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# Fully evaluated left-sequential logics

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**ABSTRACT**

We consider a family of two-valued 'fully evaluated left-sequential logics' (FELs), of which Free FEL (defined by Staudt in 2012) is weakest and immune to atomic side effects. Next is Memorising FEL, in which evaluations of subexpressions are memorised. The following stronger logic is Conditional FEL (inspired by Guzmán and Squier's Conditional logic, 1990). The strongest FEL is static FEL, a sequential version of propositional logic. We use evaluation trees as a simple, intuitive semantics and provide complete axiomatisations for closed terms. For each FEL except Static FEL, we also define its three-valued version, with a constant U for 'undefinedness' and again provide complete, independent axiomatisations, each one containing two additional axioms for U on top of the axiomatisations of the two-valued case. In this setting, the strongest FEL is equivalent to Bochvar's logic. Finally, we discuss how the family of FELs is related to the previously defined family of 'short-circuit logics'.

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## 1. Introduction

This paper has its origin in the work of Staudt (2012), in which so called 'Free Fully Evaluated Left-Sequential Logic' (FFEL) was introduced, together with 'evaluation trees' as a simple semantics and an equational axiomatisation of their equality.

We define a family of 'Fully Evaluated Left-Sequential Logics' (FELs) that is about full left-sequential evaluation (also called strict evaluation) and of which FFEL is the most distinguishing (weakest). As in Staudt (2012), we use for the Boolean connectives a dedicated notation that prescribes a full, left-sequential evaluation strategy: we consider terms (propositional expressions) that are built from atoms $a, b, c, \ldots$ (propositional variables) and constants T and F for truth and falsehood by composition with negation ($\neg$) and sequential connectives: left-sequential conjunction, notation $\mathbin{\raisebox{-0.3ex}{$\bullet$}\!\!\wedge}$, and left-sequential disjunction, notation $\mathbin{\raisebox{0.8ex}{$\bullet$}\!\!\vee}$, where the little black dot denotes that the left argument must be evaluated first, and then the right argument. For two-valued FELs we discern a hierarchy starting with FFEL, which is immune to (atomic) side effects, and ending with a sequential version of propositional logic.

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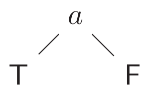
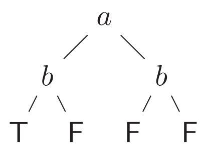
*Evaluation trees* are binary trees with internal nodes labelled by atoms and leaves labelled with either T or F, and provide a simple semantics for propositional expressions: a path from the root to a leaf models an evaluation. The left branch from an internal node indicates that its atom is evaluated *true*, and the right branch that the atom evaluates to *false*. The leaves of an evaluation tree represent evaluation results. Atom *a* has as its semantics the evaluation tree



and for example, $a \mathbin{{}_{\bullet}\!\wedge} b$ has as its semantics the evaluation tree



which is composed from the evaluation trees of atoms *a* and *b*. If in $a \mathbin{{}_{\bullet}\!\wedge} b$, atom *a* evaluates to *true*, then atom *b* is evaluated and determines the evaluation result, and if *a* evaluates to *false*, then the evaluation result is *false*, but *b* is (still) evaluated. Note that in the example tree of $a \mathbin{{}_{\bullet}\!\wedge} b$, replacing *b* by *a* has an evaluation tree that is *not* equal to that of *a*; the evaluation of the first *a* can have a side effect that changes the second evaluation result.

In Staudt (2012), evaluation trees were introduced as the basis of a relatively simple semantic framework, both for FFEL and for so-called *Free Short-Circuit Logic* (FSCL, Bergstra, Ponse, and Staudt, 2010[2013]),[1] and completeness results for both FFEL and FSCL were provided, i.e. equational axiomatisations of the equality of their evaluation trees. For example, FFEL refutes the idempotence axiom $x \mathbin{{}_{\bullet}\!\wedge} x = x$, but satisfies the associativity of the binary connectives. In Bergstra and Ponse (2015), transformations on evaluation trees were defined in order to provide a semantic basis for other short-circuit logics, see Ponse and Staudt (2018), Bergstra et al. (2021), and Bergstra and Ponse (2025a).

We define three new FELs, each of which has a counterpart in short-circuit logic and models sequential full evaluation according to a particular evaluation strategy, obtained by an adaptation of FFEL's evaluation trees. Moreover, we give complete, equational axiomatisations[2] of the equality of their evaluation trees:

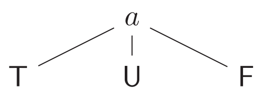
*Memorising FEL (*MFEL*).* The logic MFEL characterises the evaluation strategy in which the evaluation of each atom in an expression is memorised: no complete path in a 'memorising evaluation tree' contains multiple occurrences of the same atom. For example, *a* and $a \mathbin{{}_{\bullet}\!\wedge} a$ have the same memorising evaluation tree (that of *a*). In MFEL, atoms cannot have side effects and the evaluation order of a propositional expression is prescribed. As an example, the memorising evaluation trees of $a \mathbin{{}_{\bullet}\!\wedge} b$ and $b \mathbin{{}_{\bullet}\!\wedge} a$ are different. MFEL is axiomatised by adding one equational axiom to those of FFEL.

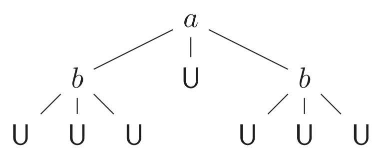
*Conditional FEL ($C\ell FEL_2$).* The name $C\ell FEL_2$ refers to Conditional logic, defined by Guzmán and Squier (1990), and the subscript 2 refers to its two-valued version.

In $C\ell FEL_2$, $\leftand$ and $\leftor$ are taken to be commutative. In Bergstra and Ponse (2025a), we show that this commutativity is a consequence of Conditional logic. $C\ell FEL_2$-evaluation trees are memorising evaluation trees in which the order of commuting atoms can be changed while preserving equivalence. $C\ell FEL_2$ is axiomatised by the axioms of MFEL and $x \leftand y = y \leftand x$.

*Static FEL (*SFEL*).* This logic is just a sequential version of propositional logic. SFEL is axiomatised by adding the axiom $x \leftand \mathsf{F} = \mathsf{F}$ to those of $C\ell FEL_2$.

Evaluation trees for full left-sequential evaluation can be easily extended to a three-valued case by including U as a leaf, which represents the truth value *undefined*. Evaluation trees with leaves in {T, F, U} were introduced in Bergstra et al. (2021), where they serve as a semantics for short-circuit logic with undefinedness. In both FEL and SCL, atom $a$ has as its semantics the left tree below, which shows that the evaluation of each atom can be *undefined*. Another example in FEL is the evaluation tree of $a \leftand (b \leftand \mathsf{U})$, shown on the right:




The general idea here is that in propositional expressions, U aborts further evaluation. This property can be equationally axiomatised by $\neg\mathsf{U} = \mathsf{U}$ and $\mathsf{U} \leftand x = \mathsf{U}$ (and by duality it follows that $\mathsf{U} \leftor x = \mathsf{U}$). A programming-oriented example is a condition `[y != 0]` $\leftand$ `[x/y > 17]`, where the atom `[x/y > 17]` can be undefined.

For each FEL except SFEL, we define the three-valued version and give equational axiomatisations consisting of those for the two-valued case and the two mentioned axioms for U. In this family, $C\ell FEL$ with undefinedness is equivalent to Bochvar's logic (1938) and U is fully absorptive. Note that SFEL cannot be extended with U: $\mathsf{F} = \mathsf{U} \leftand \mathsf{F} = \mathsf{U}$.

***Prover9* and *Mace4*.** Apart from inductive proofs, all presented derivabilities from equational axiomatisations were found by or checked with the theorem prover *Prover9*, and finite (counter)models were generated with the tool *Mace4*. For both tools, including free downloads, see McCune (2008). We used these tools on a Macbook Pro with a 2.4 GHz dual-core Intel Core i5 processor and 4 GB of RAM. Average run times are mostly given in mere seconds and rounded up, for example, 2 s (the default runtimes on our installation are 60 s). To show how we used *Prover9* and *Mace4*, we have included their output for two typical cases in Appendices 3 and 4, respectively.

***Structure of the paper.*** In Section 2, we review Free FEL (FFEL) and its equational axiomatisation. In Section 3, we define $FFEL^{\mathsf{U}}$, the extension of FFEL with undefinedness. In Section 4 we define Memorising FEL (MFEL), and in Section 5 we extend MFEL to $MFEL^{\mathsf{U}}$ with undefinedness. In Section 6, we define two-valued Conditional FEL ($C\ell FEL_2$) and its extension $C\ell FEL$ with undefinedness. In Section 7, we define and axiomatise Static FEL (SFEL), and provide for each FEL an independent axiomatisation. In Section 8, we discuss how the family of FELs is related to the previously defined family of 'short-circuit logics'. We end with a comment on expressiveness, some conclusions and a comment on future work. Appendices 1 and 2 are used for results (in most cases intermediate results) that require detailed proofs. Moreover, in Appendix 2 we generalise a result from Bergstra et al. (2021, La.3.3). and use some results about short-circuit logic in order to prove a soundness result (Lemma 4.6). In Appendix 3 we display the difficult part of a proof generated by *Prover9* (Lemma 4.7) and in Appendix 4 we show the models generated by *Mace4* that we use to prove an independence result (Theorem 7.5).

## 2. Free FEL (FFEL) and an equational axiomatisation

In this section, we review evaluation trees and the logic FFEL. We repeat the main results about FFEL, that is, its evaluation trees and axiomatisation, and the completeness proof as presented in Staudt (2012).

From this point on, we assume that $A$ is a countable set of atoms with typical elements $a, b, c$. We start with a formal definition of evaluation trees.

**Definition 2.1:** The set $\mathcal{T}_A$ of *evaluation trees* over $A$ with leaves in $\{\mathsf{T}, \mathsf{F}\}$ is defined inductively by

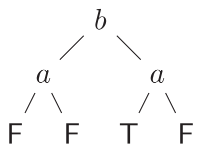
$$\mathsf{T} \in \mathcal{T}_A, \quad \mathsf{F} \in \mathcal{T}_A, \quad (X \trianglelefteq a \trianglerighteq Y) \in \mathcal{T}_A \quad \text{for any } X, Y \in \mathcal{T}_A \quad \text{and} \quad a \in A.$$

The operator $\_ \trianglelefteq a \trianglerighteq \_$ is called *tree composition over a*. In the evaluation tree $X \trianglelefteq a \trianglerighteq Y$, the root is represented by $a$, the left branch by $X$ and the right branch by $Y$. The *depth* $d : \mathcal{T}_A \to \mathbb{N}$ of an evaluation tree is defined by $d(\mathsf{T}) = d(\mathsf{F}) = 0$ and $d(Y \trianglelefteq a \trianglerighteq Z) = 1 + \max(d(Y), d(Z))$.

The leaves of an evaluation tree represent evaluation results (we use the constants T and F for *true* and *false*). Next to the formal notation for evaluation trees we also use a more pictorial representation. For example, the tree

$$(\mathsf{F} \trianglelefteq a \trianglerighteq \mathsf{F}) \trianglelefteq b \trianglerighteq (\mathsf{T} \trianglelefteq a \trianglerighteq \mathsf{F})$$

can be represented as follows, where $\trianglelefteq$ yields a left branch, and $\trianglerighteq$ a right branch:




(Picture 1)

In order to define a semantics for full evaluation of negation and the left-sequential connectives, we first define the *leaf replacement* operator, 'replacement' for short, on trees in $\mathcal{T}_A$ as follows. For $X \in \mathcal{T}_A$, the replacement of T with $Y$ and F with $Z$ in $X$, denoted

$$X[\mathsf{T} \mapsto Y, \mathsf{F} \mapsto Z]$$

is defined recursively by

$$\mathsf{T}[\mathsf{T} \mapsto Y, \mathsf{F} \mapsto Z] = Y,$$

$$\mathsf{F}[\mathsf{T} \mapsto Y, \mathsf{F} \mapsto Z] = Z,$$

$$(X_1 \trianglelefteq a \trianglerighteq X_2)[\mathsf{T} \mapsto Y, \mathsf{F} \mapsto Z] = X_1[\mathsf{T} \mapsto Y, \mathsf{F} \mapsto Z] \trianglelefteq a \trianglerighteq X_2[\mathsf{T} \mapsto Y, \mathsf{F} \mapsto Z].$$

We note that the order in which the replacements of leaves of $X$ is listed is irrelevant and adopt the convention of not listing identities inside the brackets, for example $X[\mathsf{F} \mapsto Z] = X[\mathsf{T} \mapsto \mathsf{T}, \mathsf{F} \mapsto Z]$. By structural induction it follows that repeated replacements satisfy

$$\begin{aligned} &X[\mathsf{T} \mapsto Y_1, \mathsf{F} \mapsto Z_1][\mathsf{T} \mapsto Y_2, \mathsf{F} \mapsto Z_2] \\ &\quad = X[\mathsf{T} \mapsto Y_1[\mathsf{T} \mapsto Y_2, \mathsf{F} \mapsto Z_2], \mathsf{F} \mapsto Z_1[\mathsf{T} \mapsto Y_2, \mathsf{F} \mapsto Z_2]]. \end{aligned} \tag{RR}$$

The base cases $X \in \{\mathsf{T}, \mathsf{F}\}$ are immediate, and so is the induction step because replacements distribute over $\_ \trianglelefteq a \trianglerighteq \_$ (i.e. tree composition over $a \in A$).

The set $\mathcal{SP}_A$ of (left-sequential) propositional expressions over $A$ that prescribe full left-sequential evaluation is defined by the following grammar ($a \in A$):

$$P ::= \mathsf{T} \mid \mathsf{F} \mid a \mid \neg P \mid P \mathbin{{}_{\bullet}\!\wedge} P \mid P \mathbin{{}^{\bullet}\!\vee} P,$$

and we refer to its signature by

$$\Sigma_{\mathsf{FEL}}(A) = \{{}_{\bullet}\!\wedge, {}^{\bullet}\!\vee, \neg, \mathsf{T}, \mathsf{F}, a \mid a \in A\}.$$

We interpret expressions in $\mathcal{SP}_A$ as evaluation trees by a function *fe* (abbreviating full evaluation).

**Definition 2.2:** The unary *full evaluation* function $fe : \mathcal{SP}_A \to \mathcal{T}_A$ is defined as follows, where $a \in A$:

$$\begin{aligned} &fe(\mathsf{T}) = \mathsf{T}, \quad fe(\neg P) = fe(P)[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}], \\ &fe(\mathsf{F}) = \mathsf{F}, \quad fe(P \mathbin{{}_{\bullet}\!\wedge} Q) = fe(P)[\mathsf{T} \mapsto fe(Q), \mathsf{F} \mapsto fe(Q)[\mathsf{T} \mapsto \mathsf{F}]], \\ &fe(a) = \mathsf{T} \trianglelefteq a \trianglerighteq \mathsf{F}, \quad fe(P \mathbin{{}^{\bullet}\!\vee} Q) = fe(P)[\mathsf{T} \mapsto fe(Q)[\mathsf{F} \mapsto \mathsf{T}], \mathsf{F} \mapsto fe(Q)]. \end{aligned}$$

The overloading of the symbol T in $fe(\mathsf{T}) = \mathsf{T}$ will not cause confusion (and similarly for F). As a simple example we derive the evaluation tree of $\neg b \mathbin{{}_{\bullet}\!\wedge} a$:

$$\begin{aligned} fe(\neg b \mathbin{{}_{\bullet}\!\wedge} a) &= fe(\neg b)[\mathsf{T} \mapsto fe(a), \mathsf{F} \mapsto fe(a)[\mathsf{T} \mapsto \mathsf{F}]] \\ &= (\mathsf{F} \trianglelefteq b \trianglerighteq \mathsf{T})[\mathsf{T} \mapsto fe(a), \mathsf{F} \mapsto fe(a)[\mathsf{T} \mapsto \mathsf{F}]] \\ &= (\mathsf{F} \trianglelefteq a \trianglerighteq \mathsf{F}) \trianglelefteq b \trianglerighteq (\mathsf{T} \trianglelefteq a \trianglerighteq \mathsf{F}), \end{aligned}$$

which can be visualised as in Picture 1. Also, $fe(\neg(b \mathbin{{}^{\bullet}\!\vee} \neg a)) = (\mathsf{F} \trianglelefteq a \trianglerighteq \mathsf{F}) \trianglelefteq b \trianglerighteq (\mathsf{T} \trianglelefteq a \trianglerighteq \mathsf{F})$. An evaluation tree $fe(P)$ represents full evaluation in a way that can be compared to the notion of a truth table for propositional logic in that it represents each possible evaluation of $P$. However, there are some important differences with truth tables: in $fe(P)$, the sequentiality of $P$'s evaluation is represented, the same atom may occur multiple times in $fe(P)$ with different evaluation values, and all atoms are evaluated, thus yielding evaluation trees that are perfect. Evaluation trees that are not perfect occur in the setting with *short-circuit* connectives; in that setting, the connectives ${}_{\bullet}\!\wedge$ and ${}^{\bullet}\!\vee$ are definable (we return to this in Sections 7 and 8).

**Table 1.** EqFFEL, axioms for FFEL.

| | |
|---|---|
| $\mathsf{F} = \neg\mathsf{T}$ | (FFEL1) |
| $x \mathbin{{}^{\bullet}\!\vee} y = \neg(\neg x \mathbin{{}_{\bullet}\!\wedge} \neg y)$ | (FFEL2) |
| $\neg\neg x = x$ | (FFEL3) |
| $(x \mathbin{{}_{\bullet}\!\wedge} y) \mathbin{{}_{\bullet}\!\wedge} z = x \mathbin{{}_{\bullet}\!\wedge} (y \mathbin{{}_{\bullet}\!\wedge} z)$ | (FFEL4) |
| $\mathsf{T} \mathbin{{}_{\bullet}\!\wedge} x = x$ | (FFEL5) |
| $x \mathbin{{}_{\bullet}\!\wedge} \mathsf{T} = x$ | (FFEL6) |
| $x \mathbin{{}_{\bullet}\!\wedge} \mathsf{F} = \mathsf{F} \mathbin{{}_{\bullet}\!\wedge} x$ | (FFEL7) |
| $\neg x \mathbin{{}_{\bullet}\!\wedge} \mathsf{F} = x \mathbin{{}_{\bullet}\!\wedge} \mathsf{F}$ | (FFEL8) |
| $(x \mathbin{{}_{\bullet}\!\wedge} \mathsf{F}) \mathbin{{}^{\bullet}\!\vee} y = (x \mathbin{{}^{\bullet}\!\vee} \mathsf{T}) \mathbin{{}_{\bullet}\!\wedge} y$ | (FFEL9) |
| $x \mathbin{{}^{\bullet}\!\vee} (y \mathbin{{}_{\bullet}\!\wedge} \mathsf{F}) = x \mathbin{{}_{\bullet}\!\wedge} (y \mathbin{{}^{\bullet}\!\vee} \mathsf{T})$ | (FFEL10) |

**Definition 2.3:** The binary relation $=_{fe}$ on $\mathcal{SP}_A$ defined by $P =_{fe} Q \iff fe(P) = fe(Q)$ is called *free full valuation congruence.*

**Lemma 2.4:** *The relation* $=_{fe}$ *is a congruence.*

***Proof:*** It is immediately clear that identity, symmetry and transitivity are preserved. For congruence we show only that for all $P, Q, R \in \mathcal{SP}_A$, $P =_{fe} Q$ implies $R \mathbin{{}_{\bullet}\!\wedge} P =_{fe} R \mathbin{{}_{\bullet}\!\wedge} Q$. The other cases proceed in a similar fashion. If $fe(P) = fe(Q)$ then $fe(P)[\mathsf{T} \mapsto \mathsf{F}] = fe(Q)[\mathsf{T} \mapsto \mathsf{F}]$, so

$$fe(R)[\mathsf{T} \mapsto fe(P), \mathsf{F} \mapsto fe(P)[\mathsf{T} \mapsto \mathsf{F}]] = fe(R)[\mathsf{T} \mapsto fe(Q), \mathsf{F} \mapsto fe(Q)[\mathsf{T} \mapsto \mathsf{F}]].$$

Therefore, by definition of $fe$, $R \mathbin{{}_{\bullet}\!\wedge} P =_{fe} R \mathbin{{}_{\bullet}\!\wedge} Q$. ■

**Definition 2.5:** A *Fully Evaluated Left-Sequential Logic (*FEL*)* is a logic that satisfies the consequences of $fe$-equality. *Free Fully Evaluated Left-Sequential Logic (*FFEL*)* is the fully evaluated left-sequential logic that satisfies no more consequences than those of $fe$-equality, i.e. for all $P, Q \in \mathcal{SP}_A$,

$$\mathrm{FEL} \models P = Q \Longleftarrow P =_{fe} Q \quad \text{and} \quad \mathrm{FFEL} \models P = Q \iff P =_{fe} Q.$$

In Staudt (2012), it is proven that the set EqFFEL of equational axioms in Table 1 (completely) axiomatises equality of evaluation trees; this proof is based on normal forms for $\mathcal{SP}_A$. Below and also in Appendix 1, we repeat the completeness proof of Staudt (2012) almost verbatim.

We now turn to the equational logic defined by EqFFEL, which we will show is an axiomatisation of FFEL. This set of equations was first presented by Blok (personal communication, 2011).

If two FEL-terms $s$ and $t$, possibly containing variables, are derivable in EqFFEL, we write EqFFEL $\vdash s = t$ and say that $s$ and $t$ are derivably equal. By virtue of (FFEL1) through (FFEL3), $\mathbin{{}_{\bullet}\!\wedge}$ is the dual of $\mathbin{{}^{\bullet}\!\vee}$ and T is the dual of F, and hence the duals of the equations in EqFFEL are also derivable. We will use this fact throughout our proofs.

**Lemma 2.6 (Soundness):** *For all* $P, Q \in \mathcal{SP}_A$, *if* EqFFEL $\vdash P = Q$ *then* FFEL $\models P = Q$.

***Proof:*** By Lemma 2.4, $=_{fe}$ is a congruence, so it suffices to prove the validity of the equations in EqFFEL, which is easily verified. As an example we show this for (FFEL8).

$$fe(P \mathbin{{}_{\bullet}\!\wedge} \mathsf{F}) = fe(P)[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{F}[\mathsf{T} \mapsto \mathsf{F}]] \quad \text{by definition}$$

$$
\begin{aligned}
&= fe(P)[\mathsf{T} \mapsto \mathsf{F}] \quad \text{because } \mathsf{F}[\mathsf{T} \mapsto \mathsf{F}] = \mathsf{F} \\
&= fe(P)[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}][\mathsf{T} \mapsto \mathsf{F}] \quad \text{by Equation (RR)} \\
&= fe(\neg P \mathbin{{}_{\bullet}\!\wedge} \mathsf{F}). \qquad \blacksquare
\end{aligned}
$$

The following lemma shows some useful equations illustrating the special properties of terms of the form $x \mathbin{{}_{\bullet}\!\wedge} \mathsf{F}$ and $x \mathbin{{}^{\bullet}\!\vee} \mathsf{T}$. The first is an 'extension' of axiom (FFEL8) and the others show two different ways how terms of the form $x \mathbin{{}^{\bullet}\!\vee} \mathsf{T}$, and by duality terms of the form $x \mathbin{{}_{\bullet}\!\wedge} \mathsf{F}$, can change the main connective of a term.

**Lemma 2.7:** *The following equations are consequences of* EqFFEL.

(1) $x \mathbin{{}_{\bullet}\!\wedge} (y \mathbin{{}_{\bullet}\!\wedge} F) = \neg x \mathbin{{}_{\bullet}\!\wedge} (y \mathbin{{}_{\bullet}\!\wedge} F)$,
(2) $(x \mathbin{{}^{\bullet}\!\vee} T) \mathbin{{}_{\bullet}\!\wedge} y = \neg(x \mathbin{{}^{\bullet}\!\vee} T) \mathbin{{}^{\bullet}\!\vee} y$,
(3) $x \mathbin{{}^{\bullet}\!\vee} (y \mathbin{{}_{\bullet}\!\wedge} (z \mathbin{{}^{\bullet}\!\vee} T)) = (x \mathbin{{}^{\bullet}\!\vee} y) \mathbin{{}_{\bullet}\!\wedge} (z \mathbin{{}^{\bullet}\!\vee} T)$.

***Proof:*** We derive the equations in order.

$$
\begin{aligned}
x \mathbin{{}_{\bullet}\!\wedge} (y \mathbin{{}_{\bullet}\!\wedge} \mathsf{F}) &= (x \mathbin{{}_{\bullet}\!\wedge} \mathsf{F}) \mathbin{{}_{\bullet}\!\wedge} y \quad \text{by (FFEL7) and (FFEL4)} \\
&= (\neg x \mathbin{{}_{\bullet}\!\wedge} \mathsf{F}) \mathbin{{}_{\bullet}\!\wedge} y \quad \text{by (FFEL8)} \\
&= \neg x \mathbin{{}_{\bullet}\!\wedge} (y \mathbin{{}_{\bullet}\!\wedge} \mathsf{F}), \quad \text{by (FFEL7) and (FFEL4)} \\
(x \mathbin{{}^{\bullet}\!\vee} \mathsf{T}) \mathbin{{}_{\bullet}\!\wedge} y &= (x \mathbin{{}_{\bullet}\!\wedge} \mathsf{F}) \mathbin{{}^{\bullet}\!\vee} y \quad \text{by (FFEL9)} \\
&= (\neg x \mathbin{{}_{\bullet}\!\wedge} \mathsf{F}) \mathbin{{}^{\bullet}\!\vee} y \quad \text{by (FFEL8)} \\
&= (\neg x \mathbin{{}_{\bullet}\!\wedge} \neg\mathsf{T}) \mathbin{{}^{\bullet}\!\vee} y \quad \text{by (FFEL1)} \\
&= \neg(x \mathbin{{}^{\bullet}\!\vee} \mathsf{T}) \mathbin{{}^{\bullet}\!\vee} y, \quad \text{by (FFEL3) and (FFEL2)} \\
x \mathbin{{}^{\bullet}\!\vee} (y \mathbin{{}_{\bullet}\!\wedge} (z \mathbin{{}^{\bullet}\!\vee} \mathsf{T})) &= x \mathbin{{}^{\bullet}\!\vee} (y \mathbin{{}^{\bullet}\!\vee} (z \mathbin{{}_{\bullet}\!\wedge} \mathsf{F})) \quad \text{by (FFEL10)} \\
&= (x \mathbin{{}^{\bullet}\!\vee} y) \mathbin{{}^{\bullet}\!\vee} (z \mathbin{{}_{\bullet}\!\wedge} \mathsf{F}) \quad \text{by the dual of (FFEL4)} \\
&= (x \mathbin{{}^{\bullet}\!\vee} y) \mathbin{{}_{\bullet}\!\wedge} (z \mathbin{{}^{\bullet}\!\vee} \mathsf{T}). \quad \text{by (FFEL10)} \qquad \blacksquare
\end{aligned}
$$

*FEL Normal Form.* To aid in our completeness proof we define a normal form for FEL-terms. Due to the possible presence of side effects, FFEL does not identify terms which contain different atoms or the same atoms in a different order. Because of this, common normal forms for propositional logic are not normal forms for FEL-terms. For example, rewriting a term to Conjunctive Normal Form or Disjunctive Normal Form may require duplicating some of the atoms in the term, thus yielding a term that is not derivably equal to the original. We first present the grammar for our normal form, before motivating it. The normal form we present here is an adaptation of a normal form proposed by Blok (personal communication, 2011).

**Definition 2.8:** A term $P \in \mathcal{SP}_A$ is said to be in FEL *Normal Form (FNF)* if it is generated by the following grammar.

$$P \in \text{FNF} ::= P^{\mathsf{T}} \mid P^{\mathsf{F}} \mid P^{\mathsf{T}} \mathbin{{}_{\bullet}\!\wedge} P^{*}$$

$$
\begin{aligned}
P^* &::= P^c \mid P^d\\
P^c &::= P^\ell \mid P^* \leftand P^d\\
P^d &::= P^\ell \mid P^* \leftor P^c\\
P^\ell &::= a \leftand P^{\mathsf{T}} \mid \neg a \leftand P^{\mathsf{T}}\\
P^{\mathsf{T}} &::= \mathsf{T} \mid a \leftor P^{\mathsf{T}}\\
P^{\mathsf{F}} &::= \mathsf{F} \mid a \leftand P^{\mathsf{F}},
\end{aligned}
$$

where $a \in A$. We refer to $P^*$-forms as $*$-terms, to $P^\ell$-forms as $\ell$-terms, to $P^{\mathsf{T}}$-forms as T-terms and to $P^{\mathsf{F}}$-forms as F-terms. A term of the form $P^{\mathsf{T}} \leftand P^*$ is referred to as a T-$*$-term.

We immediately note that if it were not for the presence of T and F we could define a much simpler normal form. In that case it would suffice to 'push in' or 'push down' the negations, thus obtaining a Negation Normal Form (NNF), as exists for propositional logic. Naturally if our set $A$ of atoms is empty, the truth value constants would be a normal form.

When considering the image of *fe* we note that some trees only have T-leaves, some only have F-leaves and some have both T-leaves and F-leaves. For any FEL-term $P$, $fe(P \leftor \mathsf{T})$ is a tree with only T-leaves, as can easily be seen from the definition of *fe*. All terms $P$ such that $fe(P)$ only has T-leaves are rewritten to T-terms. Similarly $fe(P \leftand \mathsf{F})$ is a tree with only F-leaves. All terms $P$ such that $fe(P)$ only has F-leaves are rewritten to F-terms. The simplest trees in the image of *fe* that have both T-leaves and F-leaves are $fe(a)$ for $a \in A$. Any (occurrence of an) atom that determines (in whole or in part) the evaluation result of a term, such as $a$ in this example, is referred to as a *determinative* (occurrence of an) atom. This as opposed to a *non-determinative* (occurrence of an) atom, such as the $a$ in $a \leftor \mathsf{T}$, which does not determine (either in whole or in part) the evaluation result of the term. Note that a term $P$ such that $fe(P)$ contains both T and F must contain at least one determinative atom.

Terms that contain at least one determinative atom will be rewritten to T-$*$-terms. In T-$*$-terms we encode each determinative atom together with the non-determinative atoms that occur between it and the next determinative atom in the term (reading from left to right) as an $\ell$-term. Observe that the first atom in an $\ell$-term is the (only) determinative atom in that $\ell$-term and that determinative atoms only occur in $\ell$-terms. Also observe that the evaluation result of an $\ell$-term is that of its determinative atom. This is intuitively convincing, because the remainder of the atoms in any $\ell$-term are non-determinative and hence do not contribute to its evaluation result. The non-determinative atoms that may occur before the first determinative atom are encoded as a T-term. A T-$*$-term is the conjunction of a T-term encoding such atoms and a $*$-term, which contains only conjunctions and disjunctions of $\ell$-terms. We could also have encoded such atoms as an F-term and then taken the disjunction with a $*$-term to obtain a term with the same semantics. We consider $\ell$-terms to be 'basic' in $*$-terms in the sense that they are the smallest grammatical unit that influences the evaluation result of the $*$-term.

In the following, $P^{\mathsf{T}}, P^{\ell}$, etc. are used both to denote grammatical categories and as variables for terms in those categories. The remainder of this section is concerned with defining and proving correct the normalisation function $f : \mathcal{SP}_A \to$ FNF. We will define $f$ recursively using the functions

$$f^n : \text{FNF} \to \text{FNF} \quad \text{and} \quad f^c : \text{FNF} \times \text{FNF} \to \text{FNF}.$$

The first of these will be used to rewrite negated FNF-terms to FNF-terms and the second to rewrite the conjunction of two FNF-terms to an FNF-term. By (FFEL2) we have no need for a dedicated function that rewrites the disjunction of two FNF-terms to an FNF-term.

We start by defining $f^n$. Analysing the semantics of T-terms and F-terms together with the definition of *fe* on negations, it becomes clear that $f^n$ must turn T-terms into F-terms and vice versa. We also remark that $f^n$ must preserve the left-associativity of the $*$-terms in T-$*$-terms, modulo the associativity within $\ell$-terms. We define $f^n$ : FNF $\to$ FNF as follows, using the auxiliary function $f_1^n : P^* \to P^*$ to 'push down' or 'push in' the negation symbols when negating a T-$*$-term. We note that there is no ambiguity between the different grammatical categories present in an FNF-term, i.e. any FNF-term is in exactly one of the grammatical categories identified in Definition 2.8.

$$f^n(\mathsf{T}) = \mathsf{F} \tag{1}$$
$$f^n(a \leftor P^{\mathsf{T}}) = a \leftand f^n(P^{\mathsf{T}}) \tag{2}$$
$$f^n(\mathsf{F}) = \mathsf{T} \tag{3}$$
$$f^n(a \leftand P^{\mathsf{F}}) = a \leftor f^n(P^{\mathsf{F}}) \tag{4}$$
$$f^n(P^{\mathsf{T}} \leftand Q^*) = P^{\mathsf{T}} \leftand f_1^n(Q^*) \tag{5}$$
$$f_1^n(a \leftand P^{\mathsf{T}}) = \neg a \leftand P^{\mathsf{T}} \tag{6}$$
$$f_1^n(\neg a \leftand P^{\mathsf{T}}) = a \leftand P^{\mathsf{T}} \tag{7}$$
$$f_1^n(P^* \leftand Q^d) = f_1^n(P^*) \leftor f_1^n(Q^d) \tag{8}$$
$$f_1^n(P^* \leftor Q^c) = f_1^n(P^*) \leftand f_1^n(Q^c) \tag{9}$$

Now we turn to defining $f^c$. These definitions have a great deal of inter-dependence so we first present the definition for $f^c$ when the first argument is a T-term. We see that the conjunction of a T-term with another term always yields a term of the same grammatical category as the second conjunct.

$$f^c(\mathsf{T}, P) = P \tag{10}$$
$$f^c(a \leftor P^{\mathsf{T}}, Q^{\mathsf{T}}) = a \leftor f^c(P^{\mathsf{T}}, Q^{\mathsf{T}}) \tag{11}$$
$$f^c(a \leftor P^{\mathsf{T}}, Q^{\mathsf{F}}) = a \leftand f^c(P^{\mathsf{T}}, Q^{\mathsf{F}}) \tag{12}$$
$$f^c(a \leftor P^{\mathsf{T}}, Q^{\mathsf{T}} \leftand R^*) = f^c(a \leftor P^{\mathsf{T}}, Q^{\mathsf{T}}) \leftand R^* \tag{13}$$

For defining $f^c$ where the first argument is an F-term we make use of (FFEL7) when dealing with conjunctions of F-terms with T-$*$-terms. The definition of $f^c$ for the arguments used in the right-hand side of (16) starts at (23). We note that despite the high level of inter-dependence in these definitions, this does not create a circular definition. We also note that the conjunction of an F-term with another term is always itself an F-term.

$$f^c(\mathsf{F}, P^\mathsf{T}) = f^n(P^\mathsf{T}) \tag{14}$$

$$f^c(\mathsf{F}, P^\mathsf{F}) = P^\mathsf{F} \tag{15}$$

$$f^c(\mathsf{F}, P^\mathsf{T} \leftand Q^*) = f^c(P^\mathsf{T} \leftand Q^*, \mathsf{F}) \tag{16}$$

$$f^c(a \leftand P^\mathsf{F}, Q) = a \leftand f^c(P^\mathsf{F}, Q) \tag{17}$$

The case where the first conjunct is a T-$*$-term is the most complicated. Therefore we first consider the case where the second conjunct is a T-term. In this case we must make the T-term part of the last (rightmost) $\ell$-term in the T-$*$-term, so that the result will again be a T-$*$-term. For this 'pushing in' of the second conjunct we define an auxiliary function $f_1^c : P^* \times P^\mathsf{T} \to P^*$.

$$f^c(P^\mathsf{T} \leftand Q^*, R^\mathsf{T}) = P^\mathsf{T} \leftand f_1^c(Q^*, R^\mathsf{T}) \tag{18}$$

$$f_1^c(a \leftand P^\mathsf{T}, Q^\mathsf{T}) = a \leftand f^c(P^\mathsf{T}, Q^\mathsf{T}) \tag{19}$$

$$f_1^c(\neg a \leftand P^\mathsf{T}, Q^\mathsf{T}) = \neg a \leftand f^c(P^\mathsf{T}, Q^\mathsf{T}) \tag{20}$$

$$f_1^c(P^* \leftand Q^d, R^\mathsf{T}) = P^* \leftand f_1^c(Q^d, R^\mathsf{T}) \tag{21}$$

$$f_1^c(P^* \leftor Q^c, R^\mathsf{T}) = P^* \leftor f_1^c(Q^c, R^\mathsf{T}) \tag{22}$$

When the second conjunct is an F-term, the result will naturally be an F-term itself. So we need to convert the T-$*$-term to an F-term. Using (FFEL4) we reduce this problem to converting a $*$-term to an F-term, for which we use the auxiliary function $f_2^c : P^* \times P^\mathsf{F} \to P^\mathsf{F}$.

$$f^c(P^\mathsf{T} \leftand Q^*, R^\mathsf{F}) = f^c(P^\mathsf{T}, f_2^c(Q^*, R^\mathsf{F})) \tag{23}$$

$$f_2^c(a \leftand P^\mathsf{T}, R^\mathsf{F}) = a \leftand f^c(P^\mathsf{T}, R^\mathsf{F}) \tag{24}$$

$$f_2^c(\neg a \leftand P^\mathsf{T}, R^\mathsf{F}) = a \leftand f^c(P^\mathsf{T}, R^\mathsf{F}) \tag{25}$$

$$f_2^c(P^* \leftand Q^d, R^\mathsf{F}) = f_2^c(P^*, f_2^c(Q^d, R^\mathsf{F})) \tag{26}$$

$$f_2^c(P^* \leftor Q^c, R^\mathsf{F}) = f_2^c(P^*, f_2^c(Q^c, R^\mathsf{F})) \tag{27}$$

Finally we are left with conjunctions and disjunctions of two T-$*$-terms, thus completing the definition of $f^c$. We use the auxiliary function $f_3^c : P^* \times P^\mathsf{T} \leftand P^* \to P^*$ to ensure that the result is a T-$*$-term.

$$f^c(P^\mathsf{T} \leftand Q^*, R^\mathsf{T} \leftand S^*) = P^\mathsf{T} \leftand f_3^c(Q^*, R^\mathsf{T} \leftand S^*) \tag{28}$$

$$f_3^c(P^*, Q^\mathsf{T} \leftand R^\ell) = f_1^c(P^*, Q^\mathsf{T}) \leftand R^\ell \tag{29}$$

$$f_3^c(P^*, Q^\mathsf{T} \leftand (R^* \leftand S^d)) = f_3^c(P^*, Q^\mathsf{T} \leftand R^*) \leftand S^d \tag{30}$$

$$f_3^c(P^*, Q^\mathsf{T} \leftand (R^* \leftor S^c)) = f_1^c(P^*, Q^\mathsf{T}) \leftand (R^* \leftor S^c) \tag{31}$$

As promised, we now define the normalisation function $f : \mathcal{SP}_A \to \text{FNF}$ recursively, using $f^n$ and $f^c$, as follows.

$$f(a) = \mathsf{T} \mathbin{\curlywedge} (a \mathbin{\curlywedge} \mathsf{T}) \tag{32}$$

$$f(\mathsf{T}) = \mathsf{T} \tag{33}$$

$$f(\mathsf{F}) = \mathsf{F} \tag{34}$$

$$f(\neg P) = f^n(f(P)) \tag{35}$$

$$f(P \mathbin{\curlywedge} Q) = f^c(f(P), f(Q)) \tag{36}$$

$$f(P \mathbin{\curlyvee} Q) = f^n(f^c(f^n(f(P)), f^n(f(Q)))) \tag{37}$$

**Theorem 2.9:** *For any $P \in \mathcal{SP}_A$, $f(P)$ terminates, $f(P) \in$ FNF and EqFFEL $\vdash f(P) = P$.*

In Appendix 1 we first prove a number of lemmas showing that the definitions $f^n$ and $f^c$ are correct and use those to prove the theorem. The main reason to use a normalisation function rather than a term rewriting system to prove the correctness of FNF is that this relieves us of the need to prove the confluence of the induced rewriting system, thus simplifying the proof.

*Tree Structure.* Below we prove that EqFFEL axiomatises FFEL by showing that for $P \in$ FNF we can invert $fe(P)$. To do this we need to prove several structural properties of the trees in the image of $fe$. In the definition of $fe$ we can see how $fe(P \mathbin{\curlywedge} Q)$ is assembled from $fe(P)$ and $fe(Q)$ and similarly for $fe(P \mathbin{\curlyvee} Q)$. To decompose these trees we introduce some notation. The trees in the image of $fe$ are all finite binary trees over $A$ with leaves in $\{\mathsf{T}, \mathsf{F}\}$, i.e. $fe[\mathcal{SP}_A] \subseteq \mathcal{T}_A$. We will now also consider the set $\mathcal{T}_{A,\triangle}$ of binary trees over $A$ with leaves in $\{\mathsf{T}, \mathsf{F}, \triangle\}$. Similarly we consider $\mathcal{T}_{A,1,2}$, the set of binary trees over $A$ with leaves in $\{\mathsf{T}, \mathsf{F}, \triangle_1, \triangle_2\}$. The $\triangle$, $\triangle_1$ and $\triangle_2$ will be used as placeholders when composing or decomposing trees. Replacement of the leaves of trees in $\mathcal{T}_{A,\triangle}$ and $\mathcal{T}_{A,1,2}$ by trees (either in $\mathcal{T}_A$, $\mathcal{T}_{A,\triangle}$ or $\mathcal{T}_{A,1,2}$) is defined analogous to replacement for trees in $\mathcal{T}_A$, adopting the same notational conventions.

For example we have by definition of $fe$ that $fe(P \mathbin{\curlywedge} Q)$ can be decomposed as

$$fe(P)[\mathsf{T} \mapsto \triangle_1, \mathsf{F} \mapsto \triangle_2][\triangle_1 \mapsto fe(Q), \triangle_2 \mapsto fe(Q)[\mathsf{T} \mapsto \mathsf{F}]],$$

where $fe(P)[\mathsf{T} \mapsto \triangle_1, \mathsf{F} \mapsto \triangle_2] \in \mathcal{T}_{A,1,2}$ and $fe(Q)$ and $fe(Q)[\mathsf{T} \mapsto \mathsf{F}]$ are in $\mathcal{T}_A$. We note that this only works because the trees in the image of $fe$, or more general, in $\mathcal{T}_A$, do not contain any triangles. Similarly, as we discussed previously, $fe(P \mathbin{\curlywedge} \mathsf{F}) = fe(P)[\mathsf{T} \mapsto \mathsf{F}]$, which we can write as $fe(P)[\mathsf{T} \mapsto \triangle][\triangle \mapsto \mathsf{F}]$. We start by analysing the $fe$-image of $\ell$-terms.

**Lemma 2.10 (Structure of $\ell$-terms):** *There is no $\ell$-term $P$ such that $fe(P)$ can be decomposed as $X[\triangle \mapsto Y]$ with $X \in \mathcal{T}_{A,\triangle}$ and $Y \in \mathcal{T}_A$, where $X \neq \triangle$, but does contain $\triangle$, and $Y$ contains occurrences of both T and F.*

***Proof:*** Let $P$ be some $\ell$-term. When we analyse the grammar of $P$ we find that one branch from the root of $fe(P)$ will only contain T and not F and the other branch vice versa. Hence if $fe(P) = X[\triangle \mapsto Y]$ and $Y$ contains occurrences of both T and F, then $Y$ must contain the root and hence $X = \triangle$. ■

By definition a $*$-term contains at least one $\ell$-term and hence for any $*$-term $P$, $fe(P)$ contains both T and F. The following lemma provides the *fe*-image of the rightmost $\ell$-term in a $*$-term $P$ to witness this fact.

**Lemma 2.11 (Determinativeness):** *For all $*$-terms $P$, $fe(P)$ can be decomposed as $X[\triangle \mapsto Y]$ with $X \in \mathcal{T}_{A,\triangle}$ and $Y \in \mathcal{T}_A$ such that $X$ contains $\triangle$ and $Y = fe(Q)$ for some $\ell$-term $Q$.*

Note that $X$ may be $\triangle$ here. Further on, we will refer to $Q$ as *a witness of Lemma* 2.11 *for P*.

***Proof:*** By induction on the complexity of $*$-terms $P$ modulo the complexity of $\ell$-terms. In the base case $P$ is an $\ell$-term and $fe(P) = \triangle[\triangle \mapsto fe(P)]$ is the desired decomposition by Lemma 2.10. For the induction we have to consider both $fe(P \mathbin{\lower0.5ex\hbox{$\bullet$}\!\!\wedge} Q)$ and $fe(P \mathbin{\raise0.5ex\hbox{$\bullet$}\!\!\vee} Q)$.

We treat only the case for $fe(P \mathbin{\lower0.5ex\hbox{$\bullet$}\!\!\wedge} Q)$, the case for $fe(P \mathbin{\raise0.5ex\hbox{$\bullet$}\!\!\vee} Q)$ is analogous. Let $X[\triangle \mapsto Y]$ be the decomposition for $fe(Q)$ which we have by induction hypothesis. Since by definition of *fe* on $\mathbin{\lower0.5ex\hbox{$\bullet$}\!\!\wedge}$ we have

$$fe(P \mathbin{\lower0.5ex\hbox{$\bullet$}\!\!\wedge} Q) = fe(P)[\mathrm{T} \mapsto fe(Q), \mathrm{F} \mapsto fe(Q)[\mathrm{T} \mapsto \mathrm{F}]],$$

we also have

$$\begin{aligned} fe(P \mathbin{\lower0.5ex\hbox{$\bullet$}\!\!\wedge} Q) &= fe(P)[\mathrm{T} \mapsto X[\triangle \mapsto Y], \mathrm{F} \mapsto fe(Q)[\mathrm{T} \mapsto \mathrm{F}]] \\ &= fe(P)[\mathrm{T} \mapsto X, \mathrm{F} \mapsto fe(Q)[\mathrm{T} \mapsto \mathrm{F}]][\triangle \mapsto Y], \end{aligned}$$

where the second equality is due to the fact that the only triangles in

$$fe(P)[\mathrm{T} \mapsto X, \mathrm{F} \mapsto fe(Q)[\mathrm{T} \mapsto \mathrm{F}]]$$

are those occurring in $X$. This gives our desired decomposition. ■

The following lemma illustrates another structural property of trees in the image of $*$-terms under *fe*, namely that the left branch of any determinative atom in such a tree is different from its right branch.

**Lemma 2.12 (Non-decomposition):** *There is no $*$-term $P$ such that $fe(P)$ can be decomposed as $X[\triangle \mapsto Y]$ with $X \in \mathcal{T}_{A,\triangle}$ and $Y \in \mathcal{T}_A$, where $X \neq \triangle$ and $X$ contains $\triangle$, but not T or F.*

***Proof:*** By induction on $P$ modulo the complexity of $\ell$-terms. The base case covers $\ell$-terms and follows immediately from Lemma 2.11 ($fe(P)$ contains occurrences of both T and F) and Lemma 2.10 (no non-trivial decomposition exists that contains both). For the induction we assume that the lemma holds for all $*$-terms with lesser complexity than $P \mathbin{\lower0.5ex\hbox{$\bullet$}\!\!\wedge} Q$ and $P \mathbin{\raise0.5ex\hbox{$\bullet$}\!\!\vee} Q$.

We start with the case for $fe(P \mathbin{{}_{\bullet}\!\wedge} Q)$. Suppose for contradiction that $fe(P \mathbin{{}_{\bullet}\!\wedge} Q) = X[\triangle \mapsto Y]$ with $X \neq \triangle$ and $X$ not containing any occurrences of T or F. Let $R$ be a witness of Lemma 2.11 for $P$. Now note that $fe(P \mathbin{{}_{\bullet}\!\wedge} Q)$ has a subtree

$$R[\mathsf{T} \mapsto fe(Q), \mathsf{F} \mapsto fe(Q)[\mathsf{T} \mapsto \mathsf{F}]].$$

Because $Y$ must contain both the occurrences of F in the one branch from the root of this subtree as well as the occurrences of $fe(Q)$ in the other (because they contain T and F), Lemma 2.10 implies that $Y$ must (strictly) contain $fe(Q)$ and $fe(Q)[\mathsf{T} \mapsto \mathsf{F}]$. Hence there is a $Z \in \mathcal{T}_A$ such that $fe(P) = X[\triangle \mapsto Z]$, which violates the induction hypothesis. The case for $fe(P \mathbin{{}^{\bullet}\!\vee} Q)$ proceeds analogously. ■

We now arrive at two crucial definitions for our completeness proof. When considering $*$-terms we already know that $fe(P \mathbin{{}_{\bullet}\!\wedge} Q)$ can be decomposed as

$$fe(P)[\mathsf{T} \mapsto \triangle_1, \mathsf{F} \mapsto \triangle_2][\triangle_1 \mapsto fe(Q), \triangle_2 \mapsto fe(Q)[\mathsf{T} \mapsto \mathsf{F}]].$$

Our goal now is to give a definition for a type of decomposition so that this is the only such decomposition for $fe(P \mathbin{{}_{\bullet}\!\wedge} Q)$. We also ensure that $fe(P \mathbin{{}^{\bullet}\!\vee} Q)$ does not have a decomposition of that type, so that we can distinguish $fe(P \mathbin{{}_{\bullet}\!\wedge} Q)$ from $fe(P \mathbin{{}^{\bullet}\!\vee} Q)$. Similarly, we define another type of decomposition so that $fe(P \mathbin{{}^{\bullet}\!\vee} Q)$ can only be decomposed as

$$fe(P)[\mathsf{T} \mapsto \triangle_1, \mathsf{F} \mapsto \triangle_2][\triangle_1 \mapsto fe(Q)[\mathsf{F} \mapsto \mathsf{T}], \triangle_2 \mapsto fe(Q)]$$

and that $fe(P \mathbin{{}_{\bullet}\!\wedge} Q)$ does not have a decomposition of that type.

**Definition 2.13:** The pair $(Y, Z) \in \mathcal{T}_{A,1,2} \times \mathcal{T}_A$ is a *candidate conjunction decomposition (ccd)* of $X \in \mathcal{T}_A$, if

- $X = Y[\triangle_1 \mapsto Z, \triangle_2 \mapsto Z[\mathsf{T} \mapsto \mathsf{F}]]$,
- $Y$ contains both $\triangle_1$ and $\triangle_2$,
- $Y$ contains neither T nor F, and
- $Z$ contains both T and F.

Similarly, $(Y, Z)$ is a *candidate disjunction decomposition (cdd)* of $X$, if

- $X = Y[\triangle_1 \mapsto Z[\mathsf{F} \mapsto \mathsf{T}], \triangle_2 \mapsto Z]$,
- $Y$ contains both $\triangle_1$ and $\triangle_2$,
- $Y$ contains neither T nor F, and
- $Z$ contains both T and F.

The ccd and cdd are not necessarily the decompositions we are looking for, because, for example, $fe((P \mathbin{{}_{\bullet}\!\wedge} Q) \mathbin{{}_{\bullet}\!\wedge} R)$ has a ccd $(fe(P)[\mathsf{T} \mapsto \triangle_1, \mathsf{F} \mapsto \triangle_2], fe(Q \mathbin{{}_{\bullet}\!\wedge} R))$, whereas the decomposition we need is $(fe(P \mathbin{{}_{\bullet}\!\wedge} Q)[\mathsf{T} \mapsto \triangle_1, \mathsf{F} \mapsto \triangle_2], fe(R))$. Therefore we refine these definitions to obtain the decompositions we seek.

**Definition 2.14:** The pair $(Y, Z) \in \mathcal{T}_{A,1,2} \times \mathcal{T}_A$ is a *conjunction decomposition (cd)* of $X \in \mathcal{T}_A$, if it is a ccd of $X$ and there is no other ccd $(Y', Z')$ of $X$ where the depth of $Z'$ is smaller than that of $Z$. Similarly, $(Y, Z)$ is a *disjunction decomposition (dd)* of $X$, if it is a cdd of $X$ and there is no other cdd $(Y', Z')$ of $X$ where the depth of $Z'$ is smaller than that of $Z$.

**Theorem 2.15:** *For any $*$-term $P \mathbin{_{\bullet}\!\wedge} Q$, i.e. with $P \in P^*$ and $Q \in P^d$, $fe(P \mathbin{_{\bullet}\!\wedge} Q)$ has the (unique) cd*

$$(fe(P)[T \mapsto \triangle_1, F \mapsto \triangle_2], fe(Q))$$

*and no dd. For any $*$-term $P \mathbin{^{\bullet}\!\vee} Q$, i.e. with $P \in P^*$ and $Q \in P^c$, $fe(P \mathbin{^{\bullet}\!\vee} Q)$ has no cd and its (unique) dd is*

$$(fe(P)[T \mapsto \triangle_1, F \mapsto \triangle_2], fe(Q)).$$

***Proof:*** We first treat the case for $P \mathbin{_{\bullet}\!\wedge} Q$ and start with cd. Note that $fe(P \mathbin{_{\bullet}\!\wedge} Q)$ has a ccd $(fe(P)[\mathrm{T} \mapsto \triangle_1, \mathrm{F} \mapsto \triangle_2], fe(Q))$ by definition of $fe$ (for the first condition) and by Lemma 2.11 (for the fourth condition). It is immediate that it satisfies the second and third conditions. It also follows that for any ccd $(Y, Z)$ either $Z$ contains or is contained in $fe(Q)$, for suppose otherwise, then $Y$ will contain an occurrence of T or of F, namely those we know by Lemma 2.11 that $fe(Q)$ has. Therefore it suffices to show that there is no ccd $(Y, Z)$ where $Z$ is strictly contained in $fe(Q)$. Suppose for contradiction that $(Y, Z)$ is such a ccd. If $Z$ is strictly contained in $fe(Q)$ we can decompose $fe(Q)$ as $fe(Q) = V[\triangle \mapsto Z]$ for some $V \in \mathcal{T}_{A,\triangle}$ that contains but is not equal to $\triangle$. By Lemma 2.12 this implies that $V$ contains T or F. But then so does $Y$, because

$$Y = fe(P)[\mathrm{T} \mapsto V[\triangle \mapsto \triangle_1], \mathrm{F} \mapsto V[\triangle \mapsto \triangle_2]],$$

and so $(Y, Z)$ is not a ccd for $fe(P \mathbin{_{\bullet}\!\wedge} Q)$. Therefore $(fe(P)[\mathrm{T} \mapsto \triangle_1, \mathrm{F} \mapsto \triangle_2], fe(Q))$ is the *unique* cd for $fe(P \mathbin{_{\bullet}\!\wedge} Q)$.

Now for the dd. It suffices to show that there is no cdd for $fe(P \mathbin{_{\bullet}\!\wedge} Q)$. Suppose for contradiction that $(Y, Z)$ is a cdd for $fe(P \mathbin{_{\bullet}\!\wedge} Q)$. We note that $Z$ cannot be contained in $fe(Q)$, for then by Lemma 2.12, $Y$ would contain T or F. So $Z$ (strictly) contains $fe(Q)$. But then because

$$Y[\triangle_1 \mapsto Z[\mathrm{F} \mapsto \mathrm{T}], \triangle_2 \mapsto Z] = fe(P \mathbin{_{\bullet}\!\wedge} Q),$$

we would have by Lemma 2.11 that $fe(P \mathbin{_{\bullet}\!\wedge} Q)$ does not contain an occurrence of $fe(Q)[\mathrm{T} \mapsto \mathrm{F}]$. But the cd of $fe(P \mathbin{_{\bullet}\!\wedge} Q)$ tells us that it does, contradiction! Therefore there is no cdd, and hence no dd, for $fe(P \mathbin{_{\bullet}\!\wedge} Q)$. The case for $fe(P \mathbin{^{\bullet}\!\vee} Q)$ proceeds analogously. ■

At this point we have the tools necessary to invert $fe$ on $*$-terms, at least down to the level of $\ell$-terms. We note that we can easily detect if a tree in the image of $fe$ is in the image of $P^\ell$, because all leaves to the left of the root are one truth value, while all the leaves to the right are the other. To invert $fe$ on T-$*$-terms we still need to be able to reconstruct $fe(P^{\mathrm{T}})$ and $fe(Q^*)$ from $fe(P^{\mathrm{T}} \mathbin{_{\bullet}\!\wedge} Q^*)$. To this end we define a T-$*$-decomposition.

**Definition 2.16:** The pair $(Y, Z) \in \mathcal{T}_{A,\triangle} \times \mathcal{T}_A$ is a T-$*$-*decomposition (tsd)* of $X \in \mathcal{T}_A$, if $X = Y[\triangle \mapsto Z]$, $Y$ does not contain T or F and there is no decomposition $(V, W) \in \mathcal{T}_{A,\triangle} \times \mathcal{T}_A$ of $Z$ such that

- $Z = V[\triangle \mapsto W]$,
- $V$ contains $\triangle$,
- $V \neq \triangle$, and
- $V$ contains neither T nor F.

**Theorem 2.17:** *For any T-term P and $*$-term Q the (unique) tsd of $fe(P \leftand Q)$ is*

$$(fe(P)[T \mapsto \triangle], fe(Q)).$$

***Proof:*** First we observe that $(fe(P)[\mathrm{T} \mapsto \triangle], fe(Q))$ is a tsd because by definition of $fe$ on $\leftand$ we have $fe(P)[\mathrm{T} \mapsto fe(Q)] = fe(P \leftand Q)$ and $fe(Q)$ is non-decomposable by Lemma 2.12.

Suppose for contradiction that there is another tsd $(Y, Z)$ of $fe(P \leftand Q)$. Now $Z$ must contain or be contained in $fe(Q)$ for otherwise $Y$ would contain T or F, i.e. the ones we know $fe(Q)$ has by Lemma 2.11.

If $Z$ is strictly contained in $fe(Q)$, then $fe(Q) = V[\triangle \mapsto Z]$ for some $V \in \mathcal{T}_{A,\triangle}$ with $V \neq \triangle$ and $V$ not containing T or F (because then $Y$ would too). But this violates Lemma 2.12, which states that no such decomposition exists. If $Z$ strictly contains $fe(Q)$, then $Z$ contains at least one atom from $P$. But the left branch of any atom in $fe(P)$ is equal to its right branch and hence $Z$ is decomposable. Therefore $(fe(P)[\mathrm{T} \mapsto \triangle], fe(Q))$ is the *unique* tsd of $fe(P \leftand Q)$. ■

*Completeness.* With the last two theorems we can prove completeness for FFEL. We define three auxiliary functions to aid in our definition of the inverse of $fe$ on FNF. Let $\mathrm{cd} : \mathcal{T}_A \to \mathcal{T}_{A,1,2} \times \mathcal{T}_A$ be the function that returns the conjunction decomposition of its argument, dd of the same type its disjunction decomposition and $\mathrm{tsd} : \mathcal{T}_A \to \mathcal{T}_{A,\triangle} \times \mathcal{T}_A$ its T-$*$-decomposition. Naturally, these functions are undefined when their argument does not have a decomposition of the specified type. Each of these functions returns a pair and we will use $\mathrm{cd}_1$ ($\mathrm{dd}_1$, $\mathrm{tsd}_1$) to denote the first element of this pair and $\mathrm{cd}_2$ ($\mathrm{dd}_2$, $\mathrm{tsd}_2$) to denote the second element.

We define $g : \mathcal{T}_A \to \mathcal{SP}_A$ using the functions $g^{\mathrm{T}} : \mathcal{T}_A \to \mathcal{SP}_A$ for inverting trees in the image of T-terms and $g^{\mathrm{F}}, g^{\ell}$ and $g^*$ of the same type for inverting trees in the image of F-terms, $\ell$-terms and $*$-terms, respectively. These functions are defined as follows.

$$g^{\mathrm{T}}(X) = \begin{cases} \mathrm{T} & \text{if } X = \mathrm{T}, \\ a \leftor g^{\mathrm{T}}(Y) & \text{if } X = Y \trianglelefteq a \trianglerighteq Z. \end{cases} \tag{38}$$

We note that we might as well have used the right branch from the root in the recursive case. We chose the left branch here to more closely mirror the definition of the corresponding function for Free short-circuit logic as defined in Staudt (2012) and Ponse and Staudt (2018).

$$g^{\mathrm{F}}(X) = \begin{cases} \mathrm{F} & \text{if } X = \mathrm{F}, \\ a \leftand g^{\mathrm{F}}(Z) & \text{if } X = Y \trianglelefteq a \trianglerighteq Z. \end{cases} \tag{39}$$

Similarly, we could have taken the left branch in this case.

$$g^{\ell}(X) = \begin{cases} a \mathbin{{}_{\bullet}\!\!\wedge} g^{\mathsf{T}}(Y) & \text{if } X = Y \trianglelefteq a \trianglerighteq Z \text{ for some } a \in A, \\ & \text{and } Y \text{ only has T-leaves,} \\ \neg a \mathbin{{}_{\bullet}\!\!\wedge} g^{\mathsf{T}}(Z) & \text{if } X = Y \trianglelefteq a \trianglerighteq Z \text{ for some } a \in A, \\ & \text{and } Z \text{ only has T-leaves.} \end{cases} \tag{40}$$

$$g^{*}(X) = \begin{cases} g^{*}(\mathrm{cd}_1(X)[\triangle_1 \mapsto \mathsf{T}, \triangle_2 \mapsto \mathsf{F}]) \mathbin{{}_{\bullet}\!\!\wedge} g^{*}(\mathrm{cd}_2(X)) & \text{if } X \text{ has a cd,} \\ g^{*}(\mathrm{dd}_1(X)[\triangle_1 \mapsto \mathsf{T}, \triangle_2 \mapsto \mathsf{F}]) \mathbin{{}^{\bullet}\!\!\vee} g^{*}(\mathrm{dd}_2(X)) & \text{if } X \text{ has a dd,} \\ g^{\ell}(X) & \text{otherwise.} \end{cases} \tag{41}$$

We can immediately see how Theorem 2.15 will be used in the correctness proof of $g^{*}$.

$$g(X) = \begin{cases} g^{\mathsf{T}}(X) & \text{if } X \text{ has only T-leaves,} \\ g^{\mathsf{F}}(X) & \text{if } X \text{ has only F-leaves,} \\ g^{\mathsf{T}}(\mathrm{tsd}_1(X)[\triangle \mapsto \mathsf{T}]) \mathbin{{}_{\bullet}\!\!\wedge} g^{*}(\mathrm{tsd}_2(X)) & \text{otherwise.} \end{cases} \tag{42}$$

Similarly, we can see how Theorem 2.17 is used in the correctness proof of $g$. It should come as no surprise that $g$ is indeed correct and inverts $fe$ on FNF.

**Theorem 2.18:** *For all $P \in$ FNF, $g(fe(P)) = P$, i.e. $g(fe(P))$ is syntactically equal to $P$ for $P \in$ FNF.*

The proof for this theorem can be found in Appendix 1. For the sake of completeness, we separately state the completeness result below.

**Theorem 2.19 (Completeness):** *For all $P, Q \in \mathcal{SP}_A$, if FFEL $\models P = Q$ then EqFFEL $\vdash P = Q$.*

***Proof:*** It suffices to show that for $P, Q \in$ FNF, $fe(P) = fe(Q)$ implies $P = Q$, i.e. $P$ and $Q$ are syntactically equal. To see this suppose that $P'$ and $Q'$ are two FEL-terms and $fe(P') = fe(Q')$. By Theorem 2.9, $P'$ is derivably equal to an FNF-term $P$, i.e. EqFFEL $\vdash P' = P$, and $Q'$ is derivably equal to an FNF-term $Q$. Lemma 2.6 then gives us $fe(P') = fe(P)$ and $fe(Q') = fe(Q)$, and thus $fe(P) = fe(Q)$. Hence by Theorem 2.18, $P = Q$, so in particular EqFFEL $\vdash P = Q$. Transitivity then gives us EqFFEL $\vdash P' = Q'$ as desired. ■

## 3. Free FEL with undefinedness: FFEL$^{\mathsf{U}}$

In this section we define FFEL$^{\mathsf{U}}$, the extension of FFEL with the truth value *undefined*, for which we use the constant U. Evaluation trees with undefinedness were introduced in Bergstra et al. (2021), where they serve as a semantics for short-circuit logic with undefinedness: the evaluation/interpretation of each atom may be undefined, i.e. not yield a classical truth value (*true* or *false*). Well-known three-valued extensions of propositional logic are Kleene's 'strong' three-valued logic (1938), in which evaluation is executed in parallel so that $\mathsf{F} \wedge x = x \wedge \mathsf{F} = \mathsf{F}$, Bochvar's 'strict' logic (1938) with a constant $N$ for 'nonsense' or 'meaningless', in which an expression has the value $N$ as soon as it has a component with that value, and McCarthy's 'sequential' logic (1963), in which evaluation proceeds sequentially from left to right so that $\mathsf{F} \wedge \mathsf{U} = \mathsf{F}$ and $\mathsf{U} \wedge \mathsf{F} = \mathsf{U}$. We refer to Bergstra, Bethke, and Rodenburg (1995, Sect.2) for a brief discussion of these logics. Here we provide equational axioms for the equality of evaluation trees for full left-sequential evaluation with undefinedness and prove a completeness result.

**Definition 3.1:** The set $\mathcal{T}_A^{\mathsf{U}}$ of ***U**-evaluation trees* over $A$ with leaves in $\{\mathsf{T}, \mathsf{F}, \mathsf{U}\}$ is defined inductively by

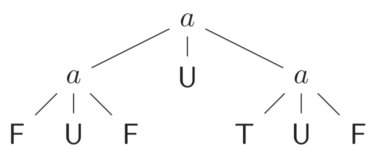
$$\mathsf{T} \in \mathcal{T}_A^{\mathsf{U}}, \quad \mathsf{F} \in \mathcal{T}_A^{\mathsf{U}}, \quad \mathsf{U} \in \mathcal{T}_A^{\mathsf{U}}, \quad (X \trianglelefteq \underline{a} \trianglerighteq Y) \in \mathcal{T}_A^{\mathsf{U}} \quad \text{for any } X, Y \in \mathcal{T}_A^{\mathsf{U}} \text{ and } a \in A.$$

The operator $\_ \trianglelefteq \underline{a} \trianglerighteq \_$ is called ***U**-tree composition over a*. In the evaluation tree $X \trianglelefteq \underline{a} \trianglerighteq Y$, the root is represented by $\underline{a}$, the left branch by $X$, the right branch by $Y$, and the underlining of the root represents a middle branch to the leaf U.

Next to the formal notation for evaluation trees we again introduce a more pictorial representation. For example, the tree

$$(\mathsf{F} \trianglelefteq \underline{a} \trianglerighteq \mathsf{F}) \trianglelefteq \underline{a} \trianglerighteq (\mathsf{T} \trianglelefteq \underline{a} \trianglerighteq \mathsf{F})$$

can be represented as follows, where $\trianglelefteq$ yields a left branch, and $\trianglerighteq$ a right branch:




Finally, observe that equation (RR) on repeated leaf replacements also applies to U-evaluation trees.

We extend the set $\mathcal{SP}_A$ to $\mathcal{SP}_A{}^{\mathsf{U}}$ of (left-sequential) propositional expressions over $A$ with U by the following grammar ($a \in A$):

$$P ::= \mathsf{T} \mid \mathsf{F} \mid \mathsf{U} \mid a \mid \neg P \mid P \leftand P \mid P \leftor P$$

and refer to its signature by $\Sigma_{\mathrm{FEL}}^{\mathsf{U}}(A) = \{\leftand, \leftor, \neg, \mathsf{T}, \mathsf{F}, \mathsf{U}, a \mid a \in A\}$.

We interpret propositional expressions in $\mathcal{SP}_A{}^{\mathsf{U}}$ as evaluation trees by extending the function *fe* (Definition 2.2).

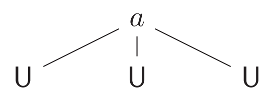
**Definition 3.2:** The unary *full evaluation function* $fe^{\mathsf{U}} : \mathcal{SP}_A{}^{\mathsf{U}} \to \mathcal{T}_A^{\mathsf{U}}$ is defined as follows, where $a \in A$:

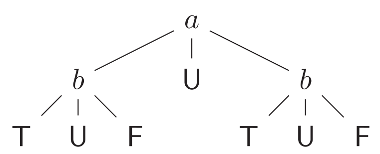
$$fe^{\mathsf{U}}(\mathsf{T}) = \mathsf{T}, \quad fe^{\mathsf{U}}(\mathsf{F}) = \mathsf{F}, \quad fe^{\mathsf{U}}(\neg P) = fe^{\mathsf{U}}(P)[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}],$$

$$fe^{\mathsf{U}}(\mathsf{U}) = \mathsf{U}, \quad fe^{\mathsf{U}}(P \leftand Q) = fe^{\mathsf{U}}(P)[\mathsf{T} \mapsto fe^{\mathsf{U}}(Q), \mathsf{F} \mapsto fe^{\mathsf{U}}(Q)[\mathsf{T} \mapsto \mathsf{F}]],$$

$$fe^{\mathsf{U}}(a) = \mathsf{T} \trianglelefteq \underline{a} \trianglerighteq \mathsf{F}, \quad fe^{\mathsf{U}}(P \leftor Q) = fe^{\mathsf{U}}(P)[\mathsf{T} \mapsto fe^{\mathsf{U}}(Q)[\mathsf{F} \mapsto \mathsf{T}], \mathsf{F} \mapsto fe^{\mathsf{U}}(Q)].$$

Two examples: the evaluation trees $fe^{\mathsf{U}}(a \mathbin{{}_{\bullet}\!\!\wedge} \mathsf{U}) = \mathsf{U} \unlhd \underline{a} \unrhd \mathsf{U}$ and $fe^{\mathsf{U}}((a \mathbin{{}^{\bullet}\!\!\vee} \mathsf{T}) \mathbin{{}_{\bullet}\!\!\wedge} b) = (\mathsf{T} \unlhd \underline{b} \unrhd \mathsf{F}) \unlhd \underline{a} \unrhd (\mathsf{T} \unlhd \underline{b} \unrhd \mathsf{F})$ can be depicted as follows:




**Definition 3.3:** The binary relation $=_{fe^{\mathsf{U}}}$ on $\mathcal{SP}_A^{\mathsf{U}}$ defined by $P =_{fe^{\mathsf{U}}} Q \iff fe^{\mathsf{U}}(P) = fe^{\mathsf{U}}(Q)$ is called *free full **U**-valuation congruence*.

**Lemma 3.4:** *The relation* $=_{fe^{U}}$ *is a congruence.*

***Proof:*** Cf. Lemma 2.4. ■

**Definition 3.5:** A *Fully Evaluated Left-Sequential Logic with undefinedness (FEL*$^{\mathsf{U}}$*)* is a logic that satisfies the consequences of $fe^{\mathsf{U}}$-equality. *Free Fully Evaluated Left-Sequential Logic with undefinedness (FFEL*$^{\mathsf{U}}$*)* is the fully evaluated left-sequential logic with undefinedness that satisfies no more consequences than those of $fe^{\mathsf{U}}$-equality, i.e. for all $P, Q \in \mathcal{SP}_A^{\mathsf{U}}$,

$$\mathrm{FEL}^{\mathsf{U}} \models P = Q \Longleftarrow P =_{fe^{\mathsf{U}}} Q \quad \text{and} \quad \mathrm{FFEL}^{\mathsf{U}} \models P = Q \iff P =_{fe^{\mathsf{U}}} Q.$$

In order to axiomatise FFEL$^{\mathsf{U}}$, we extend EqFFEL (Table 1) as follows:

$$\mathrm{EqFFEL}^{\mathsf{U}} = \mathrm{EqFFEL} \cup \{\neg \mathsf{U} = \mathsf{U}, \mathsf{U} \mathbin{{}_{\bullet}\!\!\wedge} x = \mathsf{U}\}.$$

We start with a note on duality. Write $P^{dl}$ for the dual of $P \in \mathcal{SP}_A^{\mathsf{U}}$ and define $\mathsf{U}^{dl} = \mathsf{U}$. It immediately follows that also EqFFEL$^{\mathsf{U}}$ satisfies the duality principle. Furthermore, defining $x^{dl} = x$ for each variable $x$, the duality principle extends to equations, that is, for all terms $s, t$ over $\Sigma^{\mathsf{U}}_{\mathrm{FEL}}(A)$,

$$\mathrm{EqFFEL}^{\mathsf{U}} \vdash s = t \iff \mathrm{EqFFEL}^{\mathsf{U}} \vdash s^{dl} = t^{dl}.$$

**Lemma 3.6 (Soundness):** *For all* $P, Q \in \mathcal{SP}_A^{U}$, EqFFEL$^{U}$ $\vdash P = Q \Longrightarrow$ FFEL$^{U}$ $\models P = Q$.

***Proof:*** By Lemma 3.4, the relation $=_{fe^{\mathsf{U}}}$ is a congruence on $\mathcal{SP}_A^{\mathsf{U}}$, so it suffices to show that all closed instances of the EqFFEL$^{\mathsf{U}}$-axioms satisfy $=_{fe^{\mathsf{U}}}$, which follows easily (cf. the proof of Lemma 2.6). The validity of the two new axioms of EqFFEL$^{\mathsf{U}}$ is also easily verified. ■

In the remainder of this section we prove completeness of EqFFEL$^{\mathsf{U}}$. We start with a lemma on properties of U.

**Lemma 3.7:** *The following equations are consequences of* EqFFEL$^{U}$*:*

(1) $x \mathbin{{}^{\bullet}\!\!\vee} (y \mathbin{{}_{\bullet}\!\!\wedge} U) = (x \mathbin{{}^{\bullet}\!\!\vee} y) \mathbin{{}_{\bullet}\!\!\wedge} U$,

(2) $x \leftor (y \leftand U) = x \leftand (y \leftand U)$,
(3) $\neg x \leftand (y \leftand U) = x \leftand (y \leftand U)$.

***Proof:*** Note that by duality, $\mathsf{U} \leftor x = \mathsf{U}$. We first derive two auxiliary results:

$$x \leftor \mathsf{U} = x \leftor (\mathsf{U} \leftand \mathsf{F}) \overset{\text{(FFEL10)}}{=} x \leftand (\mathsf{U} \leftor \mathsf{T}) = x \leftand \mathsf{U}, \qquad \text{(Aux1)}$$

$$\begin{aligned} \neg x \leftand \mathsf{U} &= \neg x \leftand (\mathsf{U} \leftand \mathsf{F}) = \neg x \leftand (\mathsf{F} \leftand \mathsf{U}) \\ &= (\neg x \leftand \mathsf{F}) \leftand \mathsf{U} = (x \leftand \mathsf{F}) \leftand \mathsf{U} = x \leftand \mathsf{U}. \qquad \text{(Aux2)} \end{aligned}$$

Consequence 1: $x \leftor (y \leftand \mathsf{U}) \overset{\text{(Aux1)}}{=} x \leftor (y \leftor \mathsf{U}) = (x \leftor y) \leftor \mathsf{U} \overset{\text{(Aux1)}}{=} (x \leftor y) \leftand \mathsf{U}$.

Consequence 2: $x \leftor (y \leftand \mathsf{U}) = x \leftor ((y \leftand \mathsf{U}) \leftand \mathsf{F}) \overset{\text{(FFEL10)}}{=} x \leftand ((y \leftand \mathsf{U}) \leftor \mathsf{T}) \overset{\text{(Aux1)}}{=} x \leftand ((y \leftor \mathsf{U}) \leftor \mathsf{T}) = x \leftand (y \leftor (\mathsf{U} \leftor \mathsf{T})) = x \leftand (y \leftor \mathsf{U}) \overset{\text{(Aux1)}}{=} x \leftand (y \leftand \mathsf{U})$.

Consequence 3: by consequences 1 and 2, $(x \leftor y) \leftand \mathsf{U} = x \leftand (y \leftand \mathsf{U})$, hence $\neg x \leftand (y \leftand \mathsf{U}) = \neg x \leftand (\neg y \leftand \mathsf{U}) = (\neg x \leftand \neg y) \leftand \mathsf{U} \overset{\text{(Aux2)}}{=} (x \leftor y) \leftand \mathsf{U} \overset{\text{La.3.7.2}}{=} x \leftand (y \leftand \mathsf{U})$. ■

We introduce 'U-normal forms' and prove a few more lemmas from which our completeness result follows easily.

**Definition 3.8:** For $\sigma \in A^*$ and $a \in A$, $\boldsymbol{U_\sigma}$ is defined by $\mathsf{U}_\epsilon = \mathsf{U}$ and $\mathsf{U}_{a\rho} = a \leftand \mathsf{U}_\rho$.

**Lemma 3.9:** *For all $\sigma \in A^*$, $\boldsymbol{U}_\sigma \leftand x = \boldsymbol{U}_\sigma$ and $\boldsymbol{U}_\sigma \leftor x = \boldsymbol{U}_\sigma$ are consequences of* $\mathrm{EqFFEL}^U$.

***Proof:*** The first consequence follows easily by induction on the length of $\sigma$ and associativity (FFEL4), as well as $(\mathsf{U}_\sigma)^{dl} = \mathsf{U}_\sigma$. By the latter, $\mathsf{U}_\sigma \leftor x = \mathsf{U}_\sigma$. ■

**Lemma 3.10:** *For all $\sigma \in A^*$, the following equations are consequences of* $\mathrm{EqFFEL}^U$ :

(1) $x \leftor (y \leftand U_\sigma) = (x \leftor y) \leftand \mathsf{U}_\sigma$,
(2) $x \leftor (y \leftand U_\sigma) = x \leftand (y \leftand \mathsf{U}_\sigma)$,
(3) $\neg x \leftand (y \leftand U_\sigma) = x \leftand (y \leftand \mathsf{U}_\sigma)$.

***Proof:*** By simultaneous induction on the length of $\sigma$. The base case ($\sigma = \epsilon$) is Lemma 3.7.1-3. For $\sigma = a\rho\,(a \in A)$, derive

$$\begin{aligned} x \leftor (y \leftand \mathsf{U}_{a\rho}) &= x \leftor (y \leftand (a \leftand \mathsf{U}_\rho)) \overset{\text{IH2}}{=} x \leftor (y \leftor (a \leftand \mathsf{U}_\rho)) \\ &= (x \leftor y) \leftor (a \leftand \mathsf{U}_\rho) = (x \leftor y) \leftand \mathsf{U}_{a\rho}, \\ x \leftor (y \leftand \mathsf{U}_{a\rho}) &= x \leftor ((y \leftand a) \leftand \mathsf{U}_\rho) \overset{\text{IH2}}{=} x \leftand ((y \leftand a) \leftand \mathsf{U}_\rho) = x \leftand (y \leftand \mathsf{U}_{a\rho}), \end{aligned}$$

$$\neg x \mathbin{{}_{\bullet}\!\wedge} (y \mathbin{{}_{\bullet}\!\wedge} \mathsf{U}_{a\rho}) = \neg x \mathbin{{}_{\bullet}\!\wedge} ((y \mathbin{{}_{\bullet}\!\wedge} a) \mathbin{{}_{\bullet}\!\wedge} \mathsf{U}_{\rho}) \overset{\text{IH3}}{=} x \mathbin{{}_{\bullet}\!\wedge} ((y \mathbin{{}_{\bullet}\!\wedge} a) \mathbin{{}_{\bullet}\!\wedge} \mathsf{U}_{\rho}) = x \mathbin{{}_{\bullet}\!\wedge} (y \mathbin{{}_{\bullet}\!\wedge} \mathsf{U}_{a\rho}).$$

■

To prove the forthcoming completeness result, it suffices to restrict to Negation Normal Forms (NNFs), which are defined as follows ($a \in A$):

$$P ::= \mathsf{T} \mid \mathsf{F} \mid \mathsf{U} \mid a \mid \neg a \mid P \mathbin{{}_{\bullet}\!\wedge} P \mid P \mathbin{{}^{\bullet}\!\vee} P.$$

By structural induction it immediately follows that for each $P \in \mathcal{SP}_A^{\mathsf{U}}$, there is a unique NNF $Q$ such that $\mathrm{EqFFEL}^{\mathsf{U}} \vdash P = Q$.

**Lemma 3.11:** *For each $P \in \mathcal{SP}_A^{\mathsf{U}}$ and $\sigma \in A^*$, there exists $\rho \in A^*$ such that* $\mathrm{EqFFEL}^{\mathsf{U}} \vdash P \mathbin{{}_{\bullet}\!\wedge} \mathsf{U}_\sigma = \mathsf{U}_\rho$.

***Proof:*** By structural induction, restricting to NNFs. The base cases $\mathsf{F} \mathbin{{}_{\bullet}\!\wedge} \mathsf{U}_\sigma = \mathsf{U}_\sigma$ and $\neg a \mathbin{{}_{\bullet}\!\wedge} \mathsf{U}_\sigma = a \mathbin{{}_{\bullet}\!\wedge} \mathsf{U}_\sigma = \mathsf{U}_{a\sigma}$ follow by Lemma 3.10.3, and the other base cases are trivial. For the induction there are two cases:

Case $P = Q \mathbin{{}_{\bullet}\!\wedge} R$. Derive $(Q \mathbin{{}_{\bullet}\!\wedge} R) \mathbin{{}_{\bullet}\!\wedge} \mathsf{U}_\sigma = Q \mathbin{{}_{\bullet}\!\wedge} (R \mathbin{{}_{\bullet}\!\wedge} \mathsf{U}_\sigma) \overset{\text{IH}}{=} Q \mathbin{{}_{\bullet}\!\wedge} \mathsf{U}_{\rho'} \overset{\text{IH}}{=} \mathsf{U}_\rho$ for some $\rho', \rho \in A^*$.

Case $P = Q \mathbin{{}^{\bullet}\!\vee} R$. By Lemma 3.10.1 and induction, $(Q \mathbin{{}^{\bullet}\!\vee} R) \mathbin{{}_{\bullet}\!\wedge} \mathsf{U}_\sigma = Q \mathbin{{}^{\bullet}\!\vee} (R \mathbin{{}_{\bullet}\!\wedge} \mathsf{U}_\sigma) = Q \mathbin{{}^{\bullet}\!\vee} \mathsf{U}_{\rho'}$ for some $\rho' \in A^*$, and by Lemma 3.10.2 and induction, $Q \mathbin{{}^{\bullet}\!\vee} \mathsf{U}_{\rho'} = Q \mathbin{{}_{\bullet}\!\wedge} \mathsf{U}_{\rho'} = \mathsf{U}_\rho$ for some $\rho \in A^*$. ■

**Lemma 3.12:** *For each $P \in \mathcal{SP}_A^{\mathsf{U}}$ that contains* $\mathsf{U}$*, there exists $\sigma \in A^*$ such that* $\mathrm{EqFFEL}^{\mathsf{U}} \vdash P = \mathsf{U}_\sigma$.

***Proof:*** By structural induction, restricting to NNFs. The only base case is $P = \mathsf{U} = \mathsf{U}_\epsilon$. For the induction there are two cases:

Case $P = Q \mathbin{{}_{\bullet}\!\wedge} R$. Apply a case distinction: if $\mathsf{U}$ occurs in $Q$, then by induction, $Q = \mathsf{U}_\sigma$ and by Lemma 3.9, $P = \mathsf{U}_\sigma \mathbin{{}_{\bullet}\!\wedge} R = \mathsf{U}_\sigma$; if $\mathsf{U}$ does not occur in $Q$, then $\mathsf{U}$ occurs in $R$ and by induction, $R = \mathsf{U}_\sigma$, so $P = Q \mathbin{{}_{\bullet}\!\wedge} \mathsf{U}_\sigma$. By Lemma 3.11, $Q \mathbin{{}_{\bullet}\!\wedge} \mathsf{U}_\sigma = \mathsf{U}_\rho$ for some $\rho \in A^*$.

Case $P = Q \mathbin{{}^{\bullet}\!\vee} R$. Apply a case distinction: if $\mathsf{U}$ occurs in $Q$, then by induction, $Q = \mathsf{U}_\sigma$, and by Lemma 3.9, $P = \mathsf{U}_\sigma \mathbin{{}^{\bullet}\!\vee} R = \mathsf{U}_\sigma$; if $\mathsf{U}$ does not occur in $Q$, then $\mathsf{U}$ occurs in $R$ and by induction, $R = \mathsf{U}_\sigma$, so, $P = Q \mathbin{{}^{\bullet}\!\vee} \mathsf{U}_\sigma$. By Lemmas 3.10.2 and 3.11, $Q \mathbin{{}^{\bullet}\!\vee} \mathsf{U}_\sigma = Q \mathbin{{}_{\bullet}\!\wedge} \mathsf{U}_\sigma = \mathsf{U}_\rho$ for some $\rho \in A^*$. ■

**Theorem 3.13 (Completeness):** *The logic* $\mathrm{FFEL}^{\mathsf{U}}$ *is axiomatised by* $\mathrm{EqFFEL}^{\mathsf{U}}$.

***Proof:*** By Lemma 3.6, $\mathrm{EqFFEL}^{\mathsf{U}}$ is sound. For completeness, assume $P_1 =_{fe^{\mathsf{U}}} P_2$. Then, either $P_1$ and $P_2$ do not contain $\mathsf{U}$ and by Theorem 2.19 we are done, or both $P_1$ and $P_2$ contain $\mathsf{U}$. By Lemma 3.12, there are $\mathsf{U}_{\sigma_i} \in A^*$ such that $\mathrm{EqFFEL}^{\mathsf{U}} \vdash P_i = \mathsf{U}_{\sigma_i}$. By assumption and soundness, $\sigma_1 = \sigma_2$. Hence $\mathrm{EqFFEL}^{\mathsf{U}} \vdash P_1 = \mathsf{U}_{\sigma_1} = P_2$. ■

## 4. Memorising FEL (MFEL)

In this section we define and axiomatise 'Memorising FEL' (MFEL). As noted in the Introduction, the logic MFEL is based on the evaluation strategy in which the evaluation of each atom is memorised: no complete path in a memorising evaluation tree contains multiple occurrences of the same atom. Memorising evaluation trees appear to be a fundamental tool in the investigation of stronger logics such as $C\ell FEL_2$ (Section 6) and several short-circuit logics defined in Bergstra et al. (2021) and Bergstra and Ponse (2025a). We present an equational axiomatisation of the equality of memorising evaluation trees and prove a completeness result.

**Definition 4.1:** The evaluation trees $\mathsf{T}, \mathsf{F} \in \mathcal{T}_A$ are *memorising evaluation trees*.

The evaluation tree $(X \trianglelefteq a \trianglerighteq Y) \in \mathcal{T}_A$ is a *memorising evaluation tree* over $A$ if both $X$ and $Y$ are memorising evaluation trees that do not contain the label $a$.

We interpret propositional expressions in $\mathcal{SP}_A$ as memorising evaluation trees by a function *mfe*.

**Definition 4.2:** The unary *memorising full evaluation* function $mfe : \mathcal{SP}_A \to \mathcal{T}_A$ is defined by

$$mfe(P) = m(fe(P)),$$

where the auxiliary function $m : \mathcal{T}_A \to \mathcal{T}_A$ is defined as follows, for $a \in A$:

$$m(B) = B \quad \text{for } B \in \{\mathsf{T}, \mathsf{F}\}, m(X \trianglelefteq a \trianglerighteq Y) = m(L_a(X)) \trianglelefteq a \trianglerighteq m(R_a(Y)),$$

and the auxiliary functions $L_a, R_a : \mathcal{T}_A \to \mathcal{T}_A$ are defined as follows, for $b \in A$:

$$L_a(B) = R_a(B) = B \quad \text{for } B \in \{\mathsf{T}, \mathsf{F}\},$$

$$L_a(X \trianglelefteq b \trianglerighteq Y) = \begin{cases} L_a(X) & \text{if } b = a, \\ L_a(X) \trianglelefteq b \trianglerighteq L_a(Y) & \text{otherwise.} \end{cases}$$

$$R_a(X \trianglelefteq b \trianglerighteq Y) = \begin{cases} R_a(Y) & \text{if } b = a, \\ R_a(X) \trianglelefteq b \trianglerighteq R_a(Y) & \text{otherwise.} \end{cases}$$

Let $A^s$ be the set of strings over $A$ with the property that each $\sigma \in A^s$ contains no multiple occurrences of the same atom. Hence, for all $P \in \mathcal{SP}_A$, each path in $mfe(P)$ from the root to a leaf has a sequence of labels of the form $\sigma \in A^s$. It is clear that not all memorising evaluation trees can be expressed in MFEL, a simple example is

$$(\mathsf{T} \trianglelefteq b \trianglerighteq \mathsf{F}) \trianglelefteq a \trianglerighteq (\mathsf{T} \trianglelefteq c \trianglerighteq \mathsf{F}).$$

Three examples of memorising evaluation trees in $mfe(\mathcal{SP}_A)$:

$$mfe(a \,\&\!\!\wedge\, a)$$

$$= m(fe(a \,\&\!\!\wedge\, a)) = m(fe(a)[\mathsf{T} \mapsto fe(a), \mathsf{F} \mapsto fe(a)])$$

$$= m((\mathsf{T} \trianglelefteq a \trianglerighteq \mathsf{F}) \trianglelefteq a \trianglerighteq (\mathsf{T} \trianglelefteq a \trianglerighteq \mathsf{F})) = m(L_a(\mathsf{T})) \trianglelefteq a \trianglerighteq m(R_a(\mathsf{F})) = fe(a),$$

**Table 2.** EqMFEL, axioms for MFEL.

| | |
|---|---|
| Import: EqFFEL (Table 1) | |
| $(x \,{}^{\bullet}\!\!\vee\, y) \,{}_{\bullet}\!\!\wedge\, z = (\neg x \,{}_{\bullet}\!\!\wedge\, (y \,{}_{\bullet}\!\!\wedge\, z)) \,{}^{\bullet}\!\!\vee\, (x \,{}_{\bullet}\!\!\wedge\, z)$ | (M1) |

$$mfe(a \,{}^{\bullet}\!\!\vee\, \neg a) = \mathsf{T} \trianglelefteq a \trianglerighteq \mathsf{T} = fe(a \,{}^{\bullet}\!\!\vee\, \mathsf{T}),$$

$$mfe((a \,{}_{\bullet}\!\!\wedge\, b) \,{}^{\bullet}\!\!\vee\, (\neg a \,{}_{\bullet}\!\!\wedge\, \neg b)) = (\mathsf{T} \trianglelefteq b \trianglerighteq \mathsf{F}) \trianglelefteq a \trianglerighteq (\mathsf{F} \trianglelefteq b \trianglerighteq \mathsf{T}).$$

The last example suggests that all memorising evaluation trees with complete traces $abB_i$ with $i = 1,.., 4$ and $B_i \in \{\mathsf{T}, \mathsf{F}\}$ can be expressed in MFEL and below we show that each such evaluation tree can be expressed as $mfe((a \,{}_{\bullet}\!\!\wedge\, t_1) \,{}^{\bullet}\!\!\vee\, (\neg a \,{}_{\bullet}\!\!\wedge\, t_2))$ with $t_j \in \{b, \neg b, b \,{}_{\bullet}\!\!\wedge\, \mathsf{F}, b \,{}^{\bullet}\!\!\vee\, \mathsf{T}\}$. We note that this last evaluation tree cannot be expressed in FFEL and return to expressiveness issues in Section 8.

**Definition 4.3:** The binary relation $=_{mfe}$ on $\mathcal{SP}_A$ is called *memorising full valuation congruence* and is defined by $P =_{mfe} Q \iff mfe(P) = mfe(Q)$.

The following lemma is an adaptation of Bergstra et al. (2021, La.3.5), a detailed proof is given in Appendix 2.

**Lemma 4.4:** *The relation $=_{mfe}$ is a congruence.*

**Definition 4.5:** *Memorising Fully Evaluated Left-Sequential Logic (*MFEL*)* is the fully evaluated left-sequential logic that satisfies no more consequences than those of *mfe*-equality, i.e. for all $P, Q \in \mathcal{SP}_A$,

$$\text{MFEL} \models P = Q \iff P =_{mfe} Q.$$

In Table 2 we provide a set EqMFEL of axioms for MFEL that is an extension of EqFFEL with the axiom (M1). In the remainder of this section we show that EqMFEL axiomatises MFEL.

**Lemma 4.6 (Soundness):** *For all $P, Q \in \mathcal{SP}_A$,* EqMFEL $\vdash P = Q \Longrightarrow$ MFEL $\models P = Q$.

***Proof:*** By Lemma 4.4, the relation $=_{mfe}$ is a congruence on $\mathcal{SP}_A$, so it suffices to show that all closed instances of the EqMFEL-axioms satisfy $=_{mfe}$. By Lemma 2.6 (soundness of FFEL), we only have to prove this for axiom (M1), thus for all $P, Q, R \in \mathcal{SP}_A$,

$$m(fe((P \,{}^{\bullet}\!\!\vee\, Q) \,{}_{\bullet}\!\!\wedge\, R)) = m(fe((\neg P \,{}_{\bullet}\!\!\wedge\, (Q \,{}_{\bullet}\!\!\wedge\, R)) \,{}^{\bullet}\!\!\vee\, (P \,{}_{\bullet}\!\!\wedge\, R))).$$

We prove this in detail in Appendix 2. ■

**Lemma 4.7:** *The equations in Table 3 are consequences of* EqMFEL.

***Proof:*** Each of (C3) and (C4) follows with *Prover9* with (the default) options `lpo` and `unfold` in 1s, but we could not find any short, handwritten proofs. An alternative and

**Table 3.** Consequences of EqMFEL.

| | |
|---|---|
| $x \mathbin{{}_\bullet\!\wedge} (y \mathbin{{}_\bullet\!\wedge} x) = x \mathbin{{}_\bullet\!\wedge} y$ | (C1) |
| $(x \mathbin{{}_\bullet\!\wedge} y) \mathbin{{}^\bullet\!\vee} x = x \mathbin{{}_\bullet\!\wedge} (y \mathbin{{}^\bullet\!\vee} x)$ | (C2) |
| $(x \mathbin{{}_\bullet\!\wedge} y) \mathbin{{}^\bullet\!\vee} (\neg x \mathbin{{}_\bullet\!\wedge} z) = (\neg x \mathbin{{}^\bullet\!\vee} y) \mathbin{{}_\bullet\!\wedge} (x \mathbin{{}^\bullet\!\vee} z)$ | (C3) |
| $x \mathbin{{}_\bullet\!\wedge} (y \mathbin{{}^\bullet\!\vee} z) = (x \mathbin{{}_\bullet\!\wedge} y) \mathbin{{}^\bullet\!\vee} (x \mathbin{{}_\bullet\!\wedge} z)$ | (C4) |

more readable proof by *Prover9* of (C4) that also contains a proof of (C1) is included in Appendix 3 (using options `kbo` and `fold`, and requiring 2s).

The auxiliary result $x \mathbin{{}_\bullet\!\wedge} \mathrm{F} = (x \mathbin{{}_\bullet\!\wedge} \mathrm{F}) \mathbin{{}^\bullet\!\vee} \mathrm{F} \overset{(\mathrm{M1})'}{=} (\neg x \mathbin{{}^\bullet\!\vee} (\mathrm{F} \mathbin{{}^\bullet\!\vee} \mathrm{F})) \mathbin{{}_\bullet\!\wedge} (x \mathbin{{}^\bullet\!\vee} \mathrm{F}) = \neg x \mathbin{{}_\bullet\!\wedge} x$, where (M1)′ is the dual of (M1), is used below.

(C1): $x \mathbin{{}_\bullet\!\wedge} y = x \mathbin{{}_\bullet\!\wedge} (y \mathbin{{}^\bullet\!\vee} \mathrm{F}) \overset{(\mathrm{C4})}{=} (x \mathbin{{}_\bullet\!\wedge} y) \mathbin{{}^\bullet\!\vee} (x \mathbin{{}_\bullet\!\wedge} \mathrm{F}) = (x \mathbin{{}_\bullet\!\wedge} y) \mathbin{{}^\bullet\!\vee} (\neg x \mathbin{{}_\bullet\!\wedge} \mathrm{F}) \overset{(\mathrm{C3})}{=} (\neg x \mathbin{{}^\bullet\!\vee} y) \mathbin{{}_\bullet\!\wedge} (x \mathbin{{}^\bullet\!\vee} \mathrm{F}) = (\neg x \mathbin{{}^\bullet\!\vee} y) \mathbin{{}_\bullet\!\wedge} x \overset{(\mathrm{M1})}{=} (x \mathbin{{}_\bullet\!\wedge} (y \mathbin{{}_\bullet\!\wedge} x)) \mathbin{{}^\bullet\!\vee} (\neg x \mathbin{{}_\bullet\!\wedge} x) = (x \mathbin{{}_\bullet\!\wedge} (y \mathbin{{}_\bullet\!\wedge} x)) \mathbin{{}^\bullet\!\vee} (x \mathbin{{}_\bullet\!\wedge} \mathrm{F}) \overset{(\mathrm{C4})}{=} x \mathbin{{}_\bullet\!\wedge} ((y \mathbin{{}_\bullet\!\wedge} x) \mathbin{{}^\bullet\!\vee} \mathrm{F}) = x \mathbin{{}_\bullet\!\wedge} (y \mathbin{{}_\bullet\!\wedge} x)$.

(C2): $x \mathbin{{}_\bullet\!\wedge} (y \mathbin{{}^\bullet\!\vee} x) \overset{(\mathrm{C4})}{=} (x \mathbin{{}_\bullet\!\wedge} y) \mathbin{{}^\bullet\!\vee} (x \mathbin{{}_\bullet\!\wedge} x) \overset{(\mathrm{C1})}{=} (x \mathbin{{}_\bullet\!\wedge} y) \mathbin{{}^\bullet\!\vee} x$. ■

In order to prove completeness of EqMFEL, we use normal forms.

**Definition 4.8:** Let $\sigma \in A^s$ and $a \in A$. Then $P$ is a ***σ**-normal form* if
$\sigma = \epsilon$ and $P \in \{\mathrm{T}, \mathrm{F}\}$, or if
$\sigma = a\rho$ and $P = (a \mathbin{{}_\bullet\!\wedge} P_1) \mathbin{{}^\bullet\!\vee} (\neg a \mathbin{{}_\bullet\!\wedge} P_2)$ with both $P_1$ and $P_2$ $\rho$-normal forms.

For $\sigma = a\rho \in A^s$, each $\sigma$-normal form $(a \mathbin{{}_\bullet\!\wedge} P_1) \mathbin{{}^\bullet\!\vee} (\neg a \mathbin{{}_\bullet\!\wedge} P_2)$ yields the perfect $a\rho$-tree with root $a$, left child $P_1$ and right child $P_2$. Note that there are $2^{(2^{|\sigma|})}$ $\sigma$-normal forms.

We will often denote $\sigma$-normal forms by $P_\sigma, Q_\sigma, R_\sigma$, etc. In many coming proofs we apply induction on $P_\sigma$, that is, we induct on the length of $\sigma$, often using the template $\sigma = a\rho$ with $a \in A$ for the induction step(s). We call this approach 'by induction on $\sigma$' for short.

**Definition 4.9:** Let $\sigma \in A^s$. The $\sigma$-normal forms $\mathrm{T}_\sigma$ and $\mathrm{F}_\sigma$ are defined by

$$\mathrm{T}_\epsilon = \mathrm{T}, \quad \mathrm{T}_{a\rho} = (a \mathbin{{}_\bullet\!\wedge} \mathrm{T}_\rho) \mathbin{{}^\bullet\!\vee} (\neg a \mathbin{{}_\bullet\!\wedge} \mathrm{T}_\rho), \quad \text{and}$$
$$\mathrm{F}_\epsilon = \mathrm{F}, \quad \mathrm{F}_{a\rho} = (a \mathbin{{}_\bullet\!\wedge} \mathrm{F}_\rho) \mathbin{{}^\bullet\!\vee} (\neg a \mathbin{{}_\bullet\!\wedge} \mathrm{F}_\rho).$$

We will also use terms $\widetilde{\mathbf{F}}_\sigma$ defined by $\widetilde{\mathrm{F}}_\epsilon = \mathrm{F}$ and $\widetilde{\mathrm{F}}_{a\rho} = a \mathbin{{}_\bullet\!\wedge} \widetilde{\mathrm{F}}_\rho$.

**Lemma 4.10:** *For all* $\sigma \in A^s$, EqMFEL $\vdash \neg F_\sigma = T_\sigma, \widetilde{F}_\sigma = F_\sigma$. *If* $\sigma = a\rho$, *then*

$$\text{EqMFEL} \vdash \widetilde{F}_\sigma = \neg a \mathbin{{}_\bullet\!\wedge} \widetilde{F}_\rho.$$

***Proof:*** The first consequence follows by induction on $\sigma$. If $\sigma = \epsilon$, this is trivial, and if $\sigma = a\rho$, then

$$\neg\mathsf{F}_{a\rho} = (\neg a \leftor \neg\mathsf{F}_\rho) \leftand (a \leftor \neg\mathsf{F}_\rho) \overset{\text{(C3),IH}}{=} (a \leftand \mathsf{T}_\rho) \leftor (\neg a \leftand \mathsf{T}_\rho) = \mathsf{T}_{a\rho}.$$

The last consequence follows from associativity and axiom (FFEL7): $\widetilde{\mathsf{F}}_\rho = \widetilde{\mathsf{F}}_\rho \leftand \mathsf{F} = \mathsf{F} \leftand \widetilde{\mathsf{F}}_\rho$, hence

$$\widetilde{\mathsf{F}}_{a\rho} = (a \leftand (\mathsf{F} \leftand \widetilde{\mathsf{F}}_\rho)) = ((a \leftand \mathsf{F}) \leftand \widetilde{\mathsf{F}}_\rho)) \overset{\text{(FFEL8)}}{=} ((\neg a \leftand \mathsf{F}) \leftand \widetilde{\mathsf{F}}_\rho)) = (\neg a \leftand \widetilde{\mathsf{F}}_\rho).$$

Finally, $\widetilde{\mathsf{F}}_\sigma = \mathsf{F}_\sigma$ follows by induction on $\sigma$. If $\sigma = \epsilon$ this is immediate, and if $\sigma = a\rho$,

$$\begin{aligned}
\widetilde{\mathsf{F}}_{a\rho} &= (a \leftand \widetilde{\mathsf{F}}_\rho) \leftor (\neg a \leftand \widetilde{\mathsf{F}}_\rho) \quad \text{by idempotence and the last consequence}\\
&= (a \leftand \mathsf{F}_\rho) \leftor (\neg a \leftand \mathsf{F}_\rho) \quad \text{by IH}\\
&= \mathsf{F}_{a\rho}.
\end{aligned}$$

■

**Lemma 4.11:** *Let $\sigma \in A^s$, then $P_\sigma \leftor F_\sigma = P_\sigma$ and $P_\sigma \leftand T_\sigma = P_\sigma$ are consequences of* EqMFEL.

***Proof:*** By induction on $\sigma$ The two cases for $\sigma = \epsilon$ are immediate. If $\sigma = a\rho$,

$$\begin{aligned}
&P_{a\rho} \leftor \mathsf{F}_{a\rho}\\
&\quad = ((a \leftand Q_\rho) \leftor (\neg a \leftand R_\rho)) \leftor (\neg a \leftand \widetilde{\mathsf{F}}_\rho) \quad \text{by Lemma 4.10}\\
&\quad = (a \leftand Q_\rho) \leftor ((\neg a \leftand R_\rho) \leftor (\neg a \leftand \mathsf{F}_\rho)) \quad \text{by associativity and Lemma 4.10}\\
&\quad = (a \leftand Q_\rho) \leftor (\neg a \leftand (R_\rho \leftor \mathsf{F}_\rho)) \quad \text{by (C4)}\\
&\quad = (a \leftand Q_\rho) \leftor (\neg a \leftand R_\rho) \quad \text{by IH}\\
&= P_{a\rho}.
\end{aligned}$$

Next, it follows from (C3) (in Table 3) by induction on $\nu \in A^s$ that $\neg P_\nu$ is provably equal to a $\nu$-normal form. By Lemma 4.10, $P_{a\rho} \leftand \mathsf{T}_{a\rho} = \neg(\neg P_{a\rho} \leftor \mathsf{F}_{a\rho}) = P_{a\rho}$. ■

We define an auxiliary operator $h(x, y, z)$ that preserves the property that in the evaluation of closed terms the left-right order is always respected (modulo memorising occurrences of atoms).

**Definition 4.12:** The ternary operator $\boldsymbol{h(x, y, z)}$ on terms over $\Sigma_{\mathsf{FEL}}(A)$ is defined by

$$h(x, y, z) = (x \leftand y) \leftor (\neg x \leftand z).$$

Hence, $h(a, P_\sigma, Q_\sigma)$ is a $a\sigma$-normal form (adopting the notational convention that $P_\sigma$ and $Q_\sigma$ are $\sigma$-normal forms). In particular, $\mathsf{T}_{a\rho} = h(a, \mathsf{T}_\rho, \mathsf{T}_\rho)$ and $\mathsf{F}_{a\rho} = h(a, \mathsf{F}_\rho, \mathsf{F}_\rho)$.

**Lemma 4.13:** *The following equations are consequences of* EqMFEL*:*

(1) $h(x,y,z) \mathbin{{}_\bullet\!\wedge} w = h(x, (y \mathbin{{}^\bullet\!\vee} (z \mathbin{{}_\bullet\!\wedge} F)) \mathbin{{}_\bullet\!\wedge} w, z \mathbin{{}_\bullet\!\wedge} w)$,
(2) $h(x,y,z) \mathbin{{}^\bullet\!\vee} w = h(x, (y \mathbin{{}_\bullet\!\wedge} (z \mathbin{{}^\bullet\!\vee} T)) \mathbin{{}^\bullet\!\vee} w, z \mathbin{{}^\bullet\!\vee} w)$.

***Proof:*** Consequence 1 follows from consequence 2 and $\neg h(x,y,z) = h(x,\neg y,\neg z)$ (which follows easily from (C3)). Consequence 2 follows with *Prover9* with options `lpo` and `unfold` in 2 s. ■

**Lemma 4.14:** *Let $\sigma \in A^s$, then $P_\sigma \mathbin{{}_\bullet\!\wedge} F = F_\sigma$ and $P_\sigma \mathbin{{}^\bullet\!\vee} T = T_\sigma$ are consequences of* EqMFEL.

***Proof:*** By induction on $\sigma$. The case $\sigma = \epsilon$ is immediate, and if $\sigma = a\rho$ then

$$\begin{aligned} P_{a\rho} \mathbin{{}_\bullet\!\wedge} \mathsf{F} &= h(a, Q_\rho, R_\rho) \mathbin{{}_\bullet\!\wedge} \mathsf{F} \\ &= h(a, (Q_\rho \mathbin{{}^\bullet\!\vee} (R_\rho \mathbin{{}_\bullet\!\wedge} \mathsf{F})) \mathbin{{}_\bullet\!\wedge} \mathsf{F}, R_\rho \mathbin{{}_\bullet\!\wedge} \mathsf{F}) \quad \text{by Lemma 4.13.1} \\ &= h(a, (Q_\rho \mathbin{{}^\bullet\!\vee} \mathsf{F}_\rho) \mathbin{{}_\bullet\!\wedge} \mathsf{F}, \mathsf{F}_\rho) \quad \text{by IH} \\ &= h(a, Q_\rho \mathbin{{}_\bullet\!\wedge} \mathsf{F}, \mathsf{F}_\rho) \quad \text{by Lemma 4.11} \\ &= h(a, \mathsf{F}_\rho, \mathsf{F}_\rho) \quad \text{by IH} \\ &= \mathsf{F}_{a\rho}. \end{aligned}$$

The case for $P_{a\rho} \mathbin{{}^\bullet\!\vee} \mathsf{T}$ follows in a similar way (with help of Lemma 4.13.2). ■

In order to compose $\sigma$-normal forms, we introduce some notation.

**Definition 4.15 (*str*(*P*) and $\sigma \gg \rho$):** For $P \in \mathcal{SP}_A$, the string ***str*(*P*)** $\in A^s$ is defined by $str(\mathsf{T}) = str(\mathsf{F}) = \epsilon$, $str(a) = a$ $(a \in A)$, $str(\neg P) = str(P)$, and $str(P \mathbin{{}_\bullet\!\wedge} Q) = str(P \mathbin{{}^\bullet\!\vee} Q) = str(P) \gg str(Q)$, where $\_ \gg \_ : A^s \times A^s \to A^s$ is the operation that filters out the atoms of the left argument in the right argument:

$$\sigma \gg \epsilon = \sigma \text{ and } \sigma \gg a\rho = \begin{cases} \sigma \gg \rho & \text{if } a \text{ occurs in } \sigma, \\ \sigma a \gg \rho & \text{otherwise.} \end{cases}$$

Observe that $\sigma \gg a\rho = (\sigma \gg a) \gg \rho$ and $\epsilon \gg \rho = \rho$.

**Lemma 4.16:** *In* EqMFEL, *$\sigma$-normal forms in $\mathcal{SP}_A$ are provably closed under $\mathbin{{}_\bullet\!\wedge}$ and $\mathbin{{}^\bullet\!\vee}$ composition.*

***Proof:*** We first consider $P_\sigma \mathbin{{}_\bullet\!\wedge} R_\nu$ and prove this case by induction on $\sigma$.

For $\sigma = \epsilon$, $\mathsf{T} \mathbin{{}_\bullet\!\wedge} R_\nu = R_\nu$ is immediate, and $\mathsf{F} \mathbin{{}_\bullet\!\wedge} R_\nu = \mathsf{F}_\nu$ follows from (FFEL7) and Lemma 4.14.

For $\sigma = a\rho$ it suffices to prove that

$$h(a, P_\rho, Q_\rho) \mathbin{{}_\bullet\!\wedge} R_\nu = h(a, P_\rho \mathbin{{}_\bullet\!\wedge} R_\nu, Q_\rho \mathbin{{}_\bullet\!\wedge} R_\nu)$$

because by induction, both $P_\rho \mathbin{{}_\bullet\!\wedge} R_\nu$ and $Q_\rho \mathbin{{}_\bullet\!\wedge} R_\nu$ have a provably equal $(\rho \gg \nu)$-normal form:

$$h(a, P_\rho, Q_\rho) \mathbin{{}_\bullet\!\wedge} R_\nu = h(a, (P_\rho \mathbin{{}^\bullet\!\vee} (Q_\rho \mathbin{{}_\bullet\!\wedge} \mathsf{F})) \mathbin{{}_\bullet\!\wedge} R_\nu, Q_\rho \mathbin{{}_\bullet\!\wedge} R_\nu) \quad \text{by Lemma 4.13.1}$$

$$= h(a, (P_\rho \mathbin{\dot\vee} \mathsf{F}_\rho) \mathbin{\dot\wedge} R_\nu, Q_\rho \mathbin{\dot\wedge} R_\nu). \quad \text{by Lemma 4.14}$$

$$= h(a, P_\rho \mathbin{\dot\wedge} R_\nu, Q_\rho \mathbin{\dot\wedge} R_\nu). \quad \text{by Lemma 4.11}$$

The case for $P_\sigma \mathbin{\dot\vee} R_\nu$ follows in a similar way (with help of Lemma 4.13.2). ∎

**Lemma 4.17:** *For each $P \in \mathcal{SP}_A$ there is a unique $\sigma$-normal form $Q$ such that* EqMFEL $\vdash$ $P = Q$.

***Proof:*** By structural induction on $P$, restricting to NNFs. For $P \in \{\mathsf{T}, \mathsf{F}, a, \neg a \mid a \in A\}$ this is trivial: $a = h(a, \mathsf{T}, \mathsf{F})$ and $\neg a = h(a, \mathsf{F}, \mathsf{T})$. For the cases $P = P_1 \mathbin{\dot\wedge} P_2$ and $P = P_1 \mathbin{\dot\vee} P_2$ this follows from Lemma 4.16.

Uniqueness follows from the facts that $str(P_1 \mathbin{\dot\wedge} P_2)$ is fixed and that syntactically different normal forms have different evaluation trees: for $F(x, y, z) = h(y, x, z)$, their $F$-representation mimics the tree structure. ∎

**Theorem 4.18 (Completeness):** *The logic* MFEL *is axiomatised by* EqMFEL.

***Proof:*** By Lemma 4.6, EqMFEL is sound. For completeness, assume $P_1 =_{mfe} P_2$. By Lemma 4.17 there are unique $\sigma_i$-normal forms $Q_{\sigma_i}$ such that EqMFEL $\vdash P_i =_{mfe} Q_{\sigma_i}$. By assumption and soundness, $Q_{\sigma_1} = Q_{\sigma_2}$. Hence, EqMFEL $\vdash P_1 = Q_{\sigma_1} = P_2$. ∎

## 5. MFEL with undefinedness: MFEL$^{\mathsf{U}}$

In this section we define MFEL$^{\mathsf{U}}$, the extension of MFEL with U. First, we formally define memorising U-evaluation trees. Given the detailed accounts in Sections 3 and 4, this extension is rather simple and straightforward. We define memorising U-evaluation trees, give equational axioms for the equality of these trees and prove a completeness result.

**Definition 5.1:** The evaluation trees $\mathsf{T}, \mathsf{F}, \mathsf{U} \in \mathcal{T}_A^{\mathsf{U}}$ are *memorising* U*-evaluation trees*. The evaluation tree $(X \trianglelefteq \underline{a} \trianglerighteq Y) \in \mathcal{T}_A^{\mathsf{U}}$ is a *memorising* U*-evaluation tree* over $A$ if both $X$ and $Y$ are memorising U-evaluation trees that do not contain the label $\underline{a}$.

We interpret propositional expressions in $\mathcal{SP}_A^{\mathsf{U}}$ as memorising U-evaluation trees by extending the function *mfe* (Definition 4.2).

**Definition 5.2:** The unary *memorising full evaluation* function $mfe^{\mathsf{U}} : \mathcal{SP}_A^{\mathsf{U}} \to \mathcal{T}_A^{\mathsf{U}}$ is defined by

$$mfe^{\mathsf{U}}(P) = m^{\mathsf{U}}(fe^{\mathsf{U}}(P)),$$

where the auxiliary function $m^{\mathsf{U}} : \mathcal{T}_A^{\mathsf{U}} \to \mathcal{T}_A^{\mathsf{U}}$ is defined as follows, for $a \in A$:

$$m^{\mathsf{U}}(B) = B \text{ for } B \in \{\mathsf{T}, \mathsf{F}, \mathsf{U}\}, m^{\mathsf{U}}(X \trianglelefteq \underline{a} \trianglerighteq Y) = m^{\mathsf{U}}(L_a^{\mathsf{U}}(X)) \trianglelefteq \underline{a} \trianglerighteq m^{\mathsf{U}}(R_a^{\mathsf{U}}(Y)),$$

and the auxiliary functions $L_a^{\mathsf{U}}, R_a^{\mathsf{U}} : \mathcal{T}_A^{\mathsf{U}} \to \mathcal{T}_A^{\mathsf{U}}$ are defined as follows, for $b \in A$:

$$L_a^{\mathsf{U}}(B) = R_a^{\mathsf{U}}(B) = B \quad \text{for } B \in \{\mathsf{T}, \mathsf{F}, \mathsf{U}\},$$

$$L^{\mathsf{U}}_a(X \trianglelefteq \underline{b} \trianglerighteq Y) = \begin{cases} L^{\mathsf{U}}_a(X) & \text{if } b = a, \\ L^{\mathsf{U}}_a(X) \trianglelefteq \underline{b} \trianglerighteq L^{\mathsf{U}}_a(Y) & \text{otherwise.} \end{cases}$$

$$R^{\mathsf{U}}_a(X \trianglelefteq \underline{b} \trianglerighteq Y) = \begin{cases} R^{\mathsf{U}}_a(Y) & \text{if } b = a, \\ R^{\mathsf{U}}_a(X) \trianglelefteq \underline{b} \trianglerighteq R^{\mathsf{U}}_a(Y) & \text{otherwise.} \end{cases}$$

**Definition 5.3:** The binary relation $=_{mfe^{\mathsf{U}}}$ on $\mathcal{SP}_A^{\mathsf{U}}$ is called *memorising full* U*-valuation congruence* and is defined by $P =_{mfe^{\mathsf{U}}} Q \iff mfe^{\mathsf{U}}(P) = mfe^{\mathsf{U}}(Q)$.

**Lemma 5.4:** *The relation $=_{mfe^{\mathsf{U}}}$ is a congruence.*

***Proof:*** The proof of Lemma 4.4 (in Appendix 2) requires one extra base case $X = \mathsf{U}$ for all sub-proofs, which follows trivially. ■

**Definition 5.5:** *Memorising Fully Evaluated Left-Sequential Logic with undefinedness (*$\mathrm{MFEL}^{\mathsf{U}}$*)* is the fully evaluated left-sequential logic with undefinedness that satisfies no more consequences than those of $mfe^{\mathsf{U}}$-equality, i.e. for all $P, Q \in \mathcal{SP}_A^{\mathsf{U}}$,

$$\mathrm{MFEL}^{\mathsf{U}} \models P = Q \iff P =_{mfe^{\mathsf{U}}} Q.$$

We extend EqMFEL to $\mathrm{EqMFEL}^{\mathsf{U}}$ with the two axioms $\neg\mathsf{U} = \mathsf{U}$ and $\mathsf{U} \mathbin{\blacktriangle\!\!\wedge} x = \mathsf{U}$, so all $\mathrm{EqFFEL}^{\mathsf{U}}$-results of Section 3 hold in $\mathrm{EqMFEL}^{\mathsf{U}}$.

**Lemma 5.6 (Soundness):** *For all $P, Q \in \mathcal{SP}_A^{\mathsf{U}}$, $\mathrm{EqMFEL}^{\mathsf{U}} \vdash P = Q \Longrightarrow \mathrm{MFEL}^{\mathsf{U}} \models P = Q$.*

***Proof:*** By Lemma 5.4, the relation $=_{mfe^{\mathsf{U}}}$ is a congruence on $\mathcal{SP}_A^{\mathsf{U}}$, so it suffices to show that all closed instances of the $\mathrm{EqMFEL}^{\mathsf{U}}$-axioms satisfy $=_{mfe^{\mathsf{U}}}$, which follows easily (cf. the proof of Lemma 4.6 and, for the U-axioms, the proof of Lemma 3.6). ■

In order to prove completeness of $\mathrm{EqMFEL}^{\mathsf{U}}$, we use the $\sigma$-normal forms $\mathsf{U}_\sigma$ from Definition 3.8 for $\sigma \in A^s$, thus $\mathsf{U}_\epsilon = \mathsf{U}$ and $\mathsf{U}_{a\rho} = a \mathbin{\blacktriangle\!\!\wedge} \mathsf{U}_\rho$. By induction on $\sigma$ and Lemma 3.10.2-3,

$$\text{For all } \sigma \in A^s, \mathrm{EqMFEL}^{\mathsf{U}} \vdash \neg\mathsf{U}_\sigma = \mathsf{U}_\sigma. \tag{‡}$$

We further use the ternary operator $h(x, y, z) = (x \mathbin{\blacktriangle\!\!\wedge} y) \mathbin{\blacktriangledown\!\!\vee} (\neg x \mathbin{\blacktriangle\!\!\wedge} z)$ (Definition 4.12) on terms over $\Sigma^{\mathsf{U}}_{\mathrm{FEL}}(A)$.

**Lemma 5.7:** *For each $a \in A$ and $\sigma = a\rho \in A^s$, $\mathrm{EqMFEL}^{U} \vdash U_{a\rho} = h(a, U_\rho, U_\rho)$.*

***Proof:*** Derive $a \mathbin{\blacktriangle\!\!\wedge} \mathsf{U}_\rho = (a \mathbin{\blacktriangle\!\!\wedge} \mathsf{U}_\rho) \mathbin{\blacktriangledown\!\!\vee} (a \mathbin{\blacktriangle\!\!\wedge} \mathsf{U}_\rho) \overset{\text{La.3.10.3}}{=} (a \mathbin{\blacktriangle\!\!\wedge} \mathsf{U}_\rho) \mathbin{\blacktriangledown\!\!\vee} (\neg a \mathbin{\blacktriangle\!\!\wedge} \mathsf{U}_\rho) = h(a, \mathsf{U}_\rho, \mathsf{U}_\rho)$. ■

In the following, we extend previous results from Section 4 to $\mathcal{SP}_A^{\mathsf{U}}$.

**Lemma 5.8 (extension of La.4.11):** *Let* $\sigma \in A^s$*, then* $P_\sigma \mathbin{{}^{\bullet}\!\vee} U_\sigma = U_\sigma$ *and* $P_\sigma \mathbin{{}_{\bullet}\!\wedge} U_\sigma = U_\sigma$ *are consequences of* $\mathrm{EqMFEL}^{\mathrm{U}}$.

***Proof:*** If $P_\sigma = \mathrm{U}_\sigma$, we are done by Lemma 3.9. If $P_\sigma \in \mathcal{SP}_A$, the statement follows by induction on $\sigma$: if $\sigma = \epsilon$, this is trivial. If $\sigma = a\rho$,

$$
\begin{aligned}
&P_{a\rho} \mathbin{{}^{\bullet}\!\vee} \mathrm{U}_{a\rho}\\
&\quad = ((a \mathbin{{}_{\bullet}\!\wedge} Q_\rho) \mathbin{{}^{\bullet}\!\vee} (\neg a \mathbin{{}_{\bullet}\!\wedge} R_\rho)) \mathbin{{}^{\bullet}\!\vee} (a \mathbin{{}_{\bullet}\!\wedge} \mathrm{U}_\rho)\\
&\quad = (a \mathbin{{}_{\bullet}\!\wedge} Q_\rho) \mathbin{{}^{\bullet}\!\vee} ((\neg a \mathbin{{}_{\bullet}\!\wedge} R_\rho) \mathbin{{}^{\bullet}\!\vee} (\neg a \mathbin{{}_{\bullet}\!\wedge} \mathrm{U}_\rho)) \quad \text{by associativity and Lemma 3.10.3}\\
&\quad = (a \mathbin{{}_{\bullet}\!\wedge} Q_\rho) \mathbin{{}^{\bullet}\!\vee} (\neg a \mathbin{{}_{\bullet}\!\wedge} (R_\rho \mathbin{{}^{\bullet}\!\vee} \mathrm{U}_\rho)) \quad \text{by (C4)}\\
&\quad = (a \mathbin{{}_{\bullet}\!\wedge} Q_\rho) \mathbin{{}^{\bullet}\!\vee} (a \mathbin{{}_{\bullet}\!\wedge} \mathrm{U}_\rho) \quad \text{by IH and Lemma 3.10.3}\\
&\quad = a \mathbin{{}_{\bullet}\!\wedge} (Q_\rho \mathbin{{}^{\bullet}\!\vee} \mathrm{U}_\rho) \quad \text{by (C4)}\\
&\quad = \mathrm{U}_{a\rho}. \quad \text{by IH}
\end{aligned}
$$

In the proof of Lemma 4.11 it was already argued that if $P_\sigma \in \mathcal{SP}_A$ then $\neg P_\sigma$ is provably equal to a $\sigma$-normal form. By consequence (‡), $\neg\mathrm{U}_\sigma = \mathrm{U}_\sigma$. Hence, $P_\sigma \mathbin{{}_{\bullet}\!\wedge} \mathrm{U}_\sigma = \neg(\neg P_\sigma \mathbin{{}^{\bullet}\!\vee} \mathrm{U}_\sigma) = \mathrm{U}_\sigma$. ∎

**Lemma 5.9 (extension of La.4.14):** *Let* $\sigma \in A^s$*, then* $P_\sigma \mathbin{{}_{\bullet}\!\wedge} U = U_\sigma$ *and* $P_\sigma \mathbin{{}^{\bullet}\!\vee} U = U_\sigma$ *are consequences of* $\mathrm{EqMFEL}^{U}$.

***Proof:*** If $P_\sigma = \mathrm{U}_\sigma$, this follows by Lemma 3.9. If $P_\sigma \in \mathcal{SP}_A$, this follows by induction on $\sigma$. We first consider the case $P_\sigma \mathbin{{}^{\bullet}\!\vee} \mathrm{U} = \mathrm{U}_\sigma$. If $\sigma = \epsilon$ the statement is trivial. If $\sigma = a\rho$ then

$$
\begin{aligned}
P_{a\rho} \mathbin{{}^{\bullet}\!\vee} \mathrm{U} &= h(a, Q_\rho, R_\rho) \mathbin{{}^{\bullet}\!\vee} \mathrm{U}\\
&= h(a, (Q_\rho \mathbin{{}_{\bullet}\!\wedge} (R_\rho \mathbin{{}^{\bullet}\!\vee} \mathrm{T})) \mathbin{{}^{\bullet}\!\vee} \mathrm{U}, R_\rho \mathbin{{}^{\bullet}\!\vee} \mathrm{U}) \quad \text{by Lemma 4.13.2}\\
&= h(a, (Q_\rho \mathbin{{}_{\bullet}\!\wedge} \mathrm{T}_\rho) \mathbin{{}^{\bullet}\!\vee} \mathrm{U}, \mathrm{U}_\rho) \quad \text{by Lemma 4.14 and IH}\\
&= h(a, Q_\rho \mathbin{{}^{\bullet}\!\vee} \mathrm{U}, \mathrm{U}_\rho) \quad \text{by Lemma 4.11}\\
&= h(a, \mathrm{U}_\rho, \mathrm{U}_\rho) \quad \text{by IH}\\
&= \mathrm{U}_{a\rho}. \quad \text{by Lemma 5.7}
\end{aligned}
$$

The case for $P_{a\rho} \mathbin{{}_{\bullet}\!\wedge} \mathrm{U}$ follows in a similar way (with help of Lemma 4.13.1). ∎

**Lemma 5.10 (extension of La.4.16):** *In* $\mathrm{EqMFEL}^{U}$*,* $\sigma$*-normal forms in* $\mathcal{SP}_A^{U}$ *are provably closed under* $\mathbin{{}_{\bullet}\!\wedge}$ *and* $\mathbin{{}^{\bullet}\!\vee}$ *composition, and each of these compositions determines a unique* $\nu$*-normal form.*

***Proof:*** We first consider $P_\sigma \mathbin{{}_{\bullet}\!\wedge} R_\nu$ and distinguish three cases.

Case $P_\sigma = \mathrm{U}_\sigma$. By Lemma 3.9, $\mathrm{U}_\sigma \mathbin{{}_{\bullet}\!\wedge} R_\nu = \mathrm{U}_\sigma$.

Case $P_\sigma, R_\nu \in \mathcal{SP}_A$. By Lemma 4.16, there is a $(\sigma \gg \nu)$-normal form such that $P_\sigma \mathbin{{}_{\bullet}\!\wedge} R_\nu = Q_{\sigma \gg \nu}$.

Case $P_\sigma \in \mathcal{SP}_A$ and $R_\nu = \mathsf{U}_\nu$. We prove this case by induction on $\sigma$:

If $\sigma = \epsilon$, $\mathsf{T} \mathbin{\&\!\wedge} \mathsf{U}_\nu = \mathsf{U}_\nu$ is immediate, and $\mathsf{F} \mathbin{\&\!\wedge} \mathsf{U}_\nu = \mathsf{U}_\nu$ follows from (FFEL7) and Lemma 3.9.

If $\sigma = a\rho \in A^s$, it suffices to prove that

$$h(a, P_\rho, Q_\rho) \mathbin{\&\!\wedge} \mathsf{U}_\nu = h(a, P_\rho \mathbin{\&\!\wedge} \mathsf{U}_\nu, Q_\rho \mathbin{\&\!\wedge} \mathsf{U}_\nu)$$

because by induction, both $P_\rho \mathbin{\&\!\wedge} \mathsf{U}_\nu$ and $Q_\rho \mathbin{\&\!\wedge} \mathsf{U}_\nu$ have a provably equal ($\rho \gg \nu$)-normal form:

$$\begin{aligned} h(a, P_\rho, Q_\rho) \mathbin{\&\!\wedge} \mathsf{U}_\nu &= h(a, (P_\rho \mathbin{\&\!\vee} (Q_\rho \mathbin{\&\!\wedge} \mathsf{F})) \mathbin{\&\!\wedge} \mathsf{U}_\nu, Q_\rho \mathbin{\&\!\wedge} \mathsf{U}_\nu) \quad \text{by Lemma 4.13.1} \\ &= h(a, (P_\rho \mathbin{\&\!\vee} \mathsf{F}_\rho) \mathbin{\&\!\wedge} \mathsf{U}_\nu, Q_\rho \mathbin{\&\!\wedge} \mathsf{U}_\nu). \quad \text{by Lemma 4.14} \\ &= h(a, P_\rho \mathbin{\&\!\wedge} \mathsf{U}_\nu, Q_\rho \mathbin{\&\!\wedge} \mathsf{U}_\nu). \quad \text{by Lemma 4.11} \end{aligned}$$

The proof for $P_\sigma \mathbin{\&\!\vee} R_\nu$ follows in a similar way (with help of Lemma 4.13.2). ■

**Lemma 5.11:** *For each $P \in \mathcal{SP}_A^{\mathsf{U}}$ there is a unique $\sigma$-normal form $Q$ such that* $\mathrm{EqMFEL}^{\mathsf{U}} \vdash P = Q$.

***Proof:*** By structural induction on $P$, restricting to NNFs. For $P \in \{\mathsf{T}, \mathsf{F}, \mathsf{U}, a, \neg a \mid a \in A\}$ this is trivial: $a = h(a, \mathsf{T}, \mathsf{F})$ and $\neg a = h(a, \mathsf{F}, \mathsf{T})$. For the cases $P = P_1 \mathbin{\&\!\wedge} P_2$ and $P = P_1 \mathbin{\&\!\vee} P_2$ this follows from Lemma 5.10.

Uniqueness follows from the facts that for $\sigma$-normal form $P_1$ and $\rho$-normal form $P_2$, the normal forms of $P_1 \mathbin{\&\!\wedge} P_2$ and $P_1 \mathbin{\&\!\vee} P_2$ are unique by Lemma 5.10, and that syntactically different normal forms have different evaluation trees: for $F(x, y, z) = h(y, x, z)$, their $F$-representation mimics the tree structure. ■

**Theorem 5.12 (Completeness):** *The logic* $\mathrm{MFEL}^{U}$ *is axiomatised by* $\mathrm{EqMFEL}^{U}$.

***Proof:*** By Lemma 5.6, $\mathrm{EqMFEL}^{\mathsf{U}}$ is sound. For completeness, assume $P_1 =_{mfe^{\mathsf{U}}} P_2$. By Lemma 5.11 there are unique $\sigma_i$-normal forms $Q_{\sigma_i}$ such that $\mathrm{EqMFEL}^{\mathsf{U}} \vdash P_i =_{mfe} Q_{\sigma_i}$. By assumption and soundness, $Q_{\sigma_1} = Q_{\sigma_2}$. Hence, $\mathrm{EqMFEL}^{\mathsf{U}} \vdash P_1 = Q_{\sigma_1} = P_2$. ■

## 6. Two-Valued conditional FEL (C$\ell$FEL$_2$) and C$\ell$FEL

In this section we define two-valued Conditional FEL (C$\ell$FEL$_2$) and three-valued Conditional FEL (C$\ell$FEL). Characteristic properties of C$\ell$FEL$_2$ and C$\ell$FEL are the commutativity of $\mathbin{\&\!\wedge}$ (and $\mathbin{\&\!\vee}$) and the equivalence of C$\ell$FEL$_2$ to a sequential version of propositional logic that refutes the absorption laws.[3] For both the case without U and with U, we define a semantics based on ordered memorising trees, give equational axioms for their equality, and prove a completeness result.

For $P \in \mathcal{SP}_A^{\mathsf{U}}$, its *alphabet* $\alpha(P) \subset A$ is defined by $\alpha(\mathsf{T}) = \alpha(\mathsf{F}) = \alpha(\mathsf{U}) = \emptyset$, $\alpha(a) = \{a\}$, $\alpha(\neg P) = \alpha(P)$ and $\alpha(P \mathbin{\&\!\wedge} Q) = \alpha(P \mathbin{\&\!\vee} Q) = \alpha(P) \cup \alpha(Q)$.

We further assume that the atoms in $A$ are ordered $a_1 < a_2 < \cdots < a_n < a_{n+1} \cdots$, notation $(A, <)$.

**Definition 6.1:** $A^s_o$ is the set of strings in $A^*$ whose elements are ordered according to $(A, <)$.

So, $A^s_o \subset A^s$. We reserve the symbols $\beta, \gamma$ for strings $b_1 b_2 \cdots b_k b_{k+1}$ in $A^s_o$ that satisfy $b_i < b_{i+1}$ (not necessarily neighbours in $(A, <)$), and for such a string $\beta$ we write $\{\beta\}$ for its set of atoms.

Below we define evaluation functions for $\mathrm{C}\ell\mathrm{FEL}_2$ and $\mathrm{C}\ell\mathrm{FEL}$ that use disjunctions $\widetilde{\mathrm{F}}_\beta \leftor P$ in order to identify $P$ with a term that respects $A^s_o$, where the terms $\widetilde{\mathrm{F}}_\beta$ are defined in Definition 4.9.

**Definition 6.2:** The *conditional full evaluation function clfe* : $\mathcal{SP}_A \to \mathcal{T}_A$ is defined by

$$clfe(P) = mfe(\widetilde{\mathrm{F}}_\beta \leftor P), \quad \text{where } \beta \in A^s_o \text{ satisfies } \{\beta\} = \alpha(P).$$

The *conditional full evaluation function* $clfe^{\mathrm{U}}$ : $\mathcal{SP}_A^{\mathrm{U}} \to \mathcal{T}_A^{\mathrm{U}}$ is defined by

$$clfe^{\mathrm{U}}(P) = \begin{cases} mfe^{\mathrm{U}}(\widetilde{\mathrm{F}}_\beta \leftor P) & \text{if } P \in \mathcal{SP}_A \text{ and } \beta \in A^s_o \text{ satisfies } \{\beta\} = \alpha(P), \\ \mathrm{U} & \text{otherwise.} \end{cases}$$

These evaluation functions are well-defined because $\beta$ is uniquely determined by $\alpha(P)$ and $(A, <)$. For each $P \in \mathcal{SP}_A$, the evaluation tree $clfe(P)$ can be called a '$\beta$-tree', its complete traces agree with $\beta$, and its leaves determine the evaluation result of $P$.

**Example 6.3:** Four typical examples with $\beta = ab$, thus $\widetilde{\mathrm{F}}_\beta = a \leftand (b \leftand \mathrm{F})$:

$$\begin{aligned} clfe(b \leftand a) &= mfe(\widetilde{\mathrm{F}}_\beta \leftor (b \leftand a)) = mfe(a \leftand b), \\ clfe(b \leftor a) &= mfe(\widetilde{\mathrm{F}}_\beta \leftor (b \leftor a)) = mfe(a \leftor b), \\ clfe((b \leftor a) \leftand b) &= mfe(\widetilde{\mathrm{F}}_\beta \leftor ((b \leftor a) \leftand b)) = mfe((a \leftor \mathrm{T}) \leftand b), \\ clfe(b \leftand (a \leftor b)) &= mfe(\widetilde{\mathrm{F}}_\beta \leftor (b \leftand (a \leftor b))) = mfe((a \leftor \mathrm{T}) \leftand b). \end{aligned}$$

**Definition 6.4:** The relation $=_{clfe} \subset \mathcal{SP}_A \times \mathcal{SP}_A$, *conditional full valuation congruence*, is defined by $P =_{clfe} Q$ if $clfe(P) = clfe(Q)$. The relation $=_{clfe^{\mathrm{U}}} \subset \mathcal{SP}_A^{\mathrm{U}} \times \mathcal{SP}_A^{\mathrm{U}}$, *conditional full **U**-valuation congruence*, is defined by $P =_{clfe} Q$ if $clfe^{\mathrm{U}}(P) = clfe^{\mathrm{U}}(Q)$.

**Lemma 6.5:** *The relations $=_{clfe}$ and $=_{clfe^{\mathrm{U}}}$ are both congruences.*

***Proof:*** Because $=_{mfe}$ is a congruence (Lemma 4.4), so is $=_{clfe}$. This follows from the fact that $=_{mfe}$ preserves the alphabet of its arguments, that is, if $P =_{mfe} Q$ then $\alpha(P) = \alpha(Q)$. E.g. assume for $P, Q, R \in \mathcal{SP}_A$ that $P =_{clfe} Q$, then $clfe(P \leftand R) = mfe(\widetilde{\mathrm{F}}_\beta \leftor (P \leftand R)) \stackrel{\text{La.4.4}}{=} mfe(\widetilde{\mathrm{F}}_\beta \leftor (Q \leftand R)) = clfe(Q \leftand R)$, where the last equality holds because $\beta$ satisfies $\{\beta\} = \alpha(P \leftand R) = \alpha(Q \leftand R)$.

In a similar way it follows by Lemma 5.4 that $=_{clfe^{\mathrm{U}}}$ is a congruence. ■

**Definition 6.6:** *Conditional Fully Evaluated Left-Sequential Logic* (C$\ell$FEL$_{\mathbf{2}}$) is the fully evaluated left-sequential logic that satisfies no more consequences than those of *clfe*-equality, i.e. for all $P, Q \in \mathcal{SP}_A$,

$$\mathrm{C}\ell\mathrm{FEL}_2 \models P = Q \iff P =_{clfe} Q.$$

*Conditional Fully Evaluated Left-Sequential Logic with undefinedness* (C$\ell$FEL$_{\mathbf{2}}$) is the fully evaluated left-sequential logic with undefinedness that satisfies no more consequences than those of *clfe*$^{\mathsf{U}}$-equality, i.e. for all $P, Q \in \mathcal{SP}_A{}^{\mathsf{U}}$,

$$\mathrm{C}\ell\mathrm{FEL} \models P = Q \iff P =_{clfe^{\mathsf{U}}} Q.$$

We define the following sets of axioms for C$\ell$FEL$_2$ and C$\ell$FEL:

$$\mathrm{EqC}\ell\mathrm{FEL}_2 = \mathrm{EqMFEL} \cup \{x \mathbin{{}^\bullet\!\wedge} y = y \mathbin{{}^\bullet\!\wedge} x\},$$
$$\mathrm{EqC}\ell\mathrm{FEL}^{\mathsf{U}} = \mathrm{EqMFEL}^{\mathsf{U}} \cup \{x \mathbin{{}^\bullet\!\wedge} y = y \mathbin{{}^\bullet\!\wedge} x\}.$$

In order to prove the soundness of EqC$\ell$FEL$_2$ and EqC$\ell$FEL$^{\mathsf{U}}$, we start with two auxiliary lemmas.

**Lemma 6.7:** $\mathrm{EqMFEL} \vdash (x \mathbin{{}^\bullet\!\wedge} F) \mathbin{{}^\bullet\!\vee} (x \mathbin{{}^\bullet\!\wedge} y) = (x \mathbin{{}^\bullet\!\wedge} F) \mathbin{{}^\bullet\!\vee} (y \mathbin{{}^\bullet\!\wedge} x)$.

***Proof:*** With *Prover9* with options `lpo` and `unfold`: 0.2s. An alternative and more readable proof with options `kbo` and `fold`, taking 2s, is included in Appendix 3. ■

**Lemma 6.8:** *For all $\beta \in A_o^s$ and $P \in \mathcal{SP}_A$ such that $\alpha(P) \subset \{\beta\}$,*

$$\mathrm{EqMFEL} \vdash \widetilde{F}_\beta \mathbin{{}^\bullet\!\vee} (P \mathbin{{}^\bullet\!\wedge} F) = \widetilde{F}_\beta.$$

***Proof:*** By induction on the length of $\beta$. The base case $\beta = \epsilon$ is trivial.

For $\beta = a\gamma$ $(a \in A)$, thus $\widetilde{\mathrm{F}}_\beta = a \mathbin{{}^\bullet\!\wedge} \widetilde{\mathrm{F}}_\gamma$, we apply structural induction on $P$, and we note that from the dual of Lemma 2.7.3 (take $y = \mathrm{F}$) and $\widetilde{\mathrm{F}}_\gamma = \widetilde{\mathrm{F}}_\gamma \mathbin{{}^\bullet\!\wedge} \mathrm{F}$ it follows that

$$\widetilde{\mathrm{F}}_\beta = (a \mathbin{{}^\bullet\!\wedge} \mathrm{F}) \mathbin{{}^\bullet\!\vee} \widetilde{\mathrm{F}}_\gamma. \tag{C.†}$$

The base cases: (1) the case $P \in \{\mathrm{T}, \mathrm{F}\}$ is trivial; (2) if $P = a$, then by (C.$^\dagger$) and the dual of (C1), $\widetilde{\mathrm{F}}_\beta \mathbin{{}^\bullet\!\vee} (a \mathbin{{}^\bullet\!\wedge} \mathrm{F}) = ((a \mathbin{{}^\bullet\!\wedge} \mathrm{F}) \mathbin{{}^\bullet\!\vee} \widetilde{\mathrm{F}}_\gamma) \mathbin{{}^\bullet\!\vee} (a \mathbin{{}^\bullet\!\wedge} \mathrm{F}) = \widetilde{\mathrm{F}}_\beta$; (3) if $P \in \{\gamma\}$, then by (C.$^\dagger$), $\widetilde{\mathrm{F}}_\beta \mathbin{{}^\bullet\!\vee} (P \mathbin{{}^\bullet\!\wedge} \mathrm{F}) = ((a \mathbin{{}^\bullet\!\wedge} \mathrm{F}) \mathbin{{}^\bullet\!\vee} \widetilde{\mathrm{F}}_\gamma) \mathbin{{}^\bullet\!\vee} (P \mathbin{{}^\bullet\!\wedge} \mathrm{F}) = (a \mathbin{{}^\bullet\!\wedge} \mathrm{F}) \mathbin{{}^\bullet\!\vee} (\widetilde{\mathrm{F}}_\gamma \mathbin{{}^\bullet\!\vee} (P \mathbin{{}^\bullet\!\wedge} \mathrm{F})) \overset{\mathrm{IH}}{=} (a \mathbin{{}^\bullet\!\wedge} \mathrm{F}) \mathbin{{}^\bullet\!\vee} \widetilde{\mathrm{F}}_\gamma = \widetilde{\mathrm{F}}_\beta$.

If $P = \neg P_1$, then $\widetilde{\mathrm{F}}_\beta \mathbin{{}^\bullet\!\vee} (\neg P_1 \mathbin{{}^\bullet\!\wedge} \mathrm{F}) \overset{\text{(FFEL8)}}{=} \widetilde{\mathrm{F}}_\beta \mathbin{{}^\bullet\!\vee} (P_1 \mathbin{{}^\bullet\!\wedge} \mathrm{F}) \overset{\mathrm{IH}}{=} \widetilde{\mathrm{F}}_\beta$.

If $P = P_1 \mathbin{{}^\bullet\!\wedge} P_2$, then $\widetilde{\mathrm{F}}_\beta \mathbin{{}^\bullet\!\vee} ((P_1 \mathbin{{}^\bullet\!\wedge} P_2) \mathbin{{}^\bullet\!\wedge} \mathrm{F}) = \widetilde{\mathrm{F}}_\beta \mathbin{{}^\bullet\!\vee} ((P_1 \mathbin{{}^\bullet\!\wedge} \mathrm{F}) \mathbin{{}^\bullet\!\wedge} (P_2 \mathbin{{}^\bullet\!\wedge} \mathrm{F})) \overset{\text{(C4)}'}{=} (\widetilde{\mathrm{F}}_\beta \mathbin{{}^\bullet\!\vee} (P_1 \mathbin{{}^\bullet\!\wedge} \mathrm{F})) \mathbin{{}^\bullet\!\wedge} (\widetilde{\mathrm{F}}_\beta \mathbin{{}^\bullet\!\vee} (P_2 \mathbin{{}^\bullet\!\wedge} \mathrm{F})) \overset{\mathrm{IH}}{=} \widetilde{\mathrm{F}}_\beta \mathbin{{}^\bullet\!\wedge} \widetilde{\mathrm{F}}_\beta = \widetilde{\mathrm{F}}_\beta$, where (C4)′ is the dual of (C4).

If $P = P_1 \mathbin{{}^\bullet\!\vee} P_2$, then $\widetilde{\mathrm{F}}_\beta \mathbin{{}^\bullet\!\vee} ((P_1 \mathbin{{}^\bullet\!\vee} P_2) \mathbin{{}^\bullet\!\wedge} \mathrm{F}) \overset{\text{(M1)}}{=} \widetilde{\mathrm{F}}_\beta \mathbin{{}^\bullet\!\vee} ((\neg P_1 \mathbin{{}^\bullet\!\wedge} (P_2 \mathbin{{}^\bullet\!\wedge} \mathrm{F})) \mathbin{{}^\bullet\!\vee} (P_1 \mathbin{{}^\bullet\!\wedge} \mathrm{F})) \overset{\text{La.2.7.1}}{=}$
$\widetilde{\mathrm{F}}_\beta \mathbin{{}^\bullet\!\vee} ((P_1 \mathbin{{}^\bullet\!\wedge} (P_2 \mathbin{{}^\bullet\!\wedge} \mathrm{F}) \mathbin{{}^\bullet\!\vee} (P_1 \mathbin{{}^\bullet\!\wedge} \mathrm{F})) = (\widetilde{\mathrm{F}}_\beta \mathbin{{}^\bullet\!\vee} ((P_1 \mathbin{{}^\bullet\!\wedge} P_2) \mathbin{{}^\bullet\!\wedge} \mathrm{F})) \mathbin{{}^\bullet\!\vee} (\widetilde{\mathrm{F}}_\beta \mathbin{{}^\bullet\!\vee} (P_1 \mathbin{{}^\bullet\!\wedge} \mathrm{F})) \overset{\text{(pc),IH}}{=} \widetilde{\mathrm{F}}_\beta \mathbin{{}^\bullet\!\vee} \widetilde{\mathrm{F}}_\beta = \widetilde{\mathrm{F}}_\beta 2.7$, where (pc) refers to the previous case $P = P_1 \mathbin{{}^\bullet\!\wedge} P_2$. ■

**Lemma 6.9 (Soundness):** (1) *For all* $P, Q \in \mathcal{SP}_A$,
$\mathrm{EqC}\ell\mathrm{FEL}_2 \vdash P = Q \Longrightarrow \mathrm{C}\ell\mathrm{FEL}_2 \models P = Q$, *and*
(2) *For all* $P, Q \in \mathcal{SP}_A{}^U$, $\mathrm{EqC}\ell\mathrm{FEL}^U \vdash P = Q \Longrightarrow \mathrm{C}\ell\mathrm{FEL} \models P = Q$.

***Proof:*** 1. By Lemma 6.5, the relation $=_{clfe}$ is a congruence on $\mathcal{SP}_A$, so it suffices to show that all closed instances of the $\mathrm{EqC}\ell\mathrm{FEL}_2$-axioms satisfy $=_{clfe}$. By soundness of EqMFEL (Lemma 4.6), we only have to prove this for the axiom $x \leftand y = y \leftand x$. Let $P, Q \in \mathcal{SP}_A$ and let $\beta \in A^s_o$ be such that $\{\beta\} = \alpha(P \leftand Q)$. In order to prove that $mfe(\widetilde{\mathsf{F}}_\beta \leftor (P \leftand Q)) = mfe(\widetilde{\mathsf{F}}_\beta \leftor (Q \leftand P))$, and thus $clfe(P \leftand Q) = clfe(Q \leftand P)$, it suffices to derive the following for any $\beta \in A^s_o$ that satisfies $\alpha(P) \subset \{\beta\}$:

$$
\begin{aligned}
\mathrm{EqMFEL} \vdash \widetilde{\mathsf{F}}_\beta \leftor (P \leftand Q) &= (\widetilde{\mathsf{F}}_\beta \leftor (P \leftand \mathsf{F})) \leftor (P \leftand Q) && \text{by Lemma 6.8}\\
&= \widetilde{\mathsf{F}}_\beta \leftor ((P \leftand \mathsf{F}) \leftor (P \leftand Q))\\
&= \widetilde{\mathsf{F}}_\beta \leftor ((P \leftand \mathsf{F}) \leftor (Q \leftand P)) && \text{by Lemma 6.7}\\
&= \widetilde{\mathsf{F}}_\beta \leftor (Q \leftand P). \quad \text{(as above)}
\end{aligned}
$$

2. Soundness of $\mathrm{EqC}\ell\mathrm{FEL}^{\mathsf{U}}$ follows as in 1, the additional case is that for any $P \in \mathcal{SP}_A{}^{\mathsf{U}} \setminus \mathcal{SP}_A$ and $Q \in \mathcal{SP}_A{}^{\mathsf{U}}$, $clfe(P \leftand Q) = clfe(Q \leftand P) = \mathsf{U}$. ■

We first prove completeness of $\mathrm{EqC}\ell\mathrm{FEL}_2$, and then discuss the completeness of $\mathrm{EqC}\ell\mathrm{FEL}^{\mathsf{U}}$.

**Lemma 6.10:** $\mathrm{EqC}\ell\mathrm{FEL}_2 \vdash h(x, h(y, z, u), h(y, v, w)) = h(y, h(x, z, v), h(x, u, w))$.

***Proof:*** By commutativity and associativity of $\leftand$ and $\leftor$, and (C4) (distributivity):

$$
\begin{aligned}
&h(x, h(y, z, u), h(y, v, w))\\
&\quad = (x \leftand ((y \leftand z) \leftor (\neg y \leftand u))) \leftor (\neg x \leftand ((y \leftand v) \leftor (\neg y \leftand w)))\\
&\quad = ((x \leftand (y \leftand z)) \leftor (x \leftand (\neg y \leftand u))) \leftor ((\neg x \leftand (y \leftand v)) \leftor (\neg x \leftand (\neg y \leftand w)))\\
&\quad = ((y \leftand (x \leftand z)) \leftor (\neg y \leftand (x \leftand u))) \leftor ((y \leftand (\neg x \leftand v)) \leftor (\neg y \leftand (\neg x \leftand w)))\\
&\quad = (y \leftand ((x \leftand z) \leftor (\neg x \leftand v))) \leftor (\neg y \leftand ((x \leftand u) \leftor (\neg x \leftand w)))\\
&\quad = h(y, h(x, z, v), h(x, u, w)).
\end{aligned}
$$

■

**Lemma 6.11:** *Let* $\sigma \in A^s$ *and* $\sigma'$ *be a permutation of* $\sigma$. *Then for each* $\sigma$*-normal* $P_\sigma$ *there is a* $\sigma'$*-normal form* $Q$ *such that* $\mathrm{EqC}\ell\mathrm{FEL}_2 \vdash P_\sigma = Q_{\sigma'}$.

***Proof:*** By induction on $\sigma$. If $\sigma = \epsilon$ or $\sigma = a \in A$, this is trivial.

Assume $\sigma = a_0 a_1 \cdots a_k$ and $\sigma' = b_0 b_1 \cdots b_k$. If $a_0 = b_0$, we are done by induction. Otherwise, $a_0 = b_\ell$ with $\ell \neq 0$ and by induction, $P_\sigma = h(a_0, R_{\rho'}, S_{\rho'})$ with $\rho' = b_0 \rho''$ for some $\rho'' \in A^s$. By Lemma 6.10, $h(a_0, R_{\rho'}, S_{\rho'}) = h(b_0, V_{a_0\rho''}, W_{a_0\rho''})$. Now, either we are done, or there is a permuation $p()$ of $a_0\rho''$ such that $b_0 p(a_0\rho'') = \sigma'$. By induction, $h(b_0, V_{a_0\rho''}, W_{a_0\rho''}) = h(b_0, V'_{p(a_0\rho'')}, W'_{p(a_0\rho'')}) = Q_{\sigma'}$. ■

**Definition 6.12:** A ***β****-normal form* is a $\sigma$-normal form (Definition 4.8) with $\sigma \in A^s_o$. We denote $\beta$-normal forms by $P_\beta$, $Q_\beta$, etc.

**Lemma 6.13:** *Let $\beta \in A^s_o$ and $\sigma$ a permutation of $\beta$. Then in* $\mathrm{C}\ell\mathrm{FEL}_2$*, each $\sigma$-normal form $P_\sigma$ is provably equal to a $\beta$-normal form.*

***Proof:*** By Lemma 6.11. ∎

**Theorem 6.14 (Completeness):** *The logic* $\mathrm{C}\ell\mathrm{FEL}_2$ *is axiomatised by* $\mathrm{EqC}\ell\mathrm{FEL}_2$.

***Proof:*** By Lemma 6.9, $\mathrm{EqC}\ell\mathrm{FEL}_2$ is sound. For completeness, assume $P_1 =_{clfe} P_2$, thus, for some $\beta \in A^s_o$, $\alpha(P_i) = \{\beta\}$. By assumption and Lemma 6.13 there is a $\beta$-normal form $Q_\beta$ such that $\mathrm{EqC}\ell\mathrm{FEL}_2 \vdash P_1 = Q_\beta = P_2$. ∎

The completeness of $\mathrm{EqC}\ell\mathrm{FEL}^{\mathrm{U}}$ can be proved in the same way. In comparison with $\mathrm{C}\ell\mathrm{FEL}_2$, there is only one additional $\beta$-normal form U.

**Theorem 6.15 (Completeness):** *The logic* $\mathrm{C}\ell\mathrm{FEL}$ *is axiomatised by* $\mathrm{EqC}\ell\mathrm{FEL}^{U}$.

It follows that $\mathrm{C}\ell\mathrm{FEL}$ is equivalent to Bochvar's well-known three-valued logic (1938). In Comment 7.6 (on independent axiomatisations) we discuss this equivalence, or more precisely, the equivalence of $\mathrm{EqC}\ell\mathrm{FEL}^{\mathrm{U}}$ and the equational axiomatisation of Bochvar's logic discussed in Bergstra et al. (1995).

## 7. Static FEL (SFEL), short-circuit logic, and independence

In this section we define Static FEL (SFEL), the strongest two-valued FEL we consider and which is equivalent to a sequential version of propositional logic. SFEL is axiomatised by adding $x \mathbin{\bullet\!\!\wedge} \mathrm{F} = \mathrm{F}$ to the axioms of (two-valued) $\mathrm{C}\ell\mathrm{FEL}_2$. It is immediately clear that SFEL cannot be extended with U: $\mathrm{F} = \mathrm{U} \mathbin{\bullet\!\!\wedge} \mathrm{F} = \mathrm{U}$. In SFEL, the difference between *full* and *short-circuit* evaluation has disappeared and therefore we can reuse results on *static short-circuit logic*, which we briefly introduce. Finally, we review all FEL-axiomatisations and give simple equivalent alternatives with help of *Prover9*, all of which are independent.

We assume $(A, <)$ and $A^s_o$ (Definition 6.1) and consider terms whose alphabet is constrained by some $\beta \in A^s_o$ and for which we define an evaluation funtion $sfe_\beta$.

**Definition 7.1:** For any $\beta \in A^s_o$, define $\mathcal{SP}_{A,\beta} = \{P \in \mathcal{SP}_A \mid \alpha(P) \subset \{\beta\}\}$. The *static full evaluation* function $sfe_\beta : \mathcal{SP}_{A,\beta} \to \mathcal{T}_A$ is defined by

$$sfe_\beta(P) = mfe(\widetilde{\mathrm{F}}_\beta \mathbin{\bullet\!\!\vee} P).$$

The relation $=_{sfe,\beta} \subset \mathcal{SP}_{A,\beta} \times \mathcal{SP}_{A,\beta}$ is called *static full valuation* ***β***-congruence and is defined by $P =_{sfe,\beta} Q$ if $sfe_\beta(P) = sfe_\beta(Q)$.

The crucial difference between $sfe_\beta$ and the evaluation function *clfe* (Definition 6.2) concerns their domain: for $sfe_\beta(P)$ it is only required that $\alpha(P)$ is a subset of $\{\beta\}$. The completeness result below implies that $=_{sfe,\beta}$ is a congruence.

**Example 7.2:** If $\beta = ab$, we find that

$$(\mathsf{T} \trianglelefteq b \trianglerighteq \mathsf{T}) \trianglelefteq a \trianglerighteq (\mathsf{T} \trianglelefteq b \trianglerighteq \mathsf{T}) = sfe_{ab}(\mathsf{T}) = sfe_{ab}(\neg b \mathbin{{}^{\bullet}\!\vee} b) = sfe_{ab}(b \mathbin{{}^{\bullet}\!\vee} \mathsf{T}),$$

$$(\mathsf{T} \trianglelefteq b \trianglerighteq \mathsf{T}) \trianglelefteq a \trianglerighteq (\mathsf{F} \trianglelefteq b \trianglerighteq \mathsf{F}) = sfe_{ab}(a) = sfe_{ab}((b \mathbin{{}_{\bullet}\!\wedge} \mathsf{F}) \mathbin{{}^{\bullet}\!\vee} a) = sfe_{ab}(h(b,a,a)),$$

$$(\mathsf{T} \trianglelefteq b \trianglerighteq \mathsf{F}) \trianglelefteq a \trianglerighteq (\mathsf{T} \trianglelefteq b \trianglerighteq \mathsf{F}) = sfe_{ab}(b) = sfe_{ab}(b \mathbin{{}^{\bullet}\!\vee} (a \mathbin{{}_{\bullet}\!\wedge} \mathsf{F})) = sfe_{ab}(h(a,b,b)).$$

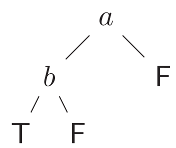
**Definition 7.3:** *Static Fully Evaluated Left-Sequential Logic (*SFEL*)* is the fully evaluated left-sequential logic that satisfies for any $\beta \in A^s_0$ no more consequences than those of $sfe_\beta$-equality, i.e. for all $P, Q \in \mathcal{SP}_{A,\beta}$,

$$\mathsf{SFEL} \models P = Q \iff P =_{sfe_\beta} Q.$$

We define the following set EqSFEL of axioms for SFEL:

$$\mathsf{EqSFEL} = \mathsf{EqC}\ell\mathsf{FEL}_2 \cup \{x \mathbin{{}_{\bullet}\!\wedge} \mathsf{F} = \mathsf{F}\}.$$

In order to prove completeness of EqSFEL, we reuse a result from *Short-Circuit Logic* (SCL), a family of logics based on the connectives $\mathbin{{}_{\circ}\!\wedge}$ and $\mathbin{{}^{\circ}\!\vee}$ that prescribe short-circuit evaluation: once the evaluation result of a propositional expression is determined, evaluation stops. Typically, $a \mathbin{{}_{\circ}\!\wedge} b$ has as its semantics the evaluation tree in which atom $b$ is only evaluated if $a$ evaluates to *true*:




The SCL-family comprises the counterparts of the FELs presented in this paper, i.e. FSCL, MSCL, C$\ell$SCL$_2$ and SSCL, and in all these logics the connective $\mathbin{{}_{\bullet}\!\wedge}$ is definable, as was noted in Staudt (2012), Bergstra et al. (2018) and Bergstra and Ponse (2025a):

$$x \mathbin{{}_{\bullet}\!\wedge} y = (x \mathbin{{}^{\circ}\!\vee} (y \mathbin{{}_{\circ}\!\wedge} \mathsf{F})) \mathbin{{}_{\circ}\!\wedge} y.$$

We return to these facts in Section 8.

**Theorem 7.4 (Completeness):** *The logic* SFEL *is axiomatised by* EqSFEL.

***Proof:*** For $\sigma \in A^s$, let $\mathcal{S}_\sigma$ be the set of closed SCL-terms with atoms in $\sigma$. In Bergstra et al. (2018), it is argued on p.21 that the static evaluation trees of all $P \in \mathcal{S}_\sigma$ are perfect binary trees, where each level characterises the evaluation of a single atom in $\sigma$, and it easily follows that $sse_\sigma(P) = sfe_\sigma(P')$ if $P'$ is obtained from $P$ by replacing the short-circuit connectives by their fully evaluated counterparts.

According to Bergstra et al. (2018), EqSSCL is the set of axioms in Table 4 that axiomatises SSCL. Three consequences of EqSSCL are $\mathsf{F} \mathbin{{}_{\circ}\!\wedge} x = \mathsf{F}$, $x \mathbin{{}_{\circ}\!\wedge} y = y \mathbin{{}_{\circ}\!\wedge} x$, and $x \mathbin{{}_{\bullet}\!\wedge} y = x \mathbin{{}_{\circ}\!\wedge} y$. We recall from Bergstra et al. (2018) a derivation of the last one: $x \mathbin{{}_{\bullet}\!\wedge} y = (x \mathbin{{}^{\circ}\!\vee} (y \mathbin{{}_{\circ}\!\wedge} \mathsf{F})) \mathbin{{}_{\circ}\!\wedge} y = (x \mathbin{{}^{\circ}\!\vee} (\mathsf{F} \mathbin{{}_{\circ}\!\wedge} y)) \mathbin{{}_{\circ}\!\wedge} y = (x \mathbin{{}^{\circ}\!\vee} \mathsf{F}) \mathbin{{}_{\circ}\!\wedge} y = x \mathbin{{}_{\circ}\!\wedge} y$.

In Bergstra et al. (2018, Thm.6.11) it is proven that for all $P, Q \in \mathcal{S}_\sigma$, $\mathsf{EqSSCL} \vdash P = Q \iff sse_\sigma(P) = sse_\sigma(Q)$. Identifying $\mathbin{{}_{\bullet}\!\wedge}$ and $\mathbin{{}_{\circ}\!\wedge}$, it suffices to show that EqSSCL and

**Table 4.** EqSSCL, a set of axioms for SSCL (axioms (Mem1)–(Mem5) are used in Appendix 2).

| | |
|---|---|
| $\mathsf{F} = \neg\mathsf{T}$ | (Mem1) |
| $x \mathbin{^\circ\!\vee} y = \neg(\neg x \mathbin{_\circ\!\wedge} \neg y)$ | (Mem2) |
| $\mathsf{T} \mathbin{_\circ\!\wedge} x = x$ | (Mem3) |
| $x \mathbin{_\circ\!\wedge} (x \mathbin{^\circ\!\vee} y) = x$ | (Mem4) |
| $(x \mathbin{^\circ\!\vee} y) \mathbin{_\circ\!\wedge} z = (\neg x \mathbin{_\circ\!\wedge} (y \mathbin{_\circ\!\wedge} z)) \mathbin{^\circ\!\vee} (x \mathbin{_\circ\!\wedge} z)$ | (Mem5) |
| $x \mathbin{_\circ\!\wedge} y = y \mathbin{_\circ\!\wedge} x$ | (Comm) |

EqSFEL are equivalent, and this follows quickly with *Prover9*: EqSSCL ⊢ EqSFEL with options `lpo` and `pass` requires 2s, and EqSFEL ⊢ EqSSCL with identical options requires 1s. Hence, for all $P, Q \in \mathcal{SP}_{A,\beta}$, EqSFEL $\vdash P = Q \iff P =_{sfe_\beta} Q$. ■

Finally, we provide independent axiomatisations for all FELs.

**Theorem 7.5:** *Let*

$$\overline{\text{EqFFEL}} = \text{EqFFEL} \setminus \{\textit{(FFEL1)}\},$$

$$\overline{\text{EqMFEL}} = \text{EqMFEL} \setminus \{\textit{(FFEL1),(FFEL5),(FFEL8),(FFEL9)}\},$$

$$\overline{\text{EqC}\ell\text{FEL}_2} = \{x \mathbin{_\bullet\!\wedge} y = y \mathbin{_\bullet\!\wedge} x\} \cup \overline{\text{EqMFEL}} \setminus \{\textit{(FFEL4), (FFEL7)}\},$$

$$\overline{\text{EqSFEL}} = \{x \mathbin{_\bullet\!\wedge} F = F\} \cup \overline{\text{EqC}\ell\text{FEL}_2}.$$

*Then each set of axioms* $\overline{\text{Ax}}$ *in the left column is equivalent to the (original) axiomatisation* Ax. *Furthermore,* $\overline{\text{EqSFEL}}$ *is independent, and for each other axiomatisation* $\overline{\text{Ax}}$, *let* $\overline{\text{Ax}}^U$ *be its extension with the two axioms* $\neg U = U$ *and* $U \mathbin{_\bullet\!\wedge} x = U$. *Then* $\overline{\text{Ax}}^U$ *is independent, and so is* $\overline{\text{Ax}}$.

***Proof:*** From $\overline{\text{EqFFEL}}$ derive $\mathsf{F} = \neg\neg\mathsf{F} = \neg(\mathsf{T} \mathbin{_\bullet\!\wedge} \neg\mathsf{F}) = \neg(\neg\neg\mathsf{T} \mathbin{_\bullet\!\wedge} \neg\mathsf{F}) = \neg\mathsf{T} \mathbin{^\bullet\!\vee} \mathsf{F}$, hence by the dual of (FFEL6), $\mathsf{F} = \neg\mathsf{T}$. According to *Mace4*, $\overline{\text{EqFFEL}}^U$ is independent.

With *Prover9* and default options `lpo` and `unfold`, $\overline{\text{EqMFEL}} \vdash$ EqMFEL (a proof requires 1s). With help of *Mace4* it quickly follows that $\overline{\text{EqMFEL}}^U$ is independent.

With *Prover9* and the default options, $\overline{\text{EqC}\ell\text{FEL}_2} \vdash \text{EqC}\ell\text{FEL}_2$ (a proof requires 1s), and according to *Mace4*, $\overline{\text{EqC}\ell\text{FEL}_2}^U$ is independent. In Appendix 4 we show the eight countermodels generated by *Mace4*.

According to *Prover9* with default options, $\overline{\text{EqSFEL}} \vdash$ EqSFEL (a proof requires 1 s). With help of *Mace4* it quickly follows that $\overline{\text{EqSFEL}}$ is independent. ■

**Comment 7.6:** Another axiomatisation of C$\ell$FEL is briefly discussed in Bergstra and Ponse (2025a, Sect.6) and stems from Bergstra et al. (1995), in which an equational axiomatisation of Bochvar's logic (1938) is introduced and proven complete that we repeat here (using full left-sequential connectives instead of ∧ and ∨):

| | | | |
|---|---|---|---|
| (S1) | $\neg\mathsf{T} = \mathsf{F}$ | (S7) | $\mathsf{T} \mathbin{{}_{\bullet}\!\wedge} x = x$ |
| (S2) | $\neg\mathsf{U} = \mathsf{U}$ | (S8) | $x \mathbin{\overset{\bullet}{\vee}} (\neg x \mathbin{{}_{\bullet}\!\wedge} y) = x \mathbin{\overset{\bullet}{\vee}} y$ |
| (S3) | $\neg\neg x = x$ | (S9) | $x \mathbin{{}_{\bullet}\!\wedge} y = y \mathbin{{}_{\bullet}\!\wedge} x$ |
| (S4) | $\neg(x \mathbin{{}_{\bullet}\!\wedge} y) = \neg x \mathbin{\overset{\bullet}{\vee}} \neg y$ | (S10) | $x \mathbin{{}_{\bullet}\!\wedge} (y \mathbin{\overset{\bullet}{\vee}} z) = (x \mathbin{{}_{\bullet}\!\wedge} y) \mathbin{\overset{\bullet}{\vee}} (x \mathbin{{}_{\bullet}\!\wedge} z)$ |
| (S6) | $(x \mathbin{{}_{\bullet}\!\wedge} y) \mathbin{{}_{\bullet}\!\wedge} z = x \mathbin{{}_{\bullet}\!\wedge} (y \mathbin{{}_{\bullet}\!\wedge} z)$ | (S11) | $\mathsf{U} \mathbin{{}_{\bullet}\!\wedge} x = \mathsf{U}$ |

The 'missing' axiom (S5) defines the connective $\bullet\!\!\longrightarrow$ (full left-sequential implication, $x \bullet\!\!\longrightarrow y = \neg x \mathbin{\overset{\bullet}{\vee}} y$) and is not relevant here. According to *Prover9*, this axiomatisation is equivalent with $\mathrm{EqC}\ell\mathrm{FEL}^{\mathsf{U}}$. However, without axiom (S9), this set is not an axiomatisation of $\mathrm{MFEL}^{\mathsf{U}}$, and therefore we prefer $\mathrm{EqC}\ell\mathrm{FEL}^{\mathsf{U}}$ and $\mathrm{EqC}\ell\mathrm{FEL}_2$. Finally, we note that according to *Prover9*, axiom (S6) is derivable from the remaining axioms, which are independent according to *Mace4*, and that the same is true for axiom (S10).

## 8. Discussion and some conclusions

In this section, we take a closer look at all FELs defined and embed the FEL-family in a larger whole, namely the family of (left-sequential) short-circuit logics, SCLs for short. We first briefly explain the SCL family and address the property that each FEL is a sublogic of its corresponding SCL (each FEL-term can be expressed as an SCL-term). We then discuss some expressiveness issues and end with some conclusions.

*Hoare's conditional connective, short-circuit evaluation and SCLs.* We could have defined the fully evaluated connectives $\mathbin{{}_{\bullet}\!\wedge}$ and $\mathbin{\overset{\bullet}{\vee}}$, and thus the FELs, with help of the ternary connective

$$x \triangleleft y \triangleright z,$$

the so-called 'conditional' that expresses '`if y then x else z`' and with which Hoare (1985) characterised the propositional calculus (also using both constants T and F).[4] The conditional connective naturally characterises short-circuit evaluation and has a notation that supports equational reasoning, which led us to define *proposition algebra*, see Bergstra and Ponse (2011), and subsequently several different SCLs in Bergstra et al. (2010[2013]) and Bergstra and Ponse (2025a). More precisely, in Bergstra and Ponse (2011), left-sequential short-circuited conjunction $\mathbin{{}_{\circ}\!\wedge}$ and disjunction $\mathbin{\overset{\circ}{\vee}}$ are defined by

$$x \mathbin{{}_{\circ}\!\wedge} y = y \triangleleft x \triangleright \mathsf{F} \quad \text{and} \quad x \mathbin{\overset{\circ}{\vee}} y = \mathsf{T} \triangleleft x \triangleright y,$$

and negation by $\neg x = \mathsf{F} \triangleleft x \triangleright \mathsf{T}$. Moreover, in that paper the following set CP of four axioms is distinguished as fundamental:

$$x \triangleleft \mathsf{T} \triangleright y = x, \tag{CP1}$$

$$x \triangleleft \mathsf{F} \triangleright y = y, \tag{CP2}$$

$$\mathsf{T} \triangleleft x \triangleright \mathsf{F} = x, \tag{CP3}$$

$$x \triangleleft (y \triangleleft z \triangleright u) \triangleright v = (x \triangleleft y \triangleright v) \triangleleft z \triangleright (x \triangleleft u \triangleright v). \tag{CP4}$$

In Bergstra et al. (2010[2013]), the short-circuit logic FSCL is defined by the equational theory of terms over the signature $\Sigma_{\mathrm{CP}} = \{\mathbin{{}_{\circ}\!\wedge}, \neg, \mathsf{T}, \mathsf{F}\}$ that is generated by the two defining axioms $x \mathbin{{}_{\circ}\!\wedge} y = y \triangleleft x \triangleright \mathsf{F}$ and $\neg x = \mathsf{F} \triangleleft x \triangleright \mathsf{T}$, and the axioms of CP. We further write $\mathrm{CP}(\mathbin{{}_{\circ}\!\wedge}, \neg)$ for the set of these six axioms.[5] As an example, the associativity of $\mathbin{{}_{\circ}\!\wedge}$ is quickly derived from $\mathrm{CP}(\mathbin{{}_{\circ}\!\wedge}, \neg)$, and thus holds in FSCL:

$$\begin{aligned}
(x \mathbin{{}_{\circ}\!\wedge} y) \mathbin{{}_{\circ}\!\wedge} z &= z \triangleleft (y \triangleleft x \triangleright \mathsf{F}) \triangleright \mathsf{F} && \text{by definition}\\
&= (z \triangleleft y \triangleright \mathsf{F}) \triangleleft x \triangleright (z \triangleleft \mathsf{F} \triangleright \mathsf{F}) && \text{by (CP4)}\\
&= (z \triangleleft y \triangleright \mathsf{F}) \triangleleft x \triangleright \mathsf{F} && \text{by (CP2)}\\
&= x \mathbin{{}_{\circ}\!\wedge} (y \mathbin{{}_{\circ}\!\wedge} z). && \text{by definition}
\end{aligned}$$

Each of the FELs discussed in this paper is related to one of the SCLs defined in Bergstra et al. (2010[2013]) and Bergstra and Ponse (2025a). These SCLs are defined in a generic way, e.g. *Memorising* SCL (MSCL) is defined as the restriction to $\Sigma_{\mathrm{CP}}$ of the equational theory of $\mathrm{CP}(\mathbin{{}_{\circ}\!\wedge}, \neg)$ extended with the 'memorising' axiom

$$x \triangleleft y \triangleright (z \triangleleft u \triangleright (v \triangleleft y \triangleright w)) = x \triangleleft y \triangleright (z \triangleleft u \triangleright w) \quad (y \text{ is 'memorised'}). \tag{43}$$

As an example, $x \mathbin{{}_{\circ}\!\wedge} x = x$ is easily derivable from these seven axioms, and so are the axioms (Mem1)–(Mem5) in Table 4. More generally, the validity of any equation in a particular SCL can be checked in the associated CP-system: either a proof can be found (often with help of *Prover9*), or *Mace4* finds a countermodel. SCLs were extended with a constant U for the third truth value *undefined* and the defining axiom $x \triangleleft \mathsf{U} \triangleright y = \mathsf{U}$ in Bergstra et al. (2021). A complete axiomatisation for $\mathrm{MSCL}^{\mathsf{U}}$ was given, but an axiomatisation of $\mathrm{FSCL}^{\mathsf{U}}$ is not yet known, not even when restricted to closed terms, see Bergstra et al. (2021, Conject.8.1).

FEL*s and* SCL*s.* As was noted in Staudt (2012), the connective $\mathbin{{}_{\bullet}\!\wedge}$ can be defined using short-circuit connectives and the constant F (due to the FSCL-law $\mathsf{F} \mathbin{{}_{\circ}\!\wedge} x = \mathsf{F}$):

$$x \mathbin{{}_{\bullet}\!\wedge} y = (x \mathbin{\overset{\circ}{\vee}} (y \mathbin{{}_{\circ}\!\wedge} \mathsf{F})) \mathbin{{}_{\circ}\!\wedge} y. \tag{44}$$

It is easy to prove that $\mathrm{CP}(\mathbin{{}_{\circ}\!\wedge}, \neg) + (44) \vdash x \mathbin{{}_{\bullet}\!\wedge} y = y \triangleleft x \triangleright (\mathsf{F} \triangleleft y \triangleright \mathsf{F})$ and the latter equation is perhaps more intuitive than (54). The addition of $x \mathbin{{}_{\bullet}\!\wedge} y = y \triangleleft x \triangleright (\mathsf{F} \triangleleft y \triangleright \mathsf{F})$ and $\neg x = \mathsf{F} \triangleleft x \triangleright \mathsf{T}$ to CP, say $\mathrm{CP}(\mathbin{{}_{\bullet}\!\wedge}, \neg)$, implies that $\mathrm{CP}(\mathbin{{}_{\bullet}\!\wedge}, \neg) \vdash \mathrm{EqFFEL}$. Similarly, the addition of axiom (43) to $\mathrm{CP}(\mathbin{{}_{\bullet}\!\wedge}, \neg)$ implies that $x \mathbin{{}_{\bullet}\!\wedge} x = x$ and (M1) in Table 2 are derivable. Like in the case of SCLs, the validity of any equation in a particular FEL can be easily checked in the associated CP-system. The addition of U is in the case of FELs simple. For $\mathrm{FFEL}^{\mathsf{U}}$ there are normal forms $\mathsf{U}_\sigma$ with $\sigma \in A^*$, and for $\mathrm{MFEL}^{\mathsf{U}}$, normal forms $\mathsf{U}_\sigma$ with $\sigma \in A^s$. In the case of C$\ell$FEL, these all reduce to U.

*Side effects and application perspective.* SCLs can be characterised by their efficiency, in the sense that atoms are not evaluated if their evaluation is not needed to determine the evaluation of a term as a whole. From that perspective FELs might seem rather inefficient, but this is not necessarily so, as was already noted in Staudt (2012), a quote:

> We find that some programming languages offer full left-sequential connectives, which motivated the initial investigation of FEL. We claim that FEL has a greater value than merely to act as means of writing certain SCL-terms using fewer symbols. The usefulness

of a full evaluation strategy lies in the increased predictability of the state of the environment after a (sub)term has been evaluated. In particular, we know that the side effects of all the atoms in the term have occurred. To determine the state of the environment after the evaluation of a FEL-term in a given environment, we need only compute how each atom in the term transforms the environment. It is not necessary to compute the evaluation of any of the atoms. With SCL-terms in general we must know to what the first atom evaluates in order to determine which atom is next to transform the environment. Thus to compute the state of the environment after the evaluation of an SCL-term we must compute the evaluation result of each atom that transforms the environment and we must compute the transformation of the environment for each atom that affects it.

For a simple example, let `v` be a programming variable that ranges over the integers, and let `e`, `e′` be simple arithmetical integer expressions that may contain `v`. As atoms we take *tests* of the form `[e > e′]` with the expected boolean evaluation and *assignments* of the form `(v: =e)` with evaluation result the boolean value of `e` (thus *false* if the value of `v` equals `0`, and otherwise *true*). Now consider the conditions

$$\texttt{(v:=v+1)} \mathbin{{}_{\circ}\!\!\wedge} \texttt{([v > 1]} \mathbin{{}_{\circ}\!\!\wedge} \texttt{(v:=v+1))} \text{ and}$$

$$\texttt{(v:=v+1)} \mathbin{{}_{\bullet}\!\!\wedge} \texttt{([v > 1]} \mathbin{{}_{\bullet}\!\!\wedge} \texttt{(v:=v+1))}.$$

For both conditions, their boolean FSCL and FFEL-evaluations are identical. However, the 'side effects' in the first condition are non-uniform and depend on the initial value of `v`, while they are uniform in the second condition. This is easy to see by inspecting the initial values of $\texttt{v} \in \{-2, -1, 0, 1\}$.

Apart from side-effects, it may be desirable for a computation or evaluation to have a predictable length or duration, e.g. to avoid timing attacks, as explained at https://en.wikipedia.org/wiki/Timing_attack,[6] where an example is also given of two pieces of C code, the first of which is short-circuited and insecure, while the second uses a bitwise (fully evaluated) operation and is secure.

*FELs and SCLs, the larger whole.* Given the relevance of FFEL and $\text{FFEL}^{\text{U}}$ as discussed above, it is natural to investigate some stronger FELs.

First, it is clear that MFEL and its evaluation trees are a useful tool in finding (and explaining) equational axiomatisations of $\text{C}\ell\text{FEL}_2$, $\text{C}\ell$FEL and SFEL, and in defining their evaluation trees.

In Bergstra and Ponse (2025a), three-valued *Conditional logic* (Guzmán & Squier, 1990) is viewed as a short-circuit logic, named $\text{C}\ell$SCL.[7] Furthermore, its two-valued variant $\text{C}\ell\text{SCL}_2$ is located in between MSCL and SSCL, and it is also noted that conditional logic defines a fully evaluated sublogic that is equivalent to the strict three-valued logic of Bochvar (1938); this is the logic we now call $\text{C}\ell$FEL. We stress that the availability of a separate notation for the fully evaluated connectives in addition to the notation $\mathbin{{}_{\circ}\!\!\wedge}$ and $\mathbin{{}^{\circ}\!\!\vee}$ for the short-circuit connectives is instrumental to this observation. As an example, it is noted in Bergstra and Ponse (2025a) that $x \mathbin{{}_{\bullet}\!\!\wedge} y = (x \mathbin{{}_{\circ}\!\!\wedge} y) \mathbin{{}^{\circ}\!\!\vee} (y \mathbin{{}_{\circ}\!\!\wedge} x)$ is a derivable consequence of the axioms for $\text{C}\ell\text{SCL}_2$. Hence, the commutativity of $\mathbin{{}_{\bullet}\!\!\wedge}$ is an immediate consequence of the axiom

$$(x \mathbin{{}_{\circ}\!\!\wedge} y) \mathbin{{}^{\circ}\!\!\vee} (y \mathbin{{}_{\circ}\!\!\wedge} x) = (y \mathbin{{}_{\circ}\!\!\wedge} x) \mathbin{{}^{\circ}\!\!\vee} (x \mathbin{{}_{\circ}\!\!\wedge} y)$$

that is typical for the axiomatisations of $\text{C}\ell$SCL and $\text{C}\ell\text{SCL}_2$. Thus, we locate $\text{C}\ell\text{FEL}_2$ in a hierarchy of two-valued FELs, and we derive its existence and motivation both from the three-valued logic C$\ell$SCL, and from the consideration to adopt a full evaluation strategy.

Finally, we observe that SFEL can be considered a sublogic of SSCL, although both these logics are obviously equivalent to propositional logic, albeit with different evaluation strategies.

*Expressiveness.* We call a perfect evaluation tree *uniform* if each complete trace contains the same sequence of atoms. By definition, each FEL defines a certain equality on the domain of uniform evaluation trees. In FFEL and $\text{FFEL}^{\text{U}}$, some uniform evaluation trees can be expressed, but for example not

$$(\mathsf{T} \trianglelefteq b \trianglerighteq \mathsf{F}) \trianglelefteq a \trianglerighteq (\mathsf{F} \trianglelefteq b \trianglerighteq \mathsf{T}),$$

which is expressible in MFEL by $h(a,b,\neg b) = (a \mathbin{{}_{\bullet}\!\wedge} b) \mathbin{{}^{\bullet}\!\vee} (\neg a \mathbin{{}_{\bullet}\!\wedge} \neg b)$, and it easily follows that *each* uniform tree can be expressed in MFEL. Of course, in SFEL, which is equivalent to SSCL, with $x \mathbin{{}_{\bullet}\!\wedge} \mathsf{F} = x \mathbin{{}_{\circ}\!\wedge} \mathsf{F} = \mathsf{F}$, each uniform evaluation tree can be compared to a truth table for propositional logic and it is possible to define a semantics for SFEL based on an equivalence $\equiv$ on evaluation trees that identifies each (sub)tree with only F-leaves with F, and each (sub)tree with only T-leaves with T. Then, any $P \in \mathcal{SP}_{A,\beta}$ has a representing evaluation tree without such subtrees, for example,

$$\begin{aligned}
&sfe_{abc}((a \mathbin{{}_{\bullet}\!\wedge} b) \mathbin{{}^{\bullet}\!\vee} (\neg a \mathbin{{}_{\bullet}\!\wedge} c))\\
&\quad = ((\mathsf{T} \trianglelefteq c \trianglerighteq \mathsf{T}) \trianglelefteq b \trianglerighteq (\mathsf{F} \trianglelefteq c \trianglerighteq \mathsf{F})) \trianglelefteq a \trianglerighteq ((\mathsf{T} \trianglelefteq c \trianglerighteq \mathsf{F}) \trianglelefteq b \trianglerighteq (\mathsf{T} \trianglelefteq c \trianglerighteq \mathsf{F}))\\
&\quad \overset{\text{La.6.10}}{=} ((\mathsf{T} \trianglelefteq c \trianglerighteq \mathsf{T}) \trianglelefteq b \trianglerighteq (\mathsf{F} \trianglelefteq c \trianglerighteq \mathsf{F})) \trianglelefteq a \trianglerighteq ((\mathsf{T} \trianglelefteq b \trianglerighteq \mathsf{T}) \trianglelefteq c \trianglerighteq (\mathsf{F} \trianglelefteq b \trianglerighteq \mathsf{F}))\\
&\quad = (\mathsf{T} \trianglelefteq b \trianglerighteq \mathsf{F}) \trianglelefteq a \trianglerighteq (\mathsf{T} \trianglelefteq c \trianglerighteq \mathsf{F}). \quad \text{(by SFEL/}\equiv\text{)}
\end{aligned}$$

In MSCL, the latter tree is the evaluation tree of $(a \mathbin{{}_{\circ}\!\wedge} b) \mathbin{{}^{\circ}\!\vee} (\neg a \mathbin{{}_{\circ}\!\wedge} c)$, see Bergstra et al. (2021) for a detailed explanation.

As a final point, we note that there are left-sequential, binary connectives for which the distinction between short-circuit evaluation and full evaluation does not exist, such as the *left-sequential biconditional* and the *left-sequential exclusive or*, defined in Cornets de Groot (2020) by $x \mathbin{\bullet\!\!\twoheadleftarrow\!\!\rightarrow} y = y \triangleleft x \triangleright (\mathsf{F} \triangleleft y \triangleright \mathsf{T})$ and $x \mathbin{\bullet\!\!<\!\!\oplus} y = (\mathsf{F} \triangleleft y \triangleright \mathsf{T}) \triangleleft x \triangleright y$, respectively. It immediately follows that $x \mathbin{\bullet\!\!<\!\!\oplus} y = x \mathbin{\bullet\!\!\twoheadleftarrow\!\!\rightarrow} \neg y$ and $\neg(x \mathbin{\bullet\!\!\twoheadleftarrow\!\!\rightarrow} y) = x \mathbin{\bullet\!\!\twoheadleftarrow\!\!\rightarrow} \neg y$. Below we use the symbols $\mathbin{\twoheadleftarrow\!\!\rightarrow}$ and $\mathbin{<\!\!\oplus}$, expressing only left-sequentiality. Regarding expressiveness, we note that in FFEL extended with $\mathbin{\twoheadleftarrow\!\!\rightarrow}$, the evaluation tree of $a \mathbin{\twoheadleftarrow\!\!\rightarrow} b$ is $(\mathsf{T} \trianglelefteq b \trianglerighteq \mathsf{F}) \trianglelefteq a \trianglerighteq (\mathsf{F} \trianglelefteq b \trianglerighteq \mathsf{T})$. However, it easily follows that not all uniform evaluation trees can be expressed in FFEL with $\mathbin{\twoheadleftarrow\!\!\rightarrow}$.[8] Of course, in MFEL and all stronger FELs, the left-sequential biconditional $x \mathbin{\twoheadleftarrow\!\!\rightarrow} y$ is definable by $h(x,y,\neg y)$.

*Conclusions and future work.* As mentioned, this paper is a continuation of Staudt (2012), in which FFEL was introduced and which contained the first completeness proof for the short-circuit logic FSCL (Free SCL). We introduced six more *fully evaluated sequential logics* (FELs), two of which are not new: C$\ell$FEL, which is equivalent to the logic of Bochvar (1938), and SFEL, which is equivalent to SSCL (Bergstra et al., 2010[2013]) and to propositional logic. We note that the work in Bergstra and Ponse (2025b), which is about 'fracterm calculus for partial meadows', is based on C$\ell$SCL.

More research on the pros and cons of programming with side effects and the use of FFEL versus FSCL (and their U-extensions) would be interesting, and could lead to new, interesting FELs that lie between FFEL and MFEL: referring to Bergstra and Ponse (2011), these could be the FELs based on *repetition-proof* and *contractive* valuation congruence.

## Notes

1. In 'short-circuit logics' (SCLs), short-circuit evaluation is prescribed by the short-circuit connectives ${}_{\circ}\!\wedge$ and ${}^{\circ}\!\vee$; in the journal paper (Ponse & Staudt, 2018), the FSCL-part of Staudt (2012) is also discussed.
2. A general reference to equational logics is Burris and Sankappanavar (2012).
3. Note that $x \,{}_{\bullet}\!\wedge\, \mathsf{F} = \mathsf{F}$ is a consequence of absorption: $x \,{}_{\bullet}\!\wedge\, \mathsf{F} = \mathsf{F} \,{}_{\bullet}\!\wedge\, x = \mathsf{F} \,{}_{\bullet}\!\wedge\, (\mathsf{F} \,{}^{\bullet}\!\vee\, x) = \mathsf{F}$. This equation plays an important role in Static FEL (see Section 7).
4. However, Church (1948) introduced the *conditioned disjunction* $[p, q, r]$ as a primitive connective for the propositional calculus, which expresses the same as Hoare's conditional $p \triangleleft q \triangleright r$, and proved the same result.
5. Short-circuit disjunction $x \,{}^{\circ}\!\vee\, y$ can be defined by $\neg(\neg x \,{}_{\circ}\!\wedge\, \neg y)$ because $\mathrm{CP}({}_{\circ}\!\wedge, \neg) \vdash \neg(\neg x \,{}_{\circ}\!\wedge\, \neg y) = \mathsf{T} \triangleleft x \triangleright y$ easily follows.
6. Last edited 4 May 2025 at time of writing.
7. The name C$\ell$SCL was chosen because CSCL is already used for 'Contractive SCL' in Bergstra et al. (2010[2013]), the short-circuit logic that (only) contracts adjacent atoms: $a \,{}_{\circ}\!\wedge\, a = a$, but $(a \,{}_{\circ}\!\wedge\, b) \,{}_{\circ}\!\wedge\, (a \,{}_{\circ}\!\wedge\, b)$ and $a \,{}_{\circ}\!\wedge\, b$ have different evaluation trees.
8. Suppose $A = \{a\}$. Then there are $5 \cdot 4^4 \cdot 3^3$ terms composed of four elements of $\{a, \neg a, a \,{}^{\bullet}\!\vee\, \mathsf{T}, a \,{}_{\bullet}\!\wedge\, \mathsf{F}\}$ and the three connectives in $\{{}_{\bullet}\!\wedge, {}^{\bullet}\!\vee, \twoheadleftarrow\!\!\!\rightarrow\}$ that have evaluation trees of depth 4. It is not hard to prove that these terms represent all closed terms with evaluation trees of depth 4. However, there are $2^{16}$ such trees.
9. The proof of clause 5 of Bergstra et al. (2021, La.3.3) (p.266) contains two errors in Case $c = a$, correct is: 'and by induction $L_a(X_1[\mathsf{T} \mapsto Y]) = L_a(X_1)[\mathsf{T} \mapsto L_a(Y)] = L_a(X)[\mathsf{T} \mapsto L_a(Y)]$' (cf. Case $b = a$ in the proof below and take $Z = \mathsf{F}$).

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## References

Bergstra, J. A., Bethke, I., & Rodenburg, P. H. (1995). A propositional logic with 4 values: True, false, divergent and meaningless. *Journal of Applied Non-Classical Logics*, *5*(2), 199–217. https://doi.org/10.1080/11663081.1995.10510855

Bergstra, J. A., & Ponse, A. (2011). Proposition algebra. *ACM Transactions on Computational Logic*, *12*(3), Article 21 (36 pages). https://doi.org/10.1145/1929954.1929958

Bergstra, J. A., & Ponse, A. (2015). *Evaluation trees for proposition algebra*. https://doi.org/10.48550/arXiv.1504.08321 [cs.LO].
Bergstra, J. A., & Ponse, A. (2025a). Conditional logic as a short-circuit logic. *Scientific Annals of Computer Science*, *35*(2), 161–196. https://doi.org/10.47743/SACS.2025.2.161
Bergstra, J. A., & Ponse, A. (2025b). *Fracterm calculus for partial meadows*. https://doi.org/10.48550/arXiv.2502.13812 [cs.LO].
Bergstra, J. A., Ponse, A., & Staudt, D. J. C. (2013). *Short-circuit logic*. (First version appeared in 2010.) https://doi.org/10.48550/arXiv.1010.3674 [cs.LO,math.LO].
Bergstra, J. A., Ponse, A., & Staudt, D. J. C. (2018). *Propositional logic with short-circuit evaluation: A non-commutative and a commutative variant*. https://doi.org/10.48550/arXiv.1810.02142 [cs.LO,math.LO].
Bergstra, J. A., Ponse, A., & Staudt, D. J. C. (2021). Non-commutative propositional logic with short-circuit evaluation. *Journal of Applied Non-Classical Logics*, *31*(3–4), 234–278. (online available). https://doi.org/10.1080/11663081.2021.2010954
Bochvar, D. A. (1938). (In Russian). On a 3-valued logical calculus and its application to the analysis of contradictions. *Matematicheskii Sbornik*, *4*(2), 287–308. Translation: Bochvar, D. A. & Bergmann, M. (1981). On a three-valued logical calculus and its application to the analysis of the paradoxes of the classical extended functional calculus. *History and Philosophy of Logic*, 2(1-2):87-112.
Burris, S. N., & Sankappanavar, H. P. (2012). *A course in universal algebra – the millennium edition*. http://www.math.uwaterloo.ca/snburris/htdocs/ualg.html.
Church, A. (1948). Conditioned disjunction as a primitive connective for the propositional calculus. *Portugaliae Mathematica*, *7*(2), 87–90.
Cornets de Groot, S. H. (2020). *Logical systems with left-sequential versions of NAND and XOR* [MSc. thesis] Logic, University of Amsterdam (June 2020). https://eprints.illc.uva.nl/id/eprint/1743/1/MoL-2020-02.text.pdf.
Guzmán, F., & Squier, C. C. (1990). The algebra of conditional logic. *Algebra Universalis*, *27*(1), 88–110. https://doi.org/10.1007/BF01190256
Hoare, C. A. R. (1985). A couple of novelties in the propositional calculus. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, *31*(2), 173–178. https://doi.org/10.1002/malq.v31:9/12
Kleene, S. C. (1938). On notation for ordinal numbers. *Journal of Symbolic Logic*, *3*(4), 150–155. https://doi.org/10.2307/2267778
McCarthy, J. (1963). A basis for a mathematical theory of computation. In P. Braffort & D. Hirschberg (Eds.), *Computer Programming and Formal Systems*, Studies in Logic and the Foundations of Mathematics (Vol. 35, pp. 33–70). Elsevier (North Holland Publishing Co.). https://doi.org/10.1016/S0049-237X(08)72018-4
McCune, W. (2008). *The GUI: Prover9 and Mace4 with a graphical user interface: Prover9-Mace4* (Version 0.5B) [Computer software]. https://www.cs.unm.edu/~mccune/prover9/gui/v05.html.
Ponse, A., & Staudt, D. J. C. (2018). An independent axiomatisation for free short-circuit logic. *Journal of Applied Non-Classical Logics*, *28*(1), 35–71. (online available). https://doi.org/10.1080/11663081.2018.1448637
Staudt, D. J. C. (2012). *Completeness for two left-sequential logics* [MSc. thesis] Logic, University of Amsterdam (May 2012). https://doi.org/10.48550/arXiv.1206.1936 [cs.LO].

# Appendices

## Appendix 1. Proofs - Section 2

*Correctness of* $\boldsymbol{f}$. To prove that $f : \mathcal{SP}_A \to \text{FNF}$ is indeed a normalization function we need to prove that for all FEL-terms $P$, $f(P)$ terminates, $f(P) \in \text{FNF}$ and $\text{EqFFEL} \vdash f(P) = P$. To arrive at this result, we prove several intermediate results about the functions $f^n$ and $f^c$, roughly in the order in which their definitions were presented in Section 2. For the sake of brevity we will not explicitly prove that these functions terminate. To see that each function terminates consider that a termination proof would closely mimic the proof structure of the lemmas dealing with the grammatical categories of the images of these functions.

**Lemma A.1:** *For any $P^F$ and $P^T$,* EqFFEL $\vdash P^F = P^F \leftand F$ *and* EqFFEL $\vdash P^T = P^T \leftor T$.

***Proof:*** We prove both claims simultaneously by induction. In the base case we have $\mathrm{F} = \mathrm{T} \leftand \mathrm{F}$ by (FFEL5), which is equal to $\mathrm{F} \leftand \mathrm{F}$ by (FFEL8) and (FFEL1). The base case for the second claim follows from that for the first claim by duality.

For the induction we have $a \leftand P^{\mathrm{F}} = a \leftand (P^{\mathrm{F}} \leftand \mathrm{F})$ by the induction hypothesis and the result follows from (FFEL4). For the second claim we again appeal to duality. ■

**Lemma A.2:** *The following equations can be derived by equational logic and* EqFFEL.

(1) $x \leftand (y \leftand (z \leftand F)) = (x \leftor y) \leftand (z \leftand F)$,
(2) $\neg x \leftand (y \leftor T) = \neg(x \leftand (y \leftor T))$.

***Proof:***

$$\begin{aligned}
x \leftand (y \leftand (z \leftand \mathrm{F})) &= x \leftand ((\neg y \leftand z) \leftand \mathrm{F}) \quad \text{by (FFEL4) and Lemma 2.7.1}\\
&= (\neg x \leftand \neg y) \leftand (z \leftand \mathrm{F}) \quad \text{by (FFEL4) and Lemma 2.7.1}\\
&= \neg(\neg x \leftand \neg y) \leftand (z \leftand \mathrm{F}) \quad \text{by Lemma 2.7.1}\\
&= (x \leftor y) \leftand (z \leftand \mathrm{F}), \quad \text{by (FFEL2)}\\
\neg x \leftand (y \leftor \mathrm{T}) &= \neg x \leftor (y \leftand \mathrm{F}) \quad \text{by (FFEL10)}\\
&= \neg(x \leftand \neg(y \leftand \mathrm{F})) \quad \text{by (FFEL2) and (FFEL3)}\\
&= \neg(x \leftand \neg(\neg y \leftand \neg\mathrm{T})) \quad \text{by (FFEL8) and (FFEL1)}\\
&= \neg(x \leftand (y \leftor \mathrm{T})). \quad \text{by (FFEL2)}
\end{aligned}$$

■

**Lemma A.3:** *For all $P \in$* FNF, *if $P$ is a T-term then $f^n(P)$ is an F-term, if it is an F-term then $f^n(P)$ is a T-term, if it is a T-$*$-term then so is $f^n(P)$, and*

$$\text{EqFFEL} \vdash f^n(P) = \neg P.$$

***Proof:*** We start with proving the claims for T-terms, by induction on $P^{\mathrm{T}}$. In the base case $f^n(\mathrm{T}) = \mathrm{F}$. It is immediate that $f^n(\mathrm{T})$ is an F-term. The claim that EqFFEL $\vdash f^n(\mathrm{T}) = \neg\mathrm{T}$ is immediate by (FFEL1). For the inductive case we have that $f^n(a \leftor P^{\mathrm{T}}) = a \leftand f^n(P^{\mathrm{T}})$, where we may assume that $f^n(P^{\mathrm{T}})$ is an F-term and that EqFFEL $\vdash f^n(P^{\mathrm{T}}) = \neg P^{\mathrm{T}}$. The grammatical claim now follows immediately from the induction hypothesis. Furthermore, noting that by the induction hypothesis we may assume that $f^n(P^{\mathrm{T}})$ is an F-term, we have:

$$\begin{aligned}
f^n(a \leftor P^{\mathrm{T}}) &= a \leftand f^n(P^{\mathrm{T}}) \quad \text{by definition}\\
&= a \leftand (f^n(P^{\mathrm{T}}) \leftand \mathrm{F}) \quad \text{by Lemma A.1}\\
&= \neg a \leftand (f^n(P^{\mathrm{T}}) \leftand \mathrm{F}) \quad \text{by Lemma 2.7.1}\\
&= \neg a \leftand f^n(P^{\mathrm{T}}) \quad \text{by Lemma A.1}\\
&= \neg a \leftand \neg P^{\mathrm{T}} \quad \text{by induction hypothesis}\\
&= \neg(a \leftor P^{\mathrm{T}}). \quad \text{by (FFEL3) and (FFEL2)}
\end{aligned}$$

For F-terms we prove our claims by induction on $P^{\mathsf{F}}$. In the base case $f^n(\mathsf{F}) = \mathsf{T}$. It is immediate that $f^n(\mathsf{F})$ is a T-term. The claim that $\mathrm{EqFFEL} \vdash f^n(\mathsf{F}) = \neg\mathsf{F}$ is immediate by the dual of (FFEL1). For the inductive case we have that $f^n(a \mathbin{{}_{\bullet}\!\wedge} P^{\mathsf{F}}) = a \mathbin{{}^{\bullet}\!\vee} f^n(P^{\mathsf{F}})$, where we may assume that $f^n(P^{\mathsf{F}})$ is a T-term and $\mathrm{EqFFEL} \vdash f^n(P^{\mathsf{F}}) = \neg P^{\mathsf{F}}$. It follows immediately from the induction hypothesis that $f^n(a \mathbin{{}_{\bullet}\!\wedge} P^{\mathsf{F}})$ is a T-term. Furthermore, noting that by the induction hypothesis we may assume that $f^n(P^{\mathsf{F}})$ is a T-term, we prove the remaining claim as follows:

$$\begin{aligned}
f^n(a \mathbin{{}_{\bullet}\!\wedge} P^{\mathsf{F}}) &= a \mathbin{{}^{\bullet}\!\vee} f^n(P^{\mathsf{F}}) && \text{by definition}\\
&= a \mathbin{{}^{\bullet}\!\vee} (f^n(P^{\mathsf{F}}) \mathbin{{}^{\bullet}\!\vee} \mathsf{T}) && \text{by Lemma A.1}\\
&= \neg a \mathbin{{}^{\bullet}\!\vee} (f^n(P^{\mathsf{F}}) \mathbin{{}^{\bullet}\!\vee} \mathsf{T}) && \text{by the dual of Lemma 2.7.1}\\
&= \neg a \mathbin{{}^{\bullet}\!\vee} f^n(P^{\mathsf{F}}) && \text{by Lemma A.1}\\
&= \neg a \mathbin{{}^{\bullet}\!\vee} \neg P^{\mathsf{F}} && \text{by induction hypothesis}\\
&= \neg(a \mathbin{{}_{\bullet}\!\wedge} P^{\mathsf{F}}). && \text{by (FFEL3) and (FFEL2)}
\end{aligned}$$

To prove the lemma for T-$*$-terms we first verify that the auxiliary function $f_1^n$ returns a $*$-term and that for any $*$-term $P$, $\mathrm{EqFFEL} \vdash f_1^n(P) = \neg P$. We show this by induction on the number of $\ell$-terms in $P$. For the base cases, i.e., for $\ell$-terms, it is immediate that $f_1^n(P)$ is a $*$-term. If $P$ is an $\ell$-term with a positive determinative atom we have:

$$\begin{aligned}
f_1^n(a \mathbin{{}_{\bullet}\!\wedge} P^{\mathsf{T}}) &= \neg a \mathbin{{}_{\bullet}\!\wedge} P^{\mathsf{T}} && \text{by definition}\\
&= \neg a \mathbin{{}_{\bullet}\!\wedge} (P^{\mathsf{T}} \mathbin{{}^{\bullet}\!\vee} \mathsf{T}) && \text{by Lemma A.1}\\
&= \neg(a \mathbin{{}_{\bullet}\!\wedge} (P^{\mathsf{T}} \mathbin{{}^{\bullet}\!\vee} \mathsf{T})) && \text{by Lemma A.2.2}\\
&= \neg(a \mathbin{{}_{\bullet}\!\wedge} P^{\mathsf{T}}). && \text{by Lemma A.1}
\end{aligned}$$

If $P$ is an $\ell$-term with a negative determinative atom the proof proceeds the same, substituting $\neg a$ for $a$ and applying (FFEL3) where needed. For the inductive step we assume that the result holds for $*$-terms with fewer $\ell$-terms than $P^* \mathbin{{}_{\bullet}\!\wedge} Q^d$ and $P^* \mathbin{{}^{\bullet}\!\vee} Q^c$. We note that each application of $f_1^n$ changes the main connective (not occurring inside an $\ell$-term) and hence the result is a $*$-term. Derivable equality is, given the induction hypothesis, an instance of (the dual of) (FFEL2).

With this result we can now see that $f^n(P^{\mathsf{T}} \mathbin{{}_{\bullet}\!\wedge} Q^*)$ is indeed a T-$*$-term. Furthermore we find that:

$$\begin{aligned}
f^n(P^{\mathsf{T}} \mathbin{{}_{\bullet}\!\wedge} Q^*) &= P^{\mathsf{T}} \mathbin{{}_{\bullet}\!\wedge} f_1^n(Q^*) && \text{by definition}\\
&= P^{\mathsf{T}} \mathbin{{}_{\bullet}\!\wedge} \neg Q^* && \text{as shown above}\\
&= (P^{\mathsf{T}} \mathbin{{}^{\bullet}\!\vee} \mathsf{T}) \mathbin{{}_{\bullet}\!\wedge} \neg Q^* && \text{by Lemma A.1}\\
&= \neg(P^{\mathsf{T}} \mathbin{{}^{\bullet}\!\vee} \mathsf{T}) \mathbin{{}^{\bullet}\!\vee} \neg Q^* && \text{by Lemma 2.7.2}\\
&= \neg P^{\mathsf{T}} \mathbin{{}^{\bullet}\!\vee} \neg Q^* && \text{by Lemma A.1}\\
&= \neg(P^{\mathsf{T}} \mathbin{{}_{\bullet}\!\wedge} Q^*). && \text{by (FFEL2) and (FFEL3)}
\end{aligned}$$

Hence for all $P \in \mathrm{FNF}$, $\mathrm{EqFFEL} \vdash f^n(P) = \neg P$. ∎

**Lemma A.4:** *For any T-term $P$ and $Q \in \mathrm{FNF}$, $f^c(P, Q)$ has the same grammatical category as $Q$ and*

$$\mathrm{EqFFEL} \vdash f^c(P, Q) = P \mathbin{{}_{\bullet}\!\wedge} Q.$$

***Proof:*** By induction on the complexity of the first argument. In the base case we see that $f^c(\mathsf{T}, P) = P$ and hence has the same grammatical category as $P$. Derivable equality follows from (FFEL5).

For the induction step we make a case distinction on the grammatical category of the second argument. If the second argument is a T-term we have that $f^c(a \mathbin{{}^{\bullet}\!\!\vee} P^{\mathrm{T}}, Q^{\mathrm{T}}) = a \mathbin{{}^{\bullet}\!\!\vee} f^c(P^{\mathrm{T}}, Q^{\mathrm{T}})$, where we assume that $f^c(P^{\mathrm{T}}, Q^{\mathrm{T}})$ is a T-term and $\mathrm{EqFFEL} \vdash f^c(P^{\mathrm{T}}, Q^{\mathrm{T}}) = P^{\mathrm{T}} \mathbin{{}_{\bullet}\!\!\wedge} Q^{\mathrm{T}}$. The grammatical claim follows immediately from the induction hypothesis. The claim about derivable equality is proved as follows:

$$\begin{aligned} f^c(a \mathbin{{}^{\bullet}\!\!\vee} P^{\mathrm{T}}, Q^{\mathrm{T}}) &= a \mathbin{{}^{\bullet}\!\!\vee} f^c(P^{\mathrm{T}}, Q^{\mathrm{T}}) && \text{by definition} \\ &= a \mathbin{{}^{\bullet}\!\!\vee} (P^{\mathrm{T}} \mathbin{{}_{\bullet}\!\!\wedge} Q^{\mathrm{T}}) && \text{by induction hypothesis} \\ &= a \mathbin{{}^{\bullet}\!\!\vee} (P^{\mathrm{T}} \mathbin{{}_{\bullet}\!\!\wedge} (Q^{\mathrm{T}} \mathbin{{}^{\bullet}\!\!\vee} \mathrm{T})) && \text{by Lemma A.1} \\ &= (a \mathbin{{}^{\bullet}\!\!\vee} P^{\mathrm{T}}) \mathbin{{}_{\bullet}\!\!\wedge} (Q^{\mathrm{T}} \mathbin{{}^{\bullet}\!\!\vee} \mathrm{T}) && \text{by Lemma 2.7.2} \\ &= (a \mathbin{{}^{\bullet}\!\!\vee} P^{\mathrm{T}}) \mathbin{{}_{\bullet}\!\!\wedge} Q^{\mathrm{T}}. && \text{by Lemma A.1} \end{aligned}$$

If the second argument is an F-term we assume that $f^c(P^{\mathrm{T}}, Q^{\mathrm{F}})$ is an F-term and that $\mathrm{EqFFEL} \vdash f^c(P^{\mathrm{T}}, Q^{\mathrm{F}}) = P^{\mathrm{T}} \mathbin{{}_{\bullet}\!\!\wedge} Q^{\mathrm{F}}$. The grammatical claim follows immediately from the induction hypothesis. Derivable equality is proved as follows:

$$\begin{aligned} f^c(a \mathbin{{}^{\bullet}\!\!\vee} P^{\mathrm{T}}, Q^{\mathrm{F}}) &= a \mathbin{{}_{\bullet}\!\!\wedge} f^c(P^{\mathrm{T}}, Q^{\mathrm{F}}) && \text{by definition} \\ &= a \mathbin{{}_{\bullet}\!\!\wedge} (P^{\mathrm{T}} \mathbin{{}_{\bullet}\!\!\wedge} Q^{\mathrm{F}}) && \text{by induction hypothesis} \\ &= a \mathbin{{}_{\bullet}\!\!\wedge} (P^{\mathrm{T}} \mathbin{{}_{\bullet}\!\!\wedge} (Q^{\mathrm{F}} \mathbin{{}_{\bullet}\!\!\wedge} \mathrm{F})) && \text{by Lemma A.1} \\ &= (a \mathbin{{}^{\bullet}\!\!\vee} P^{\mathrm{T}}) \mathbin{{}_{\bullet}\!\!\wedge} (Q^{\mathrm{F}} \mathbin{{}_{\bullet}\!\!\wedge} \mathrm{F}) && \text{by Lemma A.2.1} \\ &= (a \mathbin{{}^{\bullet}\!\!\vee} P^{\mathrm{T}}) \mathbin{{}_{\bullet}\!\!\wedge} Q^{\mathrm{F}}. && \text{by Lemma A.1} \end{aligned}$$

Finally, if the second argument is a T-$*$-term then $f^c(a \mathbin{{}^{\bullet}\!\!\vee} P^{\mathrm{T}}, Q^{\mathrm{T}} \mathbin{{}_{\bullet}\!\!\wedge} R^*) = f^c(a \mathbin{{}^{\bullet}\!\!\vee} P^{\mathrm{T}}, Q^{\mathrm{T}}) \mathbin{{}_{\bullet}\!\!\wedge} R^*$. The fact that this is a T-$*$-term follows from the fact that $f^c(a \mathbin{{}^{\bullet}\!\!\vee} P^{\mathrm{T}}, Q^{\mathrm{T}})$ is a T-term as was shown above. Derivable equality follows from the case where the second argument is a T-term and (FFEL4). ■

**Lemma A.5:** *For any T-$*$-term P and F-term Q, $f^c(P, Q)$ is an F-term and*

$$\mathrm{EqFFEL} \vdash f^c(P, Q) = P \mathbin{{}_{\bullet}\!\!\wedge} Q.$$

***Proof:*** By (FFEL4) and Lemma A.4 it suffices to show that $f_2^c(P^*, Q^{\mathrm{F}})$ is an F-term and that $\mathrm{EqFFEL} \vdash f_2^c(P^*, Q^{\mathrm{F}}) = P^* \mathbin{{}_{\bullet}\!\!\wedge} Q^{\mathrm{F}}$. We prove this by induction on the number of $\ell$-terms in $P^*$. In the base cases, i.e., $\ell$-terms, the grammatical claims follow from Lemma A.4. The claim about derivable equality in the case of $\ell$-terms with positive determinative atoms follows from Lemma A.4 and (FFEL4). For $\ell$-terms with negative determinative atoms it follows from Lemmas A.4, A.1, (FFEL7), (FFEL4) and (FFEL8).

For the induction step we assume the claims hold for any $*$-terms with fewer $\ell$-terms than $P^* \mathbin{{}_{\bullet}\!\!\wedge} Q^d$ and $P^* \mathbin{{}^{\bullet}\!\!\vee} Q^c$. In the case of conjunctions we have $f_2^c(P^* \mathbin{{}_{\bullet}\!\!\wedge} Q^d, R^{\mathrm{F}}) = f_2^c(P^*, f_2^c(Q^d, R^{\mathrm{F}}))$ and the grammatical claim follows from the induction hypothesis (applied twice). Derivable equality follows from the induction hypothesis and (FFEL4).

For disjunctions we have $f_2^c(P^* \mathbin{{}^{\bullet}\!\!\vee} Q^c, R^{\mathrm{F}}) = f_2^c(P^*, f_2^c(Q^c, R^{\mathrm{F}}))$ and the grammatical claim follows from the induction hypothesis (applied twice). The claim about derivable equality is proved as follows:

$$\begin{aligned} f_2^c(P^* \mathbin{{}^{\bullet}\!\!\vee} Q^c, R^{\mathrm{F}}) &= f_2^c(P^*, f_2^c(Q^c, R^{\mathrm{F}})) && \text{by definition} \\ &= P^* \mathbin{{}_{\bullet}\!\!\wedge} (Q^c \mathbin{{}_{\bullet}\!\!\wedge} R^{\mathrm{F}}) && \text{by induction hypothesis} \\ &= P^* \mathbin{{}_{\bullet}\!\!\wedge} (Q^c \mathbin{{}_{\bullet}\!\!\wedge} (R^{\mathrm{F}} \mathbin{{}_{\bullet}\!\!\wedge} \mathrm{F})) && \text{by Lemma A.1} \end{aligned}$$

$$= (P^* \mathbin{{}^{\bullet}\!\!\vee} Q^c) \mathbin{{}^{\bullet}\!\!\wedge} (R^{\mathsf{F}} \mathbin{{}^{\bullet}\!\!\wedge} \mathsf{F}) \quad \text{by Lemma A.2.1}$$

$$= (P^* \mathbin{{}^{\bullet}\!\!\vee} Q^c) \mathbin{{}^{\bullet}\!\!\wedge} R^{\mathsf{F}}. \quad \text{by Lemma A.1}$$

■

**Lemma A.6:** *For any F-term P and* $Q \in \mathrm{FNF}$, $f^c(P, Q)$ *is an F-term and*

$$\mathrm{EqFFEL} \vdash f^c(P, Q) = P \mathbin{{}^{\bullet}\!\!\wedge} Q.$$

***Proof:*** We make a case distinction on the grammatical category of the second argument. If the second argument is a T-term we proceed by induction on the first argument. In the base case we have $f^c(\mathsf{F}, P^{\mathsf{T}}) = f^n(P^{\mathsf{T}})$ and the result is by Lemmas A.3, A.1, (FFEL7) and (FFEL8). In the inductive case we have $f^c(a \mathbin{{}^{\bullet}\!\!\wedge} P^{\mathsf{F}}, Q^{\mathsf{T}}) = a \mathbin{{}^{\bullet}\!\!\wedge} f^c(P^{\mathsf{F}}, Q^{\mathsf{T}})$, where we assume that $f^c(P^{\mathsf{F}}, Q^{\mathsf{T}})$ is an F-term and $\mathrm{EqFFEL} \vdash f^c(P^{\mathsf{F}}, Q^{\mathsf{T}}) = P^{\mathsf{F}} \mathbin{{}^{\bullet}\!\!\wedge} Q^{\mathsf{T}}$. The result now follows from the induction hypothesis and (FFEL4).

If the second argument is an F-term the proof is almost the same, except that we need not invoke Lemma A.3 or (FFEL8) in the base case.

Finally, if the second argument is a T-$*$-term we again proceed by induction on the first argument. In the base case we have $f^c(\mathsf{F}, P^{\mathsf{T}} \mathbin{{}^{\bullet}\!\!\wedge} Q^*) = f^c(P^{\mathsf{T}} \mathbin{{}^{\bullet}\!\!\wedge} Q^*, \mathsf{F})$. The grammatical claim now follows from Lemma A.5 and derivable equality follows from Lemma A.5 and (FFEL7). For for the inductive case the results follow from the induction hypothesis and (FFEL4). ■

**Lemma A.7:** *For any T-$*$-term P and T-term Q,* $f^c(P, Q)$ *has the same grammatical category as P and*

$$\mathrm{EqFFEL} \vdash f^c(P, Q) = P \mathbin{{}^{\bullet}\!\!\wedge} Q.$$

***Proof:*** By (FFEL4) it suffices to prove the claims for $f^c_1$, i.e., that $f^c_1(P^*, Q^{\mathsf{T}})$ has the same grammatical category as $P^*$ and that $\mathrm{EqFFEL} \vdash f^c_1(P^*, Q^{\mathsf{T}}) = P^* \mathbin{{}^{\bullet}\!\!\wedge} Q^{\mathsf{T}}$. We prove this by induction on the number of $\ell$-terms in $P^*$. In the base case we deal with $\ell$-terms and the results follow from Lemma A.4 and (FFEL4).

For the inductive cases we assume that the results hold for any $*$-term with fewer $\ell$-terms than $P^* \mathbin{{}^{\bullet}\!\!\wedge} Q^d$ and $P^* \mathbin{{}^{\bullet}\!\!\vee} Q^c$. In the case of conjunctions the results follow from the induction hypothesis and (FFEL4). In the case of disjunctions the grammatical claim follows from the induction hypothesis. For derivable equality we have:

$$\begin{aligned}
f^c_1(P^* \mathbin{{}^{\bullet}\!\!\vee} Q^c, R^{\mathsf{T}}) &= P^* \mathbin{{}^{\bullet}\!\!\vee} f^c_1(Q^c, R^{\mathsf{T}}) && \text{by definition}\\
&= P^* \mathbin{{}^{\bullet}\!\!\vee} (Q^c \mathbin{{}^{\bullet}\!\!\wedge} R^{\mathsf{T}}) && \text{by induction hypothesis}\\
&= P^* \mathbin{{}^{\bullet}\!\!\vee} (Q^c \mathbin{{}^{\bullet}\!\!\wedge} (R^{\mathsf{T}} \mathbin{{}^{\bullet}\!\!\vee} \mathsf{T})) && \text{by Lemma A.1}\\
&= (P^* \mathbin{{}^{\bullet}\!\!\vee} Q^c) \mathbin{{}^{\bullet}\!\!\wedge} (R^{\mathsf{T}} \mathbin{{}^{\bullet}\!\!\vee} \mathsf{T}) && \text{by Lemma 2.7.eq:a5}\\
&= (P^* \mathbin{{}^{\bullet}\!\!\vee} Q^c) \mathbin{{}^{\bullet}\!\!\wedge} R^{\mathsf{T}}. && \text{by Lemma A.1}
\end{aligned}$$

■

**Lemma A.8:** *For any* $P, Q \in \mathrm{FNF}$, $f^c(P, Q)$ *is in* FNF *and*

$$\mathrm{EqFFEL} \vdash f^c(P, Q) = P \mathbin{{}^{\bullet}\!\!\wedge} Q.$$

***Proof:*** By the four preceding lemmas it suffices to show that $f^c(P^{\mathsf{T}} \mathbin{{}^{\bullet}\!\!\wedge} Q^*, R^{\mathsf{T}} \mathbin{{}^{\bullet}\!\!\wedge} S^*)$ is in FNF and that $\mathrm{EqFFEL} \vdash f^c(P^{\mathsf{T}} \mathbin{{}^{\bullet}\!\!\wedge} Q^*, R^{\mathsf{T}} \mathbin{{}^{\bullet}\!\!\wedge} S^*) = (P^{\mathsf{T}} \mathbin{{}^{\bullet}\!\!\wedge} Q^*) \mathbin{{}^{\bullet}\!\!\wedge} (R^{\mathsf{T}} \mathbin{{}^{\bullet}\!\!\wedge} S^*)$. By (FFEL4), in turn, it suffices to prove that $f^c_3(P^*, Q^{\mathsf{T}} \mathbin{{}^{\bullet}\!\!\wedge} R^*)$ is a $*$-term and that $\mathrm{EqFFEL} \vdash f^c_3(P^*, Q^{\mathsf{T}} \mathbin{{}^{\bullet}\!\!\wedge} R^*) = P^* \mathbin{{}^{\bullet}\!\!\wedge} (Q^{\mathsf{T}} \mathbin{{}^{\bullet}\!\!\wedge} R^*)$. We prove this by induction on the number of $\ell$-terms in $R^*$. In the base case we have that $f^c_3(P^*, Q^{\mathsf{T}} \mathbin{{}^{\bullet}\!\!\wedge} R^{\ell}) = f^c_1(P^*, Q^{\mathsf{T}}) \mathbin{{}^{\bullet}\!\!\wedge} R^{\ell}$. The results follow from Lemma A.7 and (FFEL4).

For the inductive cases we assume that the results hold for all $*$-terms with fewer $\ell$-terms than $R^* \mathbin{{}_{\bullet}\!\!\wedge} S^d$ and $R^* \mathbin{{}^{\bullet}\!\!\vee} S^c$. For conjunctions the result follows from the induction hypothesis and (FFEL4) and for disjunctions it follows from Lemma A.7 and (FFEL4). ■

**Theorem 2.9:** *For any* $P \in \mathcal{SP}_A$, $f(P)$ *terminates,* $f(P) \in \mathrm{FNF}$ *and* $\mathrm{EqFFEL} \vdash f(P) = P$.

***Proof:*** By induction on the complexity of $P$. If $P$ is an atom, the result is by (FFEL5) and (FFEL6). If $P$ is T or F the result is by identity. For the induction we assume that the result holds for all FEL-terms of lesser complexity than $P \mathbin{{}_{\bullet}\!\!\wedge} Q$ and $P \mathbin{{}^{\bullet}\!\!\vee} Q$. The result now follows from the induction hypothesis, Lemmas A.3, A.8 and (FFEL2). ■

*Correctness of* ***g***.

**Theorem 2.18:** *For all* $P \in \mathrm{FNF}$, $g(fe(P)) = P$, *i.e.,* $g(fe(P))$ *is syntactically equal to* $P$ *for* $P \in \mathrm{FNF}$.

***Proof:*** We first prove that for all T-terms $P$, $g^{\mathrm{T}}(fe(P)) = P$, by induction on $P$. In the base case $P = \mathrm{T}$ and we have $g^{\mathrm{T}}(fe(P)) = g^{\mathrm{T}}(\mathrm{T}) = \mathrm{T} = P$. For the inductive case we have $P = a \mathbin{{}^{\bullet}\!\!\vee} Q^{\mathrm{T}}$ and

$$\begin{aligned} g^{\mathrm{T}}(fe(P)) &= g^{\mathrm{T}}(fe(Q^{\mathrm{T}}) \trianglelefteq a \trianglerighteq fe(Q^{\mathrm{T}})) && \text{by definition of } fe \\ &= a \mathbin{{}^{\bullet}\!\!\vee} g^{\mathrm{T}}(fe(Q^{\mathrm{T}})) && \text{by definition of } g^{\mathrm{T}} \\ &= a \mathbin{{}^{\bullet}\!\!\vee} Q^{\mathrm{T}} && \text{by induction hypothesis} \\ &= P. \end{aligned}$$

Similarly, we see that for all F-terms $P$, $g^{\mathrm{F}}(fe(P)) = P$, by induction on $P$. In the base case $P = \mathrm{F}$ and we have $g^{\mathrm{F}}(fe(P)) = g^{\mathrm{F}}(\mathrm{F}) = \mathrm{F} = P$. For the inductive case we have $P = a \mathbin{{}_{\bullet}\!\!\wedge} Q^{\mathrm{F}}$ and

$$\begin{aligned} g^{\mathrm{F}}(fe(P)) &= g^{\mathrm{F}}(fe(Q^{\mathrm{F}}) \trianglelefteq a \trianglerighteq fe(Q^{\mathrm{F}})) && \text{by definition of } fe \\ &= a \mathbin{{}_{\bullet}\!\!\wedge} g^{\mathrm{F}}(fe(Q^{\mathrm{F}})) && \text{by definition of } g^{\mathrm{F}} \\ &= a \mathbin{{}_{\bullet}\!\!\wedge} Q^{\mathrm{F}} && \text{by induction hypothesis} \\ &= P. \end{aligned}$$

Now we check that for all $\ell$-terms $P$, $g^{\ell}(fe(P)) = P$. We observe that either $P = a \mathbin{{}_{\bullet}\!\!\wedge} Q^{\mathrm{T}}$ or $P = \neg a \mathbin{{}_{\bullet}\!\!\wedge} Q^{\mathrm{T}}$. In the first case we have

$$\begin{aligned} g^{\ell}(fe(P)) &= g^{\ell}(fe(Q^{\mathrm{T}}) \trianglelefteq a \trianglerighteq fe(Q^{\mathrm{T}})[\mathrm{T} \mapsto \mathrm{F}]) && \text{by definition of } fe \\ &= a \mathbin{{}_{\bullet}\!\!\wedge} g^{\mathrm{T}}(fe(Q^{\mathrm{T}})) && \text{by definition of } g^{\ell} \\ &= a \mathbin{{}_{\bullet}\!\!\wedge} Q^{\mathrm{T}} && \text{as shown above} \\ &= P. \end{aligned}$$

In the second case we have that

$$\begin{aligned} g^{\ell}(fe(P)) &= g^{\ell}(fe(Q^{\mathrm{T}})[\mathrm{T} \mapsto \mathrm{F}] \trianglelefteq a \trianglerighteq fe(Q^{\mathrm{T}})) && \text{by definition of } fe \\ &= \neg a \mathbin{{}_{\bullet}\!\!\wedge} g^{\mathrm{T}}(fe(Q^{\mathrm{T}})) && \text{by definition of } g_{\ell} \\ &= \neg a \mathbin{{}_{\bullet}\!\!\wedge} Q^{\mathrm{T}} && \text{as shown above} \\ &= P. \end{aligned}$$

We now prove that for all $*$-terms $P$, $g^*(fe(P)) = P$, by induction on $P$ modulo the complexity of $\ell$-terms. In the base case we are dealing with $\ell$-terms. Because an $\ell$-term has neither a cd nor a dd we have $g^*(fe(P)) = g^{\ell}(fe(P)) = P$, where the first equality is by definition of $g^*$ and the second was shown above. For the induction we have either $P = Q \leftand R$ or $P = Q \leftor R$. In the first case note that by Theorem 2.15, $fe(P)$ has a cd and no dd. So we have

$$\begin{aligned} g^*(fe(P)) &= g^*(\mathrm{cd}_1(fe(P))[\triangle_1 \mapsto \mathsf{T}, \triangle_2 \mapsto \mathsf{F}]) \leftand g^*(\mathrm{cd}_2(fe(P))) && \text{by definition of } g^* \\ &= g^*(fe(Q)) \leftand g^*(fe(R)) && \text{by Theorem 2.15} \\ &= Q \leftand R && \text{by induction hypothesis} \\ &= P. \end{aligned}$$

In the second case, again by Theorem 2.15, $fe(P)$ has a dd and no cd. So we have that

$$\begin{aligned} g^*(fe(P)) &= g^*(\mathrm{dd}_1(fe(P))[\triangle_1 \mapsto \mathsf{T}, \triangle_2 \mapsto \mathsf{F}]) \leftor g^*(\mathrm{dd}_2(fe(P))) && \text{by definition of } g^* \\ &= g^*(fe(Q)) \leftor g^*(fe(R)) && \text{by Theorem 2.15} \\ &= Q \leftor R && \text{by induction hypothesis} \\ &= P. \end{aligned}$$

Finally, we prove the theorem's statement by making a case distinction on the grammatical category of $P$. If $P$ is a T-term, then $fe(P)$ has only T-leaves and hence $g(fe(P)) = g^{\mathsf{T}}(fe(P)) = P$, where the first equality is by definition of $g$ and the second was shown above. If $P$ is an F-term, then $fe(P)$ has only F-leaves and hence $g(fe(P)) = g^{\mathsf{F}}(fe(P)) = P$, where the first equality is by definition of $g$ and the second was shown above. If $P$ is a T-$*$-term, then it has both T and F-leaves and hence, letting $P = Q \leftand R$,

$$\begin{aligned} g(fe(P)) &= g^{\mathsf{T}}(\mathrm{tsd}_1(fe(P))[\triangle \mapsto \mathsf{T}]) \leftand g^*(\mathrm{tsd}_2(fe(P))) && \text{by definition of } g \\ &= g^{\mathsf{T}}(fe(Q)) \leftand g^*(fe(R)) && \text{by Theorem 2.17} \\ &= Q \leftand R && \text{as shown above} \\ &= P, \end{aligned}$$

which completes the proof. ∎

## Appendix 2. Proofs - Section 4

In order to show that the relation $=_{mfe}$ is indeed a congruence (Lemma 4.4), we first repeat a part of Bergstra et al. (2021, La.3.3), except that we generalise its clause 5 to clause 2 below.[9]

**Lemma A.9:** *For all $a \in A$, $f \in \{L, R\}$ and $X, Y, Z \in \mathcal{T}_A$,*

(1) $m(f_a(X[T \mapsto F, F \mapsto T])) = m(f_a(X))[T \mapsto F, F \mapsto T]$,
(2) $f_a(X[\mathsf{T} \mapsto Y, \mathsf{F} \mapsto Z]) = f_a(X)[\mathsf{T} \mapsto f_a(Y), \mathsf{F} \mapsto f_a(Z)]$.

***Proof:*** Clause 1 is Lemma 3.3.4 in Bergstra et al. (2021). We prove clause 2 by induction on the structure of $X$. The base cases are trivial.

If $X = X_1 \trianglelefteq b \trianglerighteq X_2$, distinguish the cases $b = a$ and $b \neq a$:

Case $b = a$, subcase $f = L$: $L_a(X[\mathsf{T} \mapsto Y, \mathsf{F} \mapsto Z]) = L_a(X_1[\mathsf{T} \mapsto Y, \mathsf{F} \mapsto Z])$ and by induction, $L_a(X_1[\mathsf{T} \mapsto Y, \mathsf{F} \mapsto Z]) = L_a(X_1)[\mathsf{T} \mapsto L_a(Y), \mathsf{F} \mapsto L_a(Z)]$. By $L_a(X) = L_a(X_1)$, we are done. The other subcase $f = R$ follows in a similar way.

Case $b \neq a$, subcase $f = L$:

$$\begin{aligned} &L_a(X[\mathsf{T} \mapsto Y, \mathsf{F} \mapsto Z]) \\ &\quad = L_a(X_1[\mathsf{T} \mapsto Y, \mathsf{F} \mapsto Z]) \trianglelefteq b \trianglerighteq L_a(X_2[\mathsf{T} \mapsto Y, \mathsf{F} \mapsto Z]) \\ &\quad \overset{\mathrm{IH}}{=} L_a(X_1)[\mathsf{T} \mapsto L_a(Y), \mathsf{F} \mapsto L_a(Z)] \trianglelefteq b \trianglerighteq L_a(X_2)[\mathsf{T} \mapsto L_a(Y), \mathsf{F} \mapsto L_a(Z)] \end{aligned}$$

$$= L_a(X)[\mathsf{T} \mapsto L_a(Y), \mathsf{F} \mapsto L_a(Z)].$$

The other subcase $f = R$ follows in a similar way. ■

**Lemma 4.4:** *The relation $=_{mfe}$ is a congruence.*

***Proof:*** Define $\widetilde{\neg} X : \mathcal{T}_A \to \mathcal{T}_A$ and $X \mathbin{\widetilde{\leftand}} Y, X \mathbin{\widetilde{\leftor}} Y : \mathcal{T}_A \times \mathcal{T}_A \to \mathcal{T}_A$ by

$$\widetilde{\neg} X = X[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}],$$
$$X \mathbin{\widetilde{\leftand}} Y = X[\mathsf{T} \mapsto Y, \mathsf{F} \mapsto Y[\mathsf{T} \mapsto \mathsf{F}]],$$
$$X \mathbin{\widetilde{\leftor}} Y = X[\mathsf{T} \mapsto Y[\mathsf{F} \mapsto \mathsf{T}], \mathsf{F} \mapsto Y].$$

Hence, $\widetilde{\neg}(fe(P)) = fe(\neg P), fe(P) \mathbin{\widetilde{\leftand}} (Q) = fe(P \leftand Q)$, and $fe(P) \mathbin{\widetilde{\leftor}} fe(Q) = fe(P \leftor Q)$. It suffices to show that for $X, Y \in \mathcal{T}_A$, if $m(X) = m(X')$ and $m(Y) = m(Y')$, then $m(\widetilde{\neg} X) = m(\widetilde{\neg} X')$, $m(X \mathbin{\widetilde{\leftand}} Y) = m(X' \mathbin{\widetilde{\leftand}} Y')$, and $m(X \mathbin{\widetilde{\leftor}} Y) = m(X' \mathbin{\widetilde{\leftor}} Y')$.

The case for $\widetilde{\neg} X$ follows by case distinction on the form of $X$. The base cases $X \in \{\mathsf{T}, \mathsf{F}\}$ are trivial, and if $X = X_1 \trianglelefteq a \trianglerighteq X_2$, then $m(X) = m(X')$ implies that $X' = X_1' \trianglelefteq a \trianglerighteq X_2'$ for some $X_1', X_2' \in \mathcal{T}_A$, and

$$m(L_a(X_1)) = m(L_a(X_1')) \quad \text{and} \quad m(R_a(X_2)) = m(R_a(X_2')). \tag{Aux3}$$

Write **[*neg*]** for the leaf replacement $[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}]$ and derive

$$\begin{aligned} m(\widetilde{\neg} X) &= m((X_1 \trianglelefteq a \trianglerighteq X_2)[\boldsymbol{neg}]) \\ &= m(L_a(X_1[\boldsymbol{neg}])) \trianglelefteq a \trianglerighteq m(R_a(X_2[\boldsymbol{neg}])) \\ &= m(L_a(X_1'[\boldsymbol{neg}])) \trianglelefteq a \trianglerighteq m(R_a(X_2'[\boldsymbol{neg}])) \quad \text{by La.A.9.1 and (Aux3)} \\ &= m(\widetilde{\neg} X'). \end{aligned}$$

The case for $m(X \mathbin{\widetilde{\leftand}} Y) = m(X' \leftand Y')$: for readability we split the proof obligation into two parts.

(A) $m(X \mathbin{\widetilde{\leftand}} Y) = m(X' \mathbin{\widetilde{\leftand}} Y)$. This follows by induction on the depth of $X$. The base cases $X \in \{\mathsf{T}, \mathsf{F}\}$ are simple: note that if $X = \mathsf{T}$ and $m(X) = m(X')$, then $X' = \mathsf{T}$ and we are done, and similarly if $X = \mathsf{F}$.

If $X = X_1 \trianglelefteq a \trianglerighteq X_2$ and $m(X) = m(X')$, then by (Aux3), $X' = X_1' \trianglelefteq a \trianglerighteq X_2'$ for some $X_1', X_2' \in \mathcal{T}_A$ with $m(L_a(X_1)) = m(L_a(X_1'))$ and $m(R_a(X_2)) = m(R_a(X_2'))$. Write

$$\begin{aligned} &[\boldsymbol{fulland\ Y}] \text{ for } [\mathsf{T} \mapsto Y, \mathsf{F} \mapsto Y[\mathsf{T} \mapsto \mathsf{F}]], \\ &[\boldsymbol{fulland\ L_a(Y)}] \text{ for } [\mathsf{T} \mapsto L_a(Y), \mathsf{F} \mapsto L_a(Y)[\mathsf{T} \mapsto \mathsf{F}]], \\ &[\boldsymbol{fulland\ R_a(Y)}] \text{ for } [\mathsf{T} \mapsto R_a(Y), \mathsf{F} \mapsto R_a(Y)[\mathsf{T} \mapsto \mathsf{F}]]. \end{aligned}$$

Derive

$$\begin{aligned} & m(X \mathbin{\widetilde{\leftand}} Y) \\ &= m((X_1 \trianglelefteq a \trianglerighteq X_2) \mathbin{\widetilde{\leftand}} Y) \\ &= m(X_1[\boldsymbol{fulland\ Y}] \trianglelefteq a \trianglerighteq X_2[\boldsymbol{fulland\ Y}]) \\ &= m(L_a(X_1[\boldsymbol{fulland\ Y}])) \trianglelefteq a \trianglerighteq m(R_a(X_2[\boldsymbol{fulland\ Y}])) \\ &= m(L_a(X_1)[\boldsymbol{fulland\ L_a(Y)}]) \trianglelefteq a \trianglerighteq m(R_a(X_2)[\boldsymbol{fulland\ R_a(Y)}]) \quad \text{by La.A.9.2} \\ &= m(L_a(X_1) \mathbin{\widetilde{\leftand}} L_a(Y)) \trianglelefteq a \trianglerighteq m(R_a(X_2) \mathbin{\widetilde{\leftand}} R_a(Y)) \\ &= m(L_a(X_1') \mathbin{\widetilde{\leftand}} L_a(Y)) \trianglelefteq a \trianglerighteq m(R_a(X_2') \mathbin{\widetilde{\leftand}} R_a(Y)) \quad \text{by IH and (Aux3)} \\ &= \cdots = m(X' \mathbin{\widetilde{\leftand}} Y). \end{aligned}$$

(B) $m(X \mathbin{\widetilde{{}_\bullet\!\wedge}} Y) = m(X \mathbin{\widetilde{{}_\bullet\!\wedge}} Y')$. This follows by induction on the depth of $X$. The base cases $X \in \{\mathsf{T},\mathsf{F}\}$ are trivial. If $X = X_1 \trianglelefteq a \trianglerighteq X_2$ derive

$$\begin{aligned}
m(X \mathbin{\widetilde{{}_\bullet\!\wedge}} Y) &= m((X_1 \trianglelefteq a \trianglerighteq X_2) \mathbin{\widetilde{{}_\bullet\!\wedge}} Y)\\
&= m(X_1[\textbf{\textit{fulland Y}}] \trianglelefteq a \trianglerighteq X_2[\textbf{\textit{fulland Y}}])\\
&= m(L_a(X_1[\textbf{\textit{fulland Y}}])) \trianglelefteq a \trianglerighteq m(R_a(X_2[\textbf{\textit{fulland Y}}]))\\
&= m(L_a(X_1)[\textbf{\textit{fulland}}\ \boldsymbol{L_a(Y)}]) \trianglelefteq a \trianglerighteq m(R_a(X_2)[\textbf{\textit{fulland}}\ \boldsymbol{R_a(Y)}]) \quad \text{by La.A.9.2}\\
&= m(L_a(X_1) \mathbin{\widetilde{{}_\bullet\!\wedge}} L_a(Y)) \trianglelefteq a \trianglerighteq m(R_a(X_2) \mathbin{\widetilde{{}_\bullet\!\wedge}} R_a(Y))\\
&= m(L_a(X_1) \mathbin{\widetilde{{}_\bullet\!\wedge}} L_a(Y')) \trianglelefteq a \trianglerighteq m(R_a(X_2) \mathbin{\widetilde{{}_\bullet\!\wedge}} R_a(Y')) \quad \text{by IH}\\
&= \cdots = m(X \mathbin{\widetilde{{}_\bullet\!\wedge}} Y').
\end{aligned}$$

The proof for the case $m(X \mathbin{\widetilde{{}^\bullet\!\vee}} Y) = m(X' \mathbin{\widetilde{{}^\bullet\!\vee}} Y')$ is similar. ■

For the continuation of the proof of Lemma 4.6, we use results about short-circuit connectives and the memorising short-circuit evaluation function $mse(P) = m(se(P))$ defined in Bergstra et al. (2021, available online); for the function $m$ see Definition 4.2 and below we recall the definition of the short-circuit evaluation function $se$. In particular, we use the definability of the connectives $\mathbin{{}_\bullet\!\wedge}$ and $\mathbin{{}^\bullet\!\vee}$ in short-circuit logic (cf. Section 7).

Given a set of atoms $A$, the set $\mathcal{S}_A$ of closed terms with short-circuit connectives is defined by the following grammar ($a \in A$):

$$P ::= \mathsf{T} \mid \mathsf{F} \mid a \mid \neg P \mid P \mathbin{{}_\circ\!\wedge} P \mid P \mathbin{{}^\circ\!\vee} P.$$

For $a \in A$, the evaluation function $se : \mathcal{S}_A \to \mathcal{T}_A$ and the translation function $t : \mathcal{SP}_A \to \mathcal{S}_A$ are defined by

$$\begin{aligned}
&se(\mathsf{T}) = \mathsf{T}, \quad se(\mathsf{F}) = \mathsf{F}, \quad t(\mathsf{T}) = \mathsf{T}, \quad t(\mathsf{F}) = \mathsf{F},\\
&se(a) = \mathsf{T} \trianglelefteq a \trianglerighteq \mathsf{F}, \quad t(a) = a,\\
&se(\neg P) = se(P)[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}], \quad t(\neg P) = \neg(t(P)),\\
&se(P \mathbin{{}_\circ\!\wedge} Q) = se(P)[\mathsf{T} \mapsto se(Q)], \quad t(P \mathbin{{}_\bullet\!\wedge} Q) = (t(P) \mathbin{{}^\circ\!\vee} (t(Q) \mathbin{{}_\circ\!\wedge} \mathsf{F})) \mathbin{{}_\circ\!\wedge} t(Q),\\
&se(P \mathbin{{}^\circ\!\vee} Q) = se(P)[\mathsf{F} \mapsto se(Q)], \quad t(P \mathbin{{}^\bullet\!\vee} Q) = (t(P) \mathbin{{}_\circ\!\wedge} (t(Q) \mathbin{{}^\circ\!\vee} \mathsf{T})) \mathbin{{}^\circ\!\vee} t(Q).
\end{aligned}$$

**Lemma A.10:** *For all $P \in \mathcal{SP}_A$, $fe(P) = se(t(P))$.*

***Proof:*** By structural induction on $P$. The base cases are immediate, and so is $fe(\neg P) = se(t(\neg P))$.

For the case $P = Q \mathbin{{}_\bullet\!\wedge} R$, derive

$$\begin{aligned}
&se(t(P))\\
&\quad = se((t(Q) \mathbin{{}^\circ\!\vee} (t(R) \mathbin{{}_\circ\!\wedge} \mathsf{F})) \mathbin{{}_\circ\!\wedge} t(R))\\
&\quad = se(t(Q) \mathbin{{}^\circ\!\vee} (t(R) \mathbin{{}_\circ\!\wedge} \mathsf{F}))[\mathsf{T} \mapsto se(t(R))]\\
&\quad = (se(t(Q))[\mathsf{F} \mapsto se(t(R) \mathbin{{}_\circ\!\wedge} \mathsf{F})])[\mathsf{T} \mapsto se(t(R))].\\
&\quad = (se(t(Q))[\mathsf{F} \mapsto se(t(R))[\mathsf{T} \mapsto \mathsf{F}]])[\mathsf{T} \mapsto se(t(R))]\\
&\quad = se(t(Q))[\mathsf{T} \mapsto se(t(R)),\ \mathsf{F} \mapsto se(t(R))[\mathsf{T} \mapsto \mathsf{F}][\mathsf{T} \mapsto se(t(R))]]\\
&\quad = se(t(Q))[\mathsf{T} \mapsto se(t(R)),\ \mathsf{F} \mapsto se(t(R))[\mathsf{T} \mapsto \mathsf{F}]] \quad (*)\\
&\quad = fe(Q)[\mathsf{T} \mapsto fe(R), \mathsf{F} \mapsto fe(R)[\mathsf{T} \mapsto \mathsf{F}]] \quad \text{by IH}\\
&\quad = fe(P).
\end{aligned}$$

$(*)$: Note that $se(t(R))[\mathsf{T} \mapsto \mathsf{F}]$ has no T-leaves.

The case $P = Q \,{}^{\bullet}\!\!\vee\, R$ follows in a similar way. ■

For a variable $v \in \{x, y, z\}$, define $t(v) = v$. It follows with *Prover9* that

$$\text{EqMSCL} \vdash t((x \,{}^{\bullet}\!\!\vee\, y) \,{}_{\bullet}\!\!\wedge\, z) = t((\neg x \,{}_{\bullet}\!\!\wedge\, (y \,{}_{\bullet}\!\!\wedge\, z)) \,{}^{\bullet}\!\!\vee\, (x \,{}_{\bullet}\!\!\wedge\, z)), \tag{B.†}$$

where EqMSCL consists of the axioms (Mem1)–(Mem5) in Table 4, see Comment A.11 below. From Bergstra et al. (2021, Thm.4.1) it follows that for all $P, Q \in \mathcal{S}_A$,

$$\text{EqMFEL} \vdash P = Q \Longrightarrow m(se(P)) = m(se(Q)).$$

Hence, for all $P, Q, R \in \mathcal{SP}_A$,

$$m(se(t((P \,{}^{\bullet}\!\!\vee\, Q) \,{}_{\bullet}\!\!\wedge\, R))) = m(se(t((\neg P \,{}_{\bullet}\!\!\wedge\, (Q \,{}_{\bullet}\!\!\wedge\, R)) \,{}^{\bullet}\!\!\vee\, (P \,{}_{\bullet}\!\!\wedge\, R)))),$$

so, what has to be proved, i.e.,

$$m(fe((P \,{}^{\bullet}\!\!\vee\, Q) \,{}_{\bullet}\!\!\wedge\, R)) = m(fe((\neg P \,{}_{\bullet}\!\!\wedge\, (Q \,{}_{\bullet}\!\!\wedge\, R)) \,{}^{\bullet}\!\!\vee\, (P \,{}_{\bullet}\!\!\wedge\, R))),$$

follows from Lemma A.10.

**Comment A.11:** A proof with *Prover9*'s options `lpo` and `unfold`, and with as assumptions the five axioms of EqMSCL was obtained in 0.02s. The syntax of the proven goal for *Prover9*, that is, the $t()$-translation of (B.†) where `^`, `v`, `'`, `1`, `0` stand for ${}_{\circ}\!\!\wedge$, ${}^{\circ}\!\!\vee$, ¬, T, F, is:

```
(((x ^ (y v 1)) v y) v (z ^ 0)) ^ z
=
(((x' v (((y v (z ^ 0)) ^ z) ^ 0)) ^ ((y v (z ^ 0)) ^ z)) ^ (((x v (z ^ 0)) ^ z) v 1))
v
((x v (z ^ 0)) ^ z).
```

## Appendix 3. *Prover9* on Lemmas 4.7 and 6.7

In the following proof of Lemma 4.7-(C4) by *Prover9* with options `kbo` and `fold` (101 lines, the `goal` is at line 1), and the symbols `^`, `v`, `'`, `1`, `0` stand for ${}_{\bullet}\!\!\wedge$, ${}^{\bullet}\!\!\vee$, ¬, T, F, repectively. The `%%` Comments (lines 2–16) list the names of the MFEL axioms and were added by us.

Moreover, we find in this proof at line 86 a proof of Lemma 4.7-(C1), and at line 87 the difficult part of a proof of Lemma 6.7, which can be completed by a simple (C4)-application: $(x \,{}_{\bullet}\!\!\wedge\, \text{F}) \,{}^{\bullet}\!\!\vee\, (x \,{}_{\bullet}\!\!\wedge\, y) = x \,{}_{\bullet}\!\!\wedge\, (\text{F} \,{}^{\bullet}\!\!\vee\, y) = x \,{}_{\bullet}\!\!\wedge\, y$.

```
============================== PROOF =================================

% -------- Comments from original proof --------
% Proof 1 at 1.97 (+ 0.06) seconds.
% Length of proof is 101.
% Level of proof is 21.
% Maximum clause weight is 23.
% Given clauses 299.

1 x ^ (y v z) = (x ^ y) v (x ^ z) # label(non_clause) # label(goal).  [].
2 0' = 1.  [].                                                    %% (FFEL1)
3 x v y = (x' ^ y')'.  [].                                        %% (FFEL2)
4 (x' ^ y')' = x v y.  [3].
5 x'' = x.  [].                                                   %% (FFEL3)
6 (x ^ y) ^ z = x ^ (y ^ z).  [].                                 %% (FFEL4)
7 1 ^ x = x.  [].                                                 %% (FFEL5)
8 x ^ 1 = x.  [].                                                 %% (FFEL6)
9 x ^ 0 = 0 ^ x.  [].                                             %% (FFEL7)
10 0 ^ x = x ^ 0.  [9].
11 x' ^ 0 = x ^ 0.  [].                                           %% (FFEL8)
12 (x ^ 0) v y = (x v 1) ^ y.  [].                                %% (FFEL9)
13 (x v 1) ^ y = (x ^ 0) v y.  [12].
```

```
14 x v (y ^ 0) = x ^ (y v 1).  [].                              %% (FFEL10)
15 x ^ (y v 1) = x v (y ^ 0).  [14].
16 (x v y) ^ z = (x’ ^ (y ^ z)) v (x ^ z).  [].                 %% (M1)
17 (x’ ^ (y ^ z)) v (x ^ z) = (x v y) ^ z.  [16].
18 (c1 ^ c2) v (c1 ^ c3) != c1 ^ (c2 v c3).  [1].
19 0 v x = x.  [2,4,7,5].
20 x v 0 = x.  [2,4,8,5].
21 1’ = 0.  [2,5].
22 (x ^ y’)’ = x’ v y.  [5,4].
23 (x v y)’ = x’ ^ y’.  [4,5].
24 0 ^ (x ^ y) = x ^ (0 ^ y).  [10,6,6].
25 0 ^ (x ^ y) = x ^ (y ^ 0).  [10,6].
26 x’ ^ (0 ^ y) = x ^ (0 ^ y).  [11,6,6].
27 0 ^ x’ = x ^ 0.  [11,10].
28 ((x ^ 0) v y) ^ z = (x ^ 0) v (y ^ z).  [13,6,13].
29 (0 ^ (x ^ y)) v z = (x ^ (y ^ 0)) v z.  [6,13,13,10].
30 x ^ (y v (z ^ 0)) = (x ^ y) v (z ^ 0).  [15,6,15].
31 (x ^ (y ^ z)) v (x’ ^ z) = (x’ v y) ^ z.  [5,17].
32 (x’ ^ (y ^ (z ^ u))) v (x ^ u) = (x v (y ^ z)) ^ u.  [6,17].
33 ((x ^ y)’ ^ (z ^ u)) v (x ^ (y ^ u)) = ((x ^ y) v z) ^ u.  [6,17].
34 (x’ ^ y) v (x ^ y) = (x ^ 0) v y.  [7,17,13].
35 (x’ ^ y) v x = x v y.  [8,17,8,8].
36 0 ^ (x v y’) = 0 ^ (x v y).  [11,17,17,10,10].
37 (x ^ (y ^ 0)) v (x ^ 0) = 0 ^ (x’ v y).  [11,17,5,10].
38 (x ^ (0 ^ y)) v (x ^ y) = x ^ y.  [20,17,26].
39 0 ^ x’ = 0 ^ x.  [27,10].
40 (0 ^ x)’ = 1 v x.  [2,22,39].
41 (x ^ y)’ = x’ v y’.  [5,22].
42 x’ v 1 = 1 v x.  [10,22,11,41,2,2].
43 (x’ v y) v z = x’ v (y v z).  [22,22,6,41,41,5,5].
44 1 v x’ = 1 v x.  [40,41,2].
45 ((x’ v y’) ^ (z ^ u)) v (x ^ (y ^ u)) = ((x ^ y) v z) ^ u.  [33,41].
46 0 ^ (x’ ^ y) = 0 ^ (x ^ y).  [39,6,6].
47 (x ^ (0 ^ y)) v (x ^ y’) = x ^ y’.  [39,17,26,20].
48 1 v 1 = 1.  [2,42,20].
49 1 v x = x v 1.  [5,42,44].
50 (0 ^ x) v x = x.  [48,17,21,7,7,7].
51 x’ v 1 = x v 1.  [49,42].
52 (x ^ 0) v x = x.  [10,50].
53 (0 ^ x) v (y ^ 0) = x ^ (y ^ 0).  [15,24,30,15,19].
54 0 ^ (x ^ (y ^ x)) = 0 ^ (x v y).  [25,17,46,53,25,6,10].
55 x’ ^ (y ^ 0) = x ^ (0 ^ y).  [10,26].
56 (x ^ y) v x’ = x’ v y.  [5,35].
57 x’ v x = x v 1.  [8,35].
58 x’ v x = 1 v x.  [49,35,8].
59 x v x’ = x v 1.  [51,35,5,8].
60 ((0 ^ x) v y) ^ z = (x ^ 0) v (y ^ z).  [10,28].
61 (0 ^ (x’ v y)) v (x ^ y) = x ^ y.  [35,28,41,10,36,60,10,20].
62 x ^ x’ = x ^ 0.  [57,23,23,21,11,5].
63 0 ^ x = x ^ x’.  [58,23,23,21,39,5].
64 x’ ^ x = x ^ 0.  [59,23,23,21,11,5].
65 x’ v x = x v x’.  [59,35,8].
66 (x v y) ^ y’ = x ^ y’.  [62,17,55,47].
67 (x ^ x’) v x = x.  [62,52].
68 (x ^ y) v (y ^ y’) = x ^ y.  [63,17,2,7,19].
69 (x ^ (y ^ 0)) v (z ^ 0) = 0 ^ (x ^ (y ^ z)).  [29,15,15,53,25,6].
70 0 ^ (x’ v y) = 0 ^ (x v y).  [37,69,54].
71 (0 ^ (x v y)) v (x ^ y) = x ^ y.  [61,70].
72 (x v y’) ^ y = x ^ y.  [64,17,55,38].
73 x ^ x = x.  [67,35,41,5,65,72,68].
74 x ^ (x ^ y) = x ^ y.  [73,6].
75 x ^ (y ^ (x ^ y)) = x ^ y.  [73,6].
76 (x ^ 0) v y = (x v y) ^ y.  [73,17,34].
77 (x v (y ^ x)) ^ z = (x v y) ^ (x ^ z).  [74,17,32].
78 0 ^ (x ^ y) = 0 ^ (x v y).  [41,11,10,36,70,10].
79 ((x’ v y’) ^ z) v (x ^ y) = (x ^ y) v z.  [41,35].
```

```
80 (x' v y) ^ x = x ^ (y ^ x).  [35,72,5,6].
81 x ^ (y' ^ x) = x ^ y'.  [56,23,23,5,41,5,80].
82 (x' v y) ^ (z ^ (x v z')) = x ^ (y ^ (z ^ (x v z'))).  [31,66,41,5,6,41,5,6,6].
83 (x ^ 0) v (y ^ (x ^ y)) = (x v y) ^ (x ^ y).  [75,34,32,77].
84 (x v y) ^ (x ^ y) = x ^ y.  [34,76,10,78,70,71,28,83].
85 (x ^ 0) v (y ^ (x ^ y)) = x ^ y.  [83,84].
86 x ^ (y ^ x) = x ^ y.  [5,81,5].
87 (x ^ 0) v (y ^ x) = x ^ y.  [85,86].
88 (x' v y) ^ x = x ^ y.  [80,86].
89 x ^ (y ^ (z ^ x)) = x ^ (y ^ z).  [6,86].
90 x ^ (y v x') = x ^ y.  [72,86,86].
91 (x' v y) ^ (z ^ x) = x ^ (y ^ z).  [82,90,90,89].
92 (x ^ (y' ^ z)) v (x ^ y) = x ^ (y v z).  [88,17,23,5,6,89,43,88].
93 (x v y) v z = x v (y v z).  [5,43,5].
94 ((x ^ y) v z) ^ y = (x ^ y) v (z ^ y).  [73,45,79].
95 ((x ^ y) v z) ^ x = x ^ (y v z).  [86,45,91,92].
96 (x ^ 0) v ((y ^ x) v z) = (x ^ y) v z.  [87,93].
97 x ^ ((y ^ x) v z) = x ^ (y v z).  [86,95,95].
98 x ^ (y v (x' v z)) = x ^ (y v z).  [90,95,95,93].
99 (x ^ y) v (z ^ x) = x ^ (y v z).  [97,87,94,96].
100 (x ^ y) v (x ^ z) = x ^ (y v z).  [88,99,98].
101 $F.  [100,18].

============================== end of proof ==========================
```

## Appendix 4. Independence of $\overline{\mathrm{EqC}\ell\mathrm{FEL}_2}^{\mathrm{U}}$, proven with help of *Mace4*

Below we list the eight models generated by *Mace4* that prove the independence of $\overline{\mathrm{EqC}\ell\mathrm{FEL}_2}^{\mathrm{U}}$ (Theorem 7.5), each with run time "`seconds = 0`". In these models, F is always interpreted as 0 ($[\![\mathrm{F}]\!] = 0$), and T as 1 ($[\![\mathrm{T}]\!] = 1$).

Axiom (FFEL2), i.e., $x \dot\vee y = \neg(\neg x \dot\wedge \neg y)$, is not valid in the model with domain $\{0, 1\}$ and $[\![\mathrm{U}]\!] = 0$ because $1 \dot\vee 1 = 0 \neq 1 = \neg(\neg 1 \dot\wedge \neg 1)$, while the remaining axioms of $\mathrm{EqC}\ell\mathrm{FEL}_2{}^{\mathrm{U}}$ are valid:

| $\neg$ | 0 | 1 |
|---|---|---|
| | 0 | 1 |

| $\dot\wedge$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0 | 1 |

| $\dot\vee$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0 | 0 |

Axiom (FFEL3), i.e., $\neg\neg x = x$, is not valid in the model with domain $\{0, 1\}$ and $[\![\mathrm{U}]\!] = 0$ because $\neg\neg 1 = 0 \neq 1$, while the remaining axioms of $\mathrm{EqC}\ell\mathrm{FEL}_2{}^{\mathrm{U}}$ are valid:

| $\neg$ | 0 | 1 |
|---|---|---|
| | 0 | 0 |

| $\dot\wedge$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0 | 1 |

| $\dot\vee$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0 | 0 |

Axiom (FFEL6), i.e., $\mathrm{T} \dot\wedge x = x$, is not valid in the model with domain $\{0, 1\}$ and $[\![\mathrm{U}]\!] = 0$ because $1 \dot\wedge 1 = 0 \neq 1$, while the remaining axioms of $\mathrm{EqC}\ell\mathrm{FEL}_2{}^{\mathrm{U}}$ are valid:

| $\neg$ | 0 | 1 |
|---|---|---|
| | 0 | 1 |

| $\dot\wedge$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0 | 0 |

| $\dot\vee$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0 | 0 |

Axiom (FFEL10), i.e., $x \dot\vee (y \dot\wedge \mathrm{F}) = x \dot\wedge (y \dot\vee \mathrm{T})$ is not valid in the model with domain $\{0, 1\}$ and $[\![\mathrm{U}]\!] = 0$ because $1 \dot\vee (1 \dot\wedge 0) = 0 \neq 1 = 1 \dot\wedge (1 \dot\vee 1)$, while the remaining axioms of $\mathrm{EqC}\ell\mathrm{FEL}_2{}^{\mathrm{U}}$ are valid:

| $\neg$ | 0 | 1 |
|---|---|---|
| | 0 | 1 |

| $\dot\wedge$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0 | 1 |

| $\dot\vee$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0 | 1 |

Axiom (M1), i.e., $(x \dot\vee y) \dot\wedge z = (\neg x \dot\wedge (y \dot\wedge z)) \dot\vee (x \dot\wedge z)$, is not valid in the model with domain $\{0, 1, 2, 3\}$ and $[\![\mathrm{U}]\!] = 2$ because $(0 \dot\vee 0) \dot\wedge 3 = 3 \neq 2 = (\neg 0 \dot\wedge (0 \dot\wedge 3)) \dot\vee (0 \dot\wedge 3)$, while the remaining axioms of $\mathrm{EqC}\ell\mathrm{FEL}_2{}^{\mathrm{U}}$ are valid:

| $\neg$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| | 1 | 0 | 2 | 3 |

| $\leftand$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 2 | 3 |
| 1 | 0 | 1 | 2 | 3 |
| 2 | 2 | 2 | 2 | 2 |
| 3 | 3 | 3 | 2 | 2 |

| $\leftor$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 1 | 1 | 2 | 3 |
| 2 | 2 | 2 | 2 | 2 |
| 3 | 3 | 3 | 2 | 2 |

The axiom $x \leftand y = y \leftand x$ is not valid in the model with domain $\{0, 1\}$ and $[\![\mathsf{U}]\!] = 0$ because $0 \leftand 1 = 0 \neq 1 = 1 \leftand 0$, while the remaining axioms of $\mathrm{EqC}\ell\mathrm{FEL}_2{}^{\mathsf{U}}$ are valid:

| $\neg$ | 0 | 1 |
|---|---|---|
| | 0 | 1 |

| $\leftand$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 1 | 1 |

| $\leftor$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 1 | 1 |

The axiom $\neg\mathsf{U} = \mathsf{U}$ is not valid in the model with domain $\{0, 1\}$ and $[\![\mathsf{U}]\!] = 0$ because $\neg 0 = 1 \neq 0$, while the remaining axioms of $\mathrm{EqC}\ell\mathrm{FEL}_2{}^{\mathsf{U}}$ are valid:

| $\neg$ | 0 | 1 |
|---|---|---|
| | 1 | 0 |

| $\leftand$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0 | 1 |

| $\leftor$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 1 | 1 |

Finally, the axiom $\mathsf{U} \leftand x = \mathsf{U}$ is not valid in the model with domain $\{0, 1, 2, 3\}$ and $[\![\mathsf{U}]\!] = 2$ because $2 \leftand 3 = 3 \neq 2$, while the remaining axioms of $\mathrm{EqC}\ell\mathrm{FEL}_2{}^{\mathsf{U}}$ are valid:

| $\neg$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| | 1 | 0 | 2 | 3 |

| $\leftand$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 2 | 3 |
| 1 | 0 | 1 | 2 | 3 |
| 2 | 2 | 2 | 2 | 3 |
| 3 | 3 | 3 | 3 | 3 |

| $\leftor$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 1 | 1 | 2 | 3 |
| 2 | 2 | 2 | 2 | 3 |
| 3 | 3 | 3 | 3 | 3 |