Given a partial algebra A, its ⊥-enlargement $\mathsf{Enl}_\bot(A)$ is obtained by extending the domain with a new element, also denoted ⊥, that serves as the interpretation of ⊥. The notation $\mathsf{Enl}_\bot(A)$ is taken from [11].

In the opposite direction, given a total algebra B that contains an absorptive element ⊥, and such that $B \models \exists x. x \neq \bot$, the operation $\mathsf{Pdt}_\bot$ (partial data type) as introduced in [11] creates a partial algebra $\mathsf{Pdt}_\perp(A)$ with $\perp$ removed from the domain and each operation which produces $\perp$ on some arguments made partial on these arguments. The name $\mathsf{Pdt}_\perp$ suggests that the resulting algebra is minimal, a requirement on all data types. Such was the intention in [11]. We will use the same notation also in the more general case where the resulting structure need not be minimal.

Proposition 6. *If* $\perp \notin \Sigma(A)$ *and* $\perp \notin |A|$ *then* $\mathsf{Pdt}_\perp(\mathsf{Enl}_\perp(A)) = A$.

Proposition 7. *If* $\perp \in \Sigma(A)$ *and* $\perp \in |A|$ *with* $\mathsf{card}(|A|) > 1$ *then*

$$\mathsf{Enl}_\perp(\mathsf{Pdt}_\perp(A)) = A.$$

4.2 Reformulating the Semantics of $\mathsf{Eq}(\mathsf{L}_{\mathsf{sfol}}(\Sigma))$ in First Order Terms

Assuming $\perp \notin \Sigma$, let $\Sigma_\perp = \Sigma \cup \{\perp\}$. A pair of transformations ψ_{true} and ψ_{false} translates formulae in $\mathsf{L}_{\mathsf{sfol}}(\Sigma)$ to first order formulae over $\Sigma_\perp$, i.e. to $\mathsf{L}_{\mathsf{fol}}(\Sigma_\perp)$. The transformation ψ_{true} translates formulae that are assumed to evaluate to true, and ψ_{false} is used as an auxiliary operator to deal with negation (i.e. $\psi_{\mathsf{true}}(\neg\phi) \equiv \psi_{\mathsf{false}}(\phi)$):

(1) $\psi_{\mathsf{true}}(\mathsf{T}) \equiv \mathsf{T}$ and $\psi_{\mathsf{true}}(\mathsf{F}) \equiv \mathsf{F}$,
(2) $\psi_{\mathsf{false}}(\mathsf{T}) \equiv \mathsf{F}$ and $\psi_{\mathsf{false}}(\mathsf{F}) \equiv \mathsf{T}$,
(3) $\psi_{\mathsf{true}}(t =_{\mathsf{p}} r) \equiv t \neq \perp \wedge r \neq \perp \wedge t = r$ (where $x \neq y$ abbreviates $\neg(x = y)$),
(4) $\psi_{\mathsf{false}}(t =_{\mathsf{p}} r) \equiv t = \perp \vee r = \perp \vee t \neq r$,
(5) $\psi_{\mathsf{true}}(\neg\phi) \equiv \psi_{\mathsf{false}}(\phi)$ (hence, $\psi_{\mathsf{true}}(t \neq_{\mathsf{p}} r) \equiv \psi_{\mathsf{false}}(t =_{\mathsf{p}} r)$),
(6) $\psi_{\mathsf{false}}(\neg\phi) \equiv \psi_{\mathsf{true}}(\phi)$ (hence, $\psi_{\mathsf{false}}(t \neq_{\mathsf{p}} r) \equiv \psi_{\mathsf{true}}(t =_{\mathsf{p}} r)$),
(7) $\psi_{\mathsf{true}}(\phi_1 \mathbin{{}^{\circ}\!\vee} \phi_2) \equiv \psi_{\mathsf{true}}(\phi_1) \vee (\psi_{\mathsf{false}}(\phi_1) \wedge \psi_{\mathsf{true}}(\phi_2))$,
(8) $\psi_{\mathsf{false}}(\phi_1 \mathbin{{}^{\circ}\!\vee} \phi_2) \equiv \psi_{\mathsf{false}}(\phi_1) \wedge \psi_{\mathsf{false}}(\phi_2)$,
(9) $\psi_{\mathsf{true}}(\forall_{\mathsf{p}} x.\phi) \equiv \forall x.(x \neq \perp \rightarrow \psi_{\mathsf{true}}(\phi))$,
(10) $\psi_{\mathsf{false}}(\forall_{\mathsf{p}} x.\phi) \equiv \exists x.(x \neq \perp \wedge \psi_{\mathsf{false}}(\phi)) \wedge \forall x.\psi_{\mathsf{true}}(\phi \mathbin{{}^{\circ}\!\vee} \neg\phi)$.

It follows easily that

$$\begin{aligned}
&\psi_{\mathsf{true}}(\phi_1 \mathbin{{}_{\circ}\!\wedge} \phi_2) \equiv \psi_{\mathsf{true}}(\phi_1) \wedge \psi_{\mathsf{true}}(\phi_2),\\
&\psi_{\mathsf{false}}(\phi_1 \mathbin{{}_{\circ}\!\wedge} \phi_2) \equiv \psi_{\mathsf{false}}(\phi_1) \vee (\psi_{\mathsf{true}}(\phi_1) \wedge \psi_{\mathsf{false}}(\phi_2)),\\
&\psi_{\mathsf{true}}(\exists_{\mathsf{p}} x.\phi) \equiv \exists x.(x \neq \perp \wedge \psi_{\mathsf{true}}(\phi)) \wedge \forall x.\psi_{\mathsf{true}}(\phi \mathbin{{}^{\circ}\!\vee} \neg\phi),\\
&\psi_{\mathsf{false}}(\exists_{\mathsf{p}} x.\phi) \equiv \forall x.(x \neq \perp \rightarrow \psi_{\mathsf{false}}(\phi)).
\end{aligned}$$

In order to formulate key properties of the operator ψ_{true}, a consequence relation must be chosen.

4.3 Consequence Relations

Various consequence relations can be contemplated in the context of 3-valued logics. For an extensive discussion of these options we refer to [20]. We will consider the so-called *strong validity* consequence relation, notation $\models_{\mathsf{ss}}$, where both in the assumptions and in the conclusion the formulae (sentences) are considered "true" if these are valid under all valuations. Alternatively one may consider $\models_{\mathsf{sw}}$, $\models_{\mathsf{ws}}$, and $\models_{\mathsf{ww}}$ where w indicates weak validity, that is for no valuation a formula is false. We propose that in the case of elementary arithmetic, the use of $\models_{\mathsf{ss}}$ is preferable to the three alternatives just mentioned. Below we will write $\models_{\Sigma}$ instead of $\models_{\mathsf{ss}}$ in order to highlight the role and relevance of the signature involved.

Definition 3. *Define $\phi_1, ..., \phi_n \models_{\Sigma} \phi$ if, and only if, for each Σ-structure A: if for all valuations σ into $|A|$ it is the case that $A, \sigma \models_{\Sigma} \phi_1$, ..., $A, \sigma \models_{\Sigma} \phi_n$, then for all valuations σ into $|A|$, $A, \sigma \models_{\Sigma} \phi$.*

A connection between the various satisfaction relations and the transformations ψ_{true} and ψ_{false} is found under some restrictions.

Proposition 8. *For any partial Σ-algebra A, for each $\mathsf{L}_{\mathsf{sfol}}(\Sigma)$ formula ϕ and for each valuation σ taking values in $|A|$: $A, \sigma \models_{\Sigma} \phi$ if, and only if, $\mathsf{Enl}_{\bot}(A), \sigma \models \psi_{\mathsf{true}}(\phi)$.*
Proof. Straightforward by induction on the structure of ϕ. □

In the following, let A be a partial Σ-algebra and B a total Σ-algebra with $\bot \in \Sigma(B)$.

Proposition 9. *$A \models_{\Sigma} \phi$ if, and only if, $\mathsf{Enl}_{\bot}(A) \models \psi_{\mathsf{true}}(\phi)$.*
Proof. Immediate using Proposition 8. □
Proposition 10. *Let $c \in \Sigma(B)$, and assume that $B \models c \neq \bot$. Then $\mathsf{Pdt}_{\bot}(B) \models_{\Sigma} \phi$ if, and only if, $B \models \psi_{\mathsf{true}}(\phi)$.*
Proposition 11. *$\phi_1, ..., \phi_n \models_{\Sigma} \phi$ if, and only if,*

$$T_{\bot} \cup \{\psi_{\mathsf{true}}(\phi_1), ..., \psi_{\mathsf{true}}(\phi_n)\} \cup \{\exists x.x \neq \bot\} \models \psi_{\mathsf{true}}(\phi).$$

Proof. For "if", assume that $A \models_{\Sigma} \phi_1, ..., A \models_{\Sigma} \phi_n$, then by Proposition 9,

$$\mathsf{Enl}_{\bot}(A) \models \psi_{\mathsf{true}}(\phi_1), ..., \mathsf{Enl}_{\bot}(A) \models \psi_{\mathsf{true}}(\phi_n).$$

Because A has a non-empty domain, $A \models_{\Sigma} \exists_{\mathsf{p}} x.x \neq \bot$. Because $\mathsf{Enl}_{\bot}(A) \models T_{\bot}$ it follows with $T_{\bot} \cup \{\psi_{\mathsf{true}}(\phi_1), ..., \psi_{\mathsf{true}}(\phi_n)\} \models \psi_{\mathsf{true}}(\phi)$ that $\mathsf{Enl}_{\bot}(A) \models \psi_{\mathsf{true}}(\phi)$. Now using Proposition 9, $A \models_{\Sigma} \phi$.

For the other direction assume that $B \models T_{\bot} \cup \{\psi_{\mathsf{true}}(\phi_1), ..., \psi_{\mathsf{true}}(\phi_n)\} \cup \{\exists x.x \neq \bot\}$. Then $\mathsf{Pdt}_{\bot}(B)$ has a non-empty domain so that $A = \mathsf{Pdt}(B)$ is well-defined, and with Proposition 7, $B = \mathsf{Enl}_{\bot}(A)$. It follows with Proposition 8 that $A \models_{\Sigma} \phi_1, ..., A \models_{\Sigma} \phi_n$ so that $A \models_{\Sigma} \phi$ from which one obtains $B \models \psi_{\mathsf{true}}(\phi)$ with Proposition 9. □
Proposition 12. *The consequence relation $\phi_1, ..., \phi_n \models_{\Sigma} \phi$ is semi-computable.*
Proof. From Proposition 11 it follows that the consequence at hand is effectively 1-1 reducible to an instance of consequence from a semi-computable first order theory, which is known to be semi-computable. □

5 Fracterm Calculus for Common Meadows

In Sect. 5.1, we recall *common meadows* and a fracterm calculus for these, FTCcm. In Sect. 5.2, we establish that the axiomatisation of FTCcm is equivalent to the transformation $\psi_{\mathsf{true}}(\mathsf{AxFTCpm})$ with $\mathsf{AxFTCpm}$ as shown in Table 4.

5.1 FTCcm, a Specification

Fracterm calculus for common meadows, FTCcm, starts by involving the absorptive element $\bot$ and by assuming $\frac{x}{0} = \bot$.

Following [12], we adopt a modular approach and first consider $\mathsf{Enl}_\bot(R)$, the enlargement of a commutative unital ring R with $\bot$, with axioms in Table 8.

Table 8. Specification of commutative unital $\bot$-rings, with a set $E_{wcr,\bot}$ of axioms

$$
\begin{aligned}
\mathsf{signature} &: \Sigma_{wcr,\bot} = \{ \\
\mathsf{sort} &: \mathsf{Number} \\
\mathsf{constants} &: 0, 1, \bot \colon \mathsf{Number} \\
\mathsf{total\ functions} &: \_ + \_\,,\ \_ \cdot \_ : \mathsf{Number} \times \mathsf{Number} \to \mathsf{Number}; \\
&\quad - \_ \colon \mathsf{Number} \to \mathsf{Number} \\
\mathsf{equality\ relation} &: \_ = \_ \subseteq \mathsf{Number} \times \mathsf{Number}\} \\
\mathsf{variables} &: x, y, z \colon \mathsf{Number}
\end{aligned}
$$

$$(x + y) + z = x + (y + z) \quad \text{(c1)}$$
$$x + y = y + x \quad \text{(c2)}$$
$$x + 0 = x \quad \text{(c3)}$$
$$x + (-x) = 0 \cdot x \quad \text{(c4)}$$
$$(x \cdot y) \cdot z = x \cdot (y \cdot z) \quad \text{(c5)}$$
$$x \cdot y = y \cdot x \quad \text{(c6)}$$
$$1 \cdot x = x \quad \text{(c7)}$$
$$x \cdot (y + z) = (x \cdot y) + (x \cdot z) \quad \text{(c8)}$$
$$-(-x) = x \quad \text{(c9)}$$
$$x + \bot = \bot \quad \text{(c10)}$$
$$0 \cdot (x \cdot x) = 0 \cdot x \quad \text{(c11)}$$

Since $\bot$ is absorptive, it follows that $-\bot = \bot + x = \bot \cdot x = \bot$, so the familiar ring identities $0 \cdot x = 0$ and $x + (-x) = 0$ are not valid, but the weaker identities

$0 \cdot (x \cdot x) = 0 \cdot x$ and $x + (-x) = 0 \cdot x$ are. In [12, Thm.2.1], the set of axioms $E_{wcr,\perp}$ is defined and it is shown that for each equation $t = r$ over $\Sigma(R) \cup \{\perp\}$,

$$E_{wcr,\perp} \vdash t = r \iff \mathsf{Enl}_\perp(R) \models t = r.$$

For example, $E_{wcr,\perp} \vdash 0 \cdot (x+y) = 0 \cdot (x \cdot y)$ (with $0 \cdot (x+y) = 0 \cdot ((x+y) \cdot (x+y))$ this follows easily). With *Mace4* [23], it quickly follows that the axioms of E_{wcr} are logically independent.

Table 9. Specification of the fracterm calculus of common meadows, with a set AxFTCcm of axioms

$$\begin{aligned} \mathsf{import} &: E_{wcr,\perp} \setminus \{(\text{c11})\} \text{ (Table 8)} \\ \mathsf{signature} &: \Sigma^d_{md,\perp} = \Sigma_{wcr,\perp} \cup \{ \\ \mathsf{total\ functions} &: \frac{-}{-} : \mathsf{Number} \times \mathsf{Number} \to \mathsf{Number}\} \\ \mathsf{variables} &: x, y : \mathsf{Number} \end{aligned}$$

$$\frac{x}{y} = x \cdot \frac{1}{y} \tag{cm1}$$

$$\frac{x}{x} = 1 + \frac{0}{x} \tag{cm2}$$

$$\frac{1}{x \cdot y} = \frac{1}{x} \cdot \frac{1}{y} \tag{cm3}$$

$$\frac{1}{1 + (0 \cdot x)} = 1 + (0 \cdot x) \tag{cm4}$$

$$\perp = \frac{1}{0} \tag{cm5}$$

Definition 4. *A* ***common meadow*** *is an enlargement $F_\perp$ of a field F, which results by first extending the domain with an absorptive element $\perp$ and then expanding the structure thus obtained with a constant $\perp$ (for said absorptive element) and a division function which is made total by adopting*

$$\frac{x}{0} = \frac{x}{\perp} = \frac{\perp}{x} = \perp.$$

A common meadow provides arguably the most straightforward way to turn division into a total operator. The fracterm calculus of common meadows (as discussed in [4] and in [5]) has many different axiomatisations, see e.g. [12]. Here we combine axioms (c1)–(c10) of $E_{wcr,\perp}$ (Table 8) and axioms (cm1)–(cm5) of Table 9 that define division as a total function for each structure containing $\perp$

as an absorptive element. Table 9 lists a set AxFTCcm of axioms for FTCcm following the presentation of [4], though using fracterms instead of inverse notation x^{-1} for $\frac{1}{x}$.

With *Prover9* [23] it quickly follows that axiom (c11) is derivable from AxFTCcm, which, according to *Mace4* [23], is a set of independent axioms. Also with *Prover9*, the following familiar consequences of AxFTCcm are quickly derived:

$$\frac{x}{1} = x,\ -\frac{x}{y} = \frac{-x}{y} = \frac{x}{-y},\ \frac{x}{y} \cdot \frac{u}{v} = \frac{x \cdot u}{y \cdot v},\ \text{ and }\ \frac{x}{y} + \frac{u}{v} = \frac{x \cdot v + y \cdot u}{y \cdot v}\ .$$

The axioms of AxFTCcm allow fracterm flattening: each expression can be proven equal to a flat fracterm. This was first shown in [4, Prop.2.2.3] with inverse notation x^{-1} for $\frac{1}{x}$ (conversely, division can be defined by $\frac{x}{y} = x \cdot y^{-1}$).

5.2 ⊥-Enlargement: Application of $\psi_{\mathsf{true}}()$ to FTCpm

First, we establish some properties of common meadows. By Definition 4,

$$0 \neq \bot \quad \text{and} \quad 1 \neq \bot, \tag{4}$$

$$-(\bot) = \bot,\ x + \bot = \bot + x = \bot,\ x \cdot \bot = \bot \cdot x = \bot,\ \text{and}\ \frac{x}{\bot} = \frac{\bot}{x} = \bot, \tag{5}$$

$$\frac{x}{0} = \bot. \tag{6}$$

Moreover, since ⊥ is absorbing, it easily follows that

$$\begin{aligned} &-x = \bot \rightarrow x = \bot,\\ &x + y = \bot \rightarrow (x = \bot \vee y = \bot), \quad \text{and}\\ &x \cdot y = \bot \rightarrow (x = \bot \vee y = \bot). \end{aligned} \tag{7}$$

The following theorem is proven in [7, Thm.5.2.1] and implies that our axiomatisation of AxFTCpm of partial meadows is sufficiently strong. We write $\mathsf{AxFTCpm}^{\mathsf{cl}}$ for the axioms of partial meadows as represented in Table 4, thus with all universal quantifications made explicit, e.g.

$$\forall_{\mathsf{p}} x. \forall_{\mathsf{p}} y. \forall_{\mathsf{p}} z. (x + y) + z =_{\mathsf{p}} x + (y + z).$$

Theorem 3. *With* (4)–(7) *it follows that* $\psi_{\mathsf{true}}(\mathsf{AxFTCpm}^{\mathsf{cl}}) \vdash \mathsf{AxFTCcm}$ *and that* $\psi_{\mathsf{true}}(\mathsf{AxFTCpm}^{\mathsf{cl}})$ *axiomatises a common meadow.*

6 Concluding Remarks

Short-circuit logics (SCLs) were introduced in [8] and are distinguished by sequential connectives that prescribe left-to-right (sequential) evaluation of their operands, in particular, $\mathsf{F} \mathbin{\circ\!\!\wedge} x = \mathsf{F}$, while $x \mathbin{\circ\!\!\wedge} \mathsf{F} = \mathsf{F}$ is not necessarily true (in the case of a partial meadow, take $(\frac{1}{0} =_{\mathsf{p}} 0)$ for x).

Depending on the strength of possible atomic side-effects, different *short-circuit* logics were defined and axiomatised, both for the two-valued and three-valued case, see [9,24]. In [6], three-valued CℓSCL is introduced, which differs from Guzmán and Squier's Conditional logic [18] only by the use of sequential connectives. CℓSCL with only the constants T and F and none for the value undefined (CℓSCL$_2$) is also introduced in [6] and most closely resembles propositional logic: side-effects are not modelled and full left-sequential conjuction $\mathbin{\bullet\!\!\wedge}$, definable by $x \mathbin{\bullet\!\!\wedge} y = (x \mathbin{\circ\!\!\wedge} y) \mathbin{\circ\!\!\vee} (y \mathbin{\circ\!\!\wedge} x)$, is commutative. Moreover, adding $x \mathbin{\circ\!\!\wedge} \mathsf{F} = \mathsf{F}$ to CℓSCL$_2$ yields a sequential version of propositional logic and excludes the use of a third truth value undefined (because with a constant U for undefined, it would follow that $\mathsf{U} = \mathsf{U} \mathbin{\circ\!\!\wedge} \mathsf{F} = \mathsf{F}$).

Starting from CℓSCL$_2$ and partial equality ($=_{\mathsf{p}}$), we here introduced partial meadows together with axioms and rules split into a part for general, partial Σ-algebras (using rules for weak substitution from [3]) and a part specific to partial meadows (with signature Σ_m^{pd}). We have taken a pragmatic approach and included only axioms and rules that were used in our proofs. The new quantifiers $\forall_{\mathsf{p}}$ and $\exists_{\mathsf{p}}$ were introduced for readability and comprehensibility, but could have been replaced by their familiar counterparts $\forall$ and $\exists$.

It is an open question whether the axioms of FTCpm (AxFTCpm in Table 5) are independent. It is certainly the case that axioms (pm1)–(pm7) are independent (*Mace4*) and imply $x + y =_{\mathsf{p}} y + x$ (Proposition 2), and it seems that the two axioms (pm8) and (pm9) for division, i.e.

$$y \neq_{\mathsf{p}} 0 \mathrel{\circ\!\!\rightarrow} \frac{x}{y} =_{\mathsf{p}} x \cdot \frac{1}{y} \quad \text{and} \quad x \neq_{\mathsf{p}} 0 \mathrel{\circ\!\!\rightarrow} \frac{x}{x} =_{\mathsf{p}} 1,$$

are both mutually independent and also from the first seven. Axioms (pm10) and (pm11), i.e. $0 \neq_{\mathsf{p}} 1$ and $(x \neq_{\mathsf{p}} 0 \mathbin{\circ\!\!\wedge} y \neq_{\mathsf{p}} 0) \mathrel{\circ\!\!\rightarrow} x \cdot y \neq_{\mathsf{p}} 0$, express that a partial meadow is an expansion of a field, and their independence is not clear.

We expect that an extension of AxFTCpm can be designed, including its proof system, for which a suitable completeness theorem can be obtained so that it becomes rewarding to investigate the model theory of these axioms, perhaps in a manner comparable to [15,16] where the model theory of the axioms for common meadows has been worked out in considerable detail.

Partial data types for arithmetic arise in different ways, for instance the transreals of Anderson et al. [1,25] can be turned into partial transreals where division is more often defined than in a partial meadow ($\frac{1}{0} = +\infty$, and $\frac{-1}{0} = -\infty$, while $\frac{0}{0}$ is undefined) and where addition and multiplication are partial: $\infty + (-\infty)$ and $0 \cdot \infty$ are undefined (while in transreals $\infty + (-\infty) = 0 \cdot \infty = \Phi$). Just as in a common meadow where $\frac{1}{0} = \bot$ which we replace by undefined to obtain a partial meadow, viewing Φ as a representation of undefinedness leads to partial transreals and to its substructure of partial transrationals.

The entropic transreals of [13] are a modification of transreals which can be turned into partial entropic transreals where division and multiplication are total while addition is partial. Again the idea is to have t undefined (in the partial entropic transreals) in case $t = \bot$ in entropic transreals. In the partial entropic transreals one has $\frac{1}{0} = +\infty$, $\frac{0}{0} = 0 \cdot \infty = 0$, while $\infty + (-\infty)$ is undefined.

Wheels (see Carlström's [14]) can be made partial also by having t undefined if $t = \bot$ in a wheel. In a partial wheel division, addition, and multiplication are each partial. We notice that a wheel contains a single unsigned infinite element ∞ rather than a pair of signed infinite elements $+\infty$ and $-\infty$ (as in transreals and entropic transreals) and that in a partial wheel, $\infty + \infty$ is undefined.

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