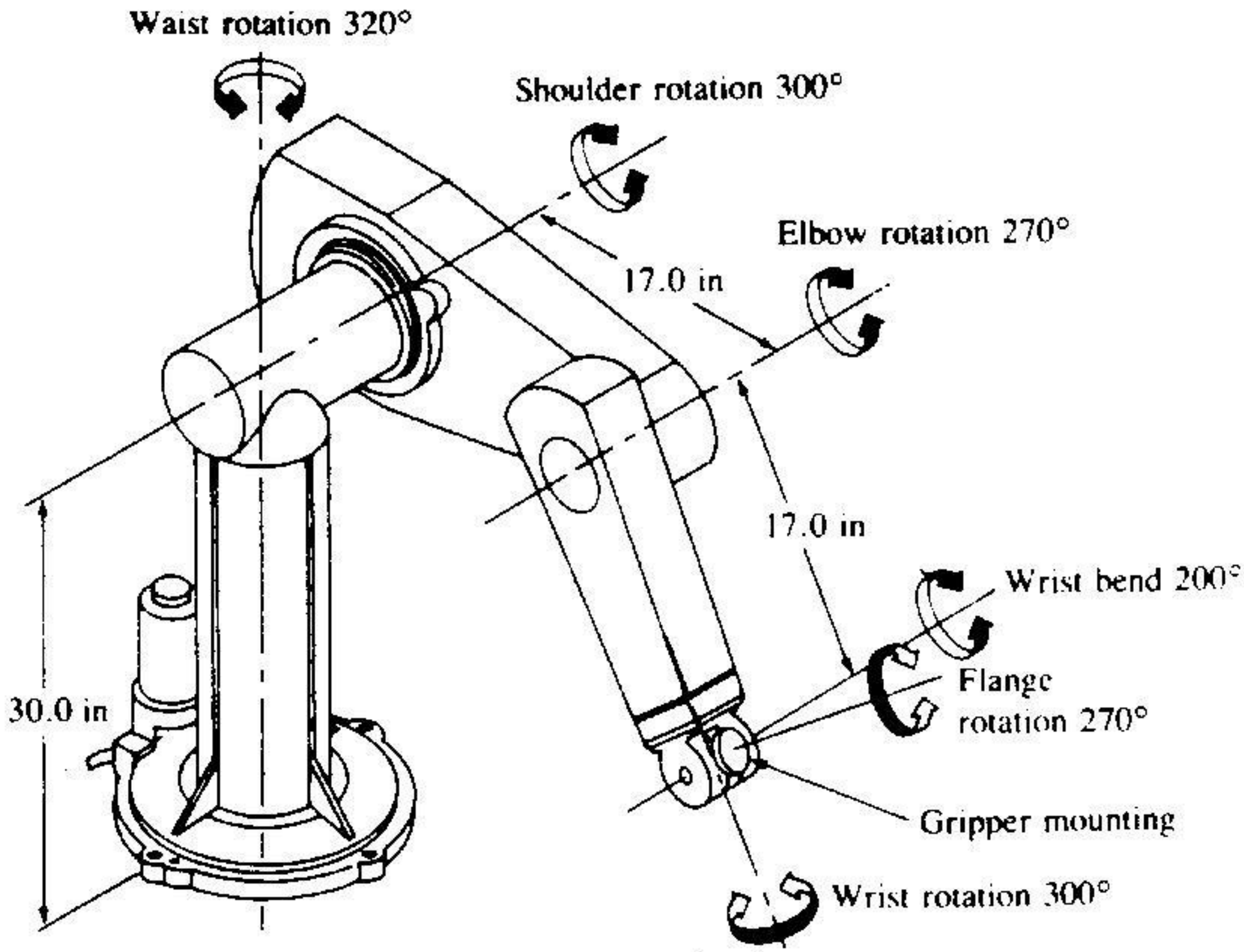
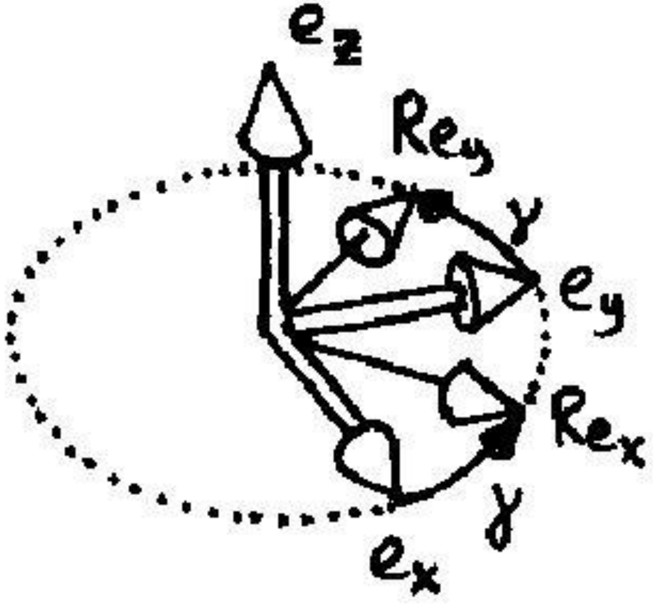
Poses and Kinematics

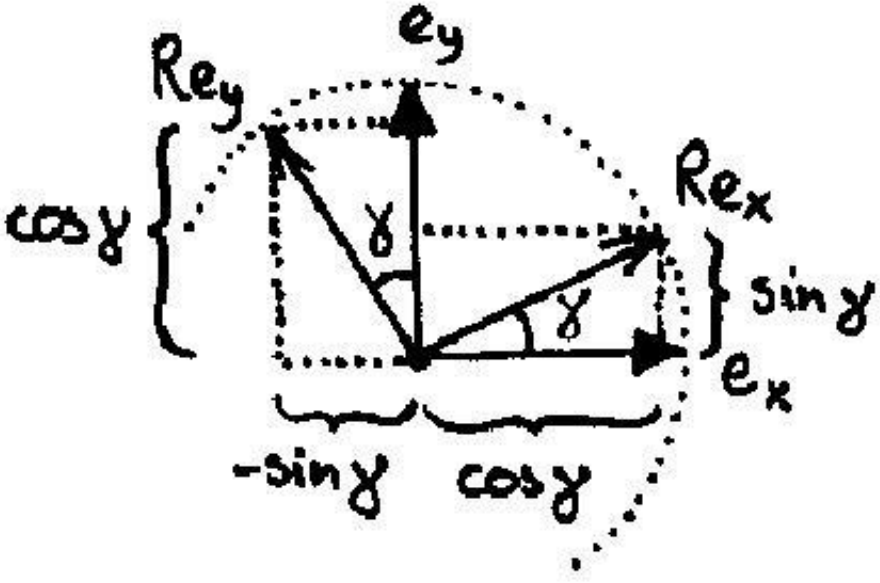
Leo Dorst


Waist rotation 320°
Shoulder rotation 300°
Elbow rotation 270°
17.0 in
17.0 in
Wrist bend 200°
Flange
rotation 270°
30.0 in
Gripper mounting
Wrist rotation 300°

Z-rotation

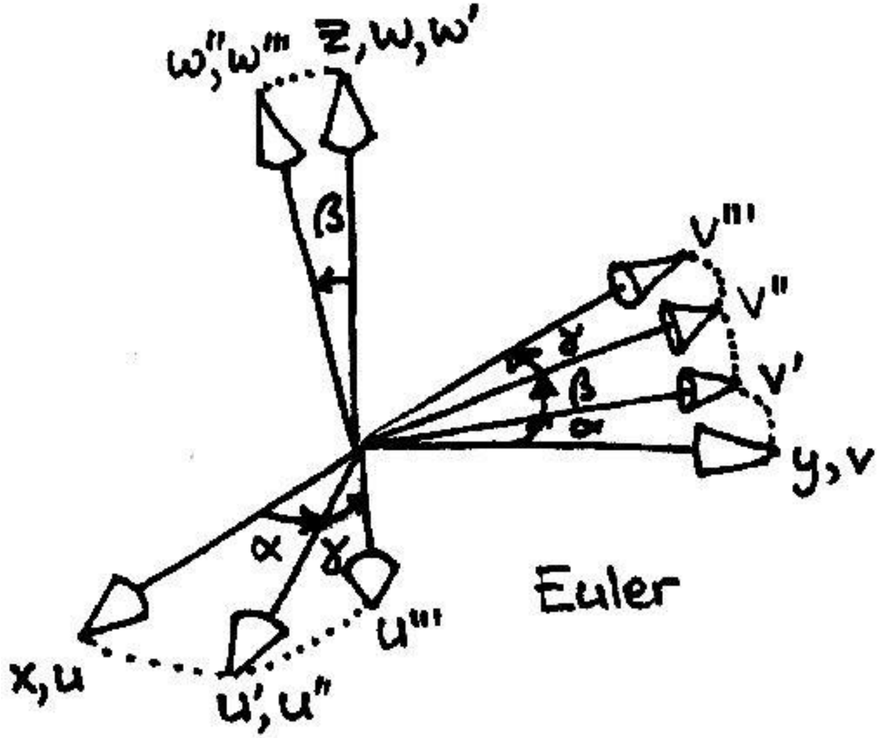


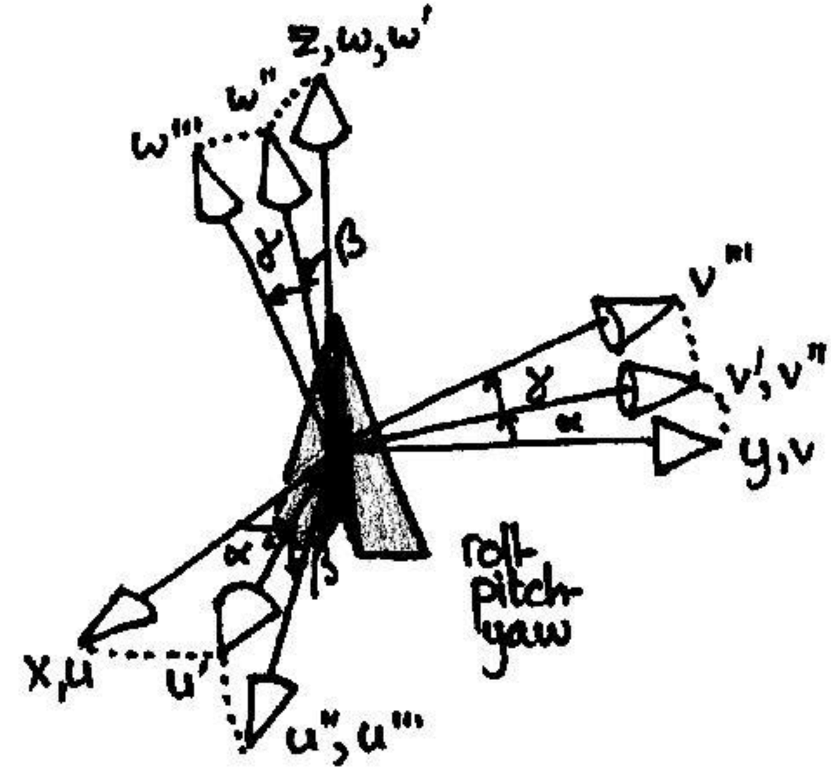
e_z
Re_y
γ
e_y
Re_x
γ
e_x



e_y
Re_y
$\cos\gamma$
γ
Re_x
γ
$\sin\gamma$
e_x
$-\sin\gamma$
$\cos\gamma$

angles for rotation specification





Roll-pitch-yaw rotation matrix

$$
\begin{aligned}
[\mathbf{R}] &= [\mathbf{R}_{u,\gamma} \circ \mathbf{R}_{v,\beta} \circ \mathbf{R}_{w,\alpha}] = [\mathbf{R}_{z,\alpha}][\mathbf{R}_{y,\beta}][\mathbf{R}_{x,\gamma}] \\
&= \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix} \\
&= \begin{bmatrix} c\beta c\alpha & (s\gamma s\beta c\alpha - c\gamma s\alpha) & (c\gamma s\beta c\alpha + s\gamma s\alpha) \\ c\beta s\alpha & (s\gamma s\beta s\alpha + c\gamma c\alpha) & (c\gamma s\beta s\alpha - s\gamma c\alpha) \\ -s\beta & s\gamma c\beta & c\gamma c\beta \end{bmatrix} \qquad (4.66)
\end{aligned}
$$

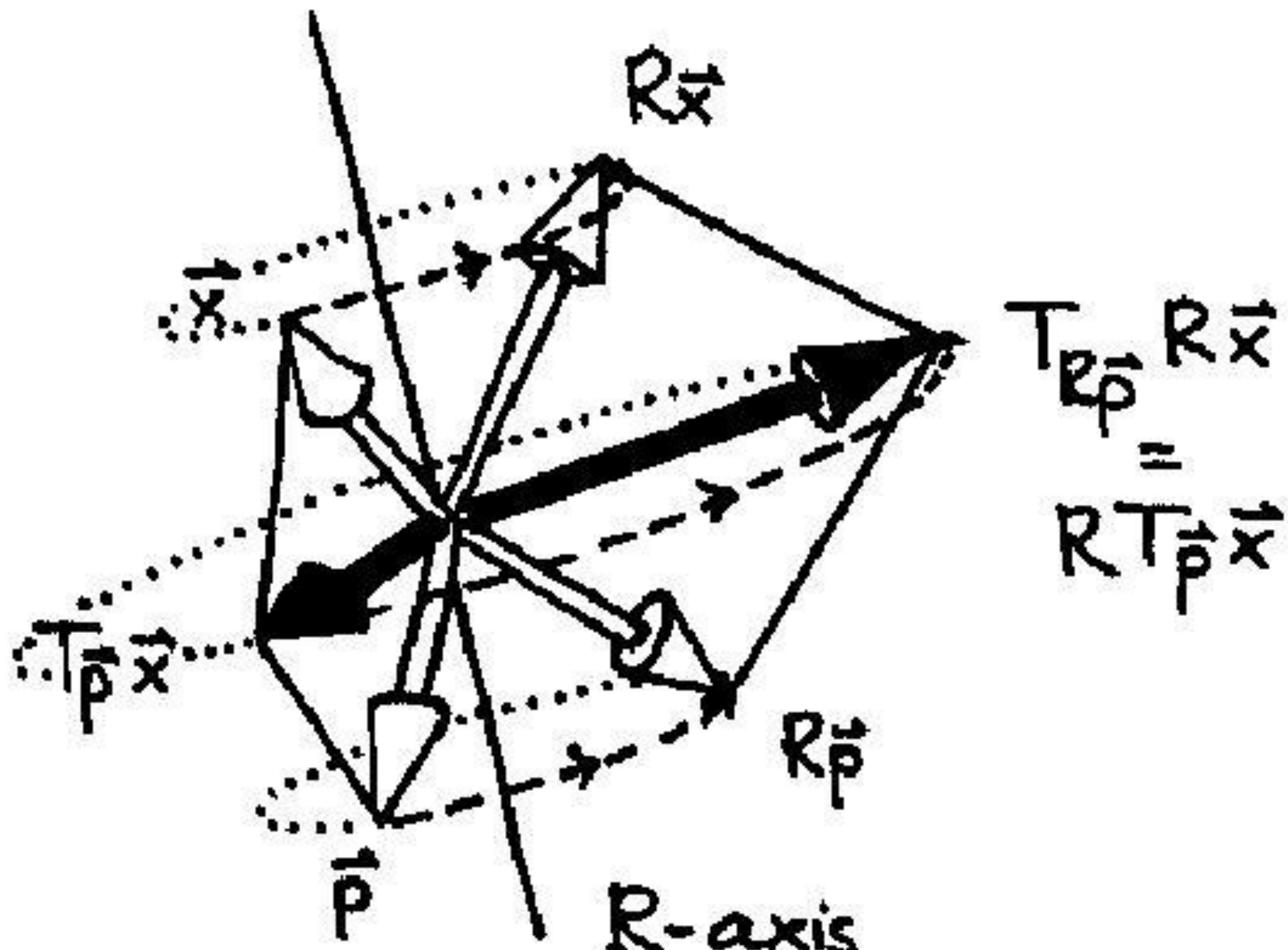
Rodrigues' rotation formula

3. Show that the rotation over an angle ϕ around an arbitrary axis through the origin, given by the unit vector $(r_x\ r_y\ r_z)^T$, is given by:

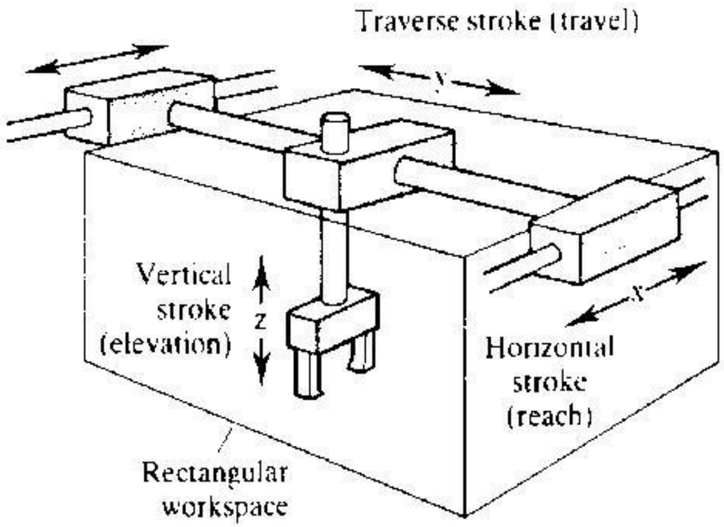
$$\begin{bmatrix} r_x^2(1-c\phi)+c\phi & r_xr_y(1-c\phi)-r_zs\phi & r_xr_z(1-c\phi)+r_ys\phi \\ r_xr_y(1-c\phi)+r_zs\phi & r_y^2(1-c\phi)+c\phi & r_yr_z(1-c\phi)-r_xs\phi \\ r_xr_z(1-c\phi)-r_ys\phi & r_yr_z(1-c\phi)+r_xs\phi & r_z^2(1-c\phi)+c\phi \end{bmatrix} \tag{4.64}$$

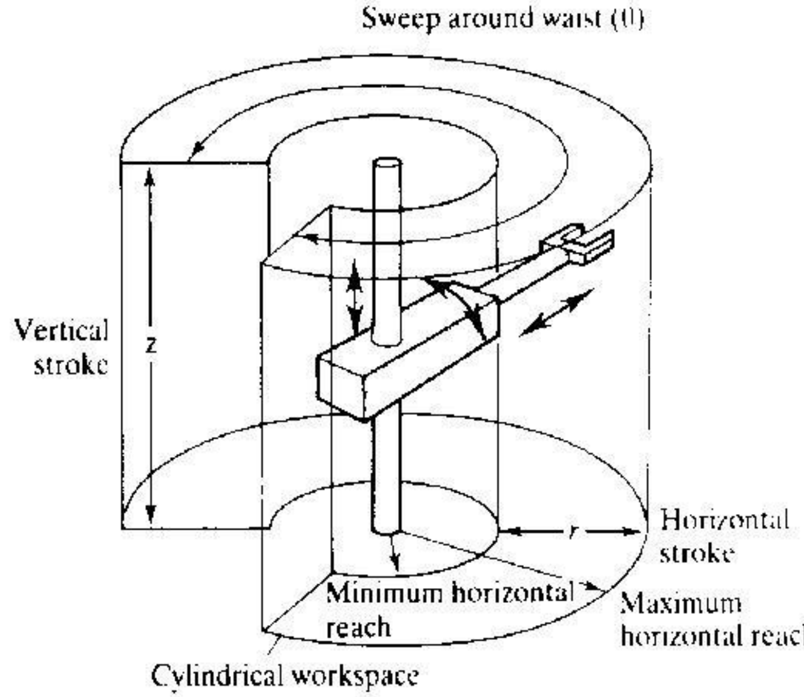
In terms of matrices, for unit axis **k** and angle θ:

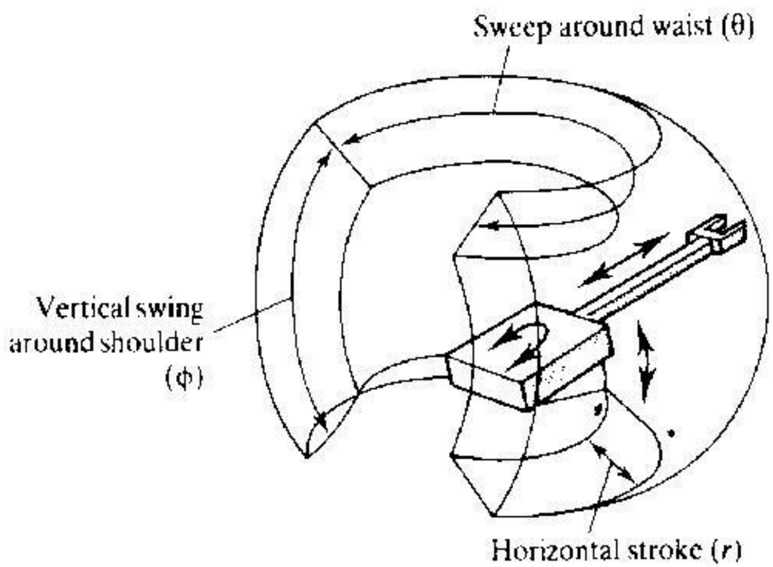
$$R = I\cos\theta + [\mathbf{k}]_\times \sin\theta + (1-\cos\theta)\mathbf{k}\mathbf{k}^\top.$$

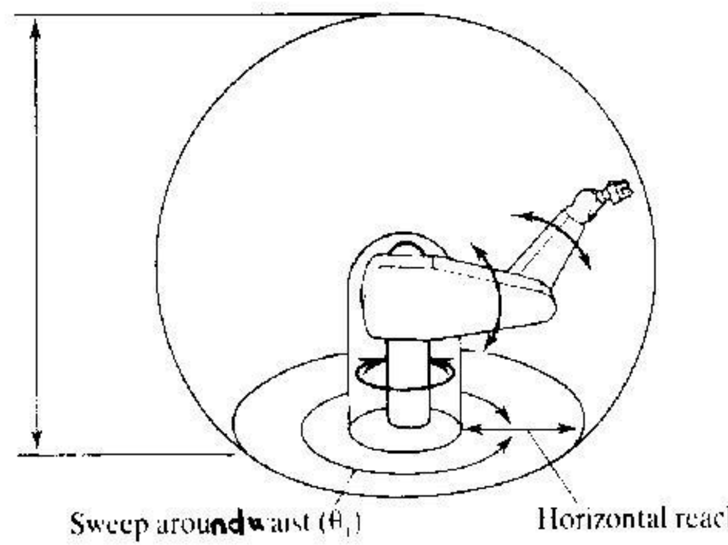


$R\vec{x}$
$\vec{x}$
$T_{R\vec{p}} R\vec{x} = R T_{\vec{p}} \vec{x}$
$T_{\vec{p}}\vec{x}$
$R\vec{p}$
$\vec{p}$
R-axis

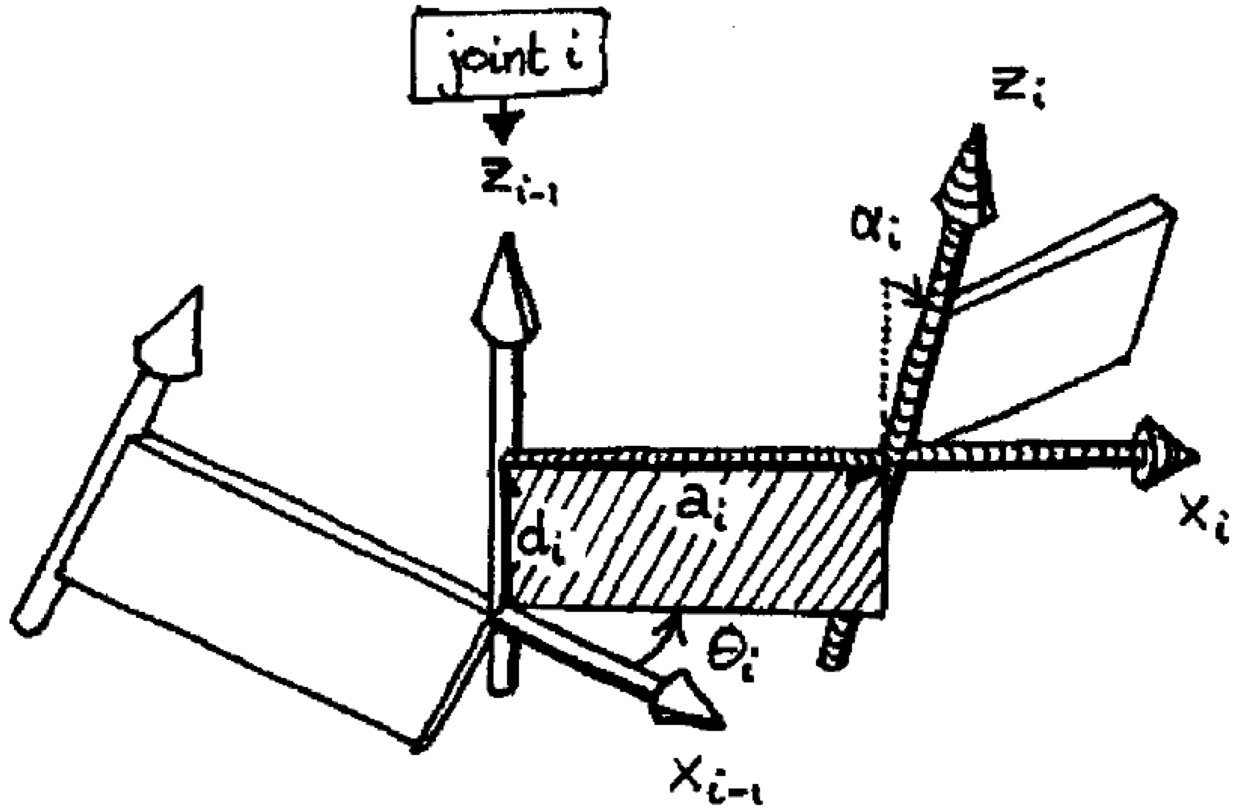

Traverse stroke (travel)
y
Vertical
stroke
(elevation)
z
x
Horizontal
stroke
(reach)
Rectangular
workspace


Sweep around waist (θ)
Vertical
stroke
z
r
Horizontal
stroke
Minimum horizontal
reach
Maximum
horizontal reach
Cylindrical workspace


Sweep around waist (θ)
Vertical swing
around shoulder
(ϕ)
Horizontal stroke (r)

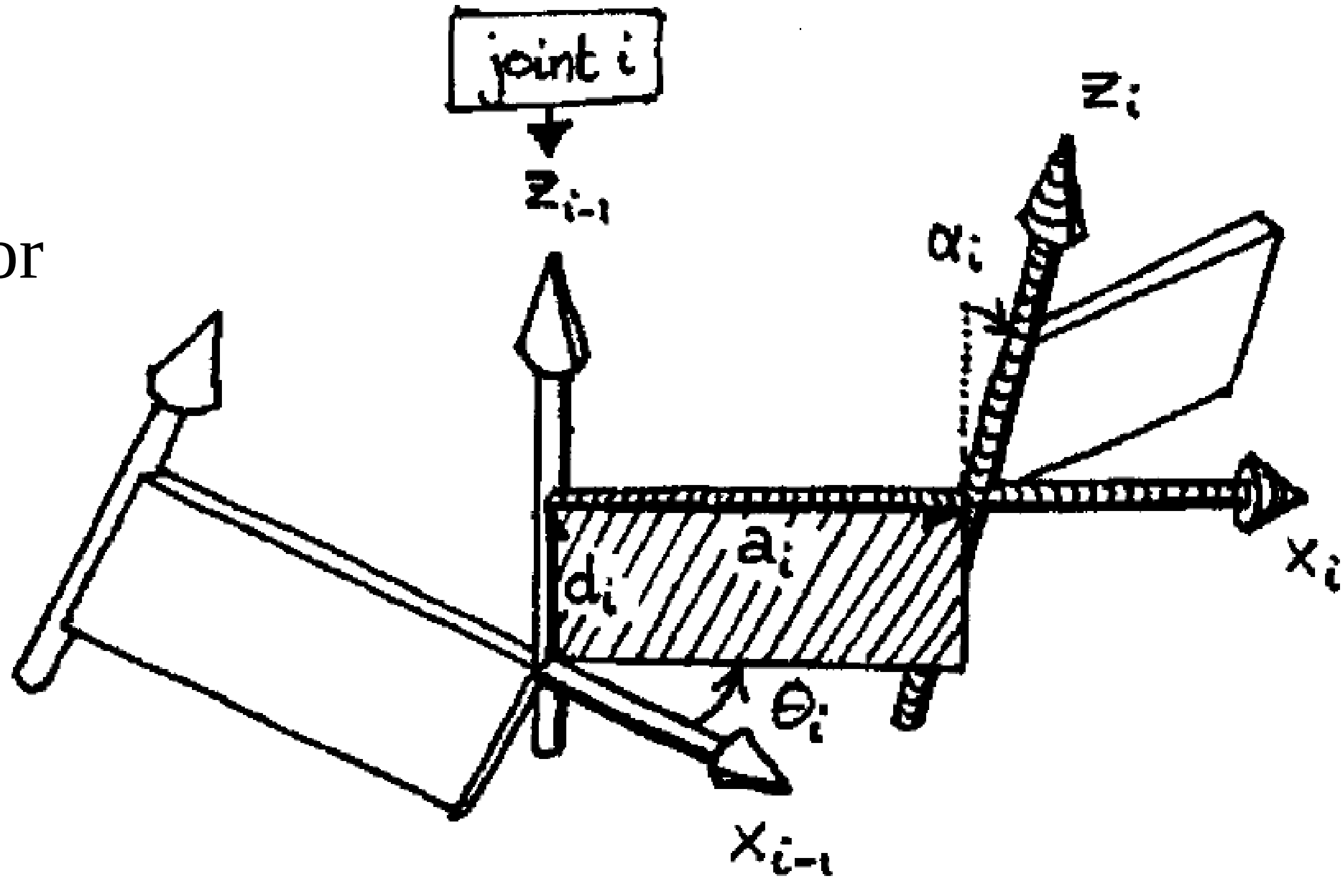

Sweep around waist (θ_1)
Horizontal reach

Denavit-Hartenberg convention for robot joint parameters

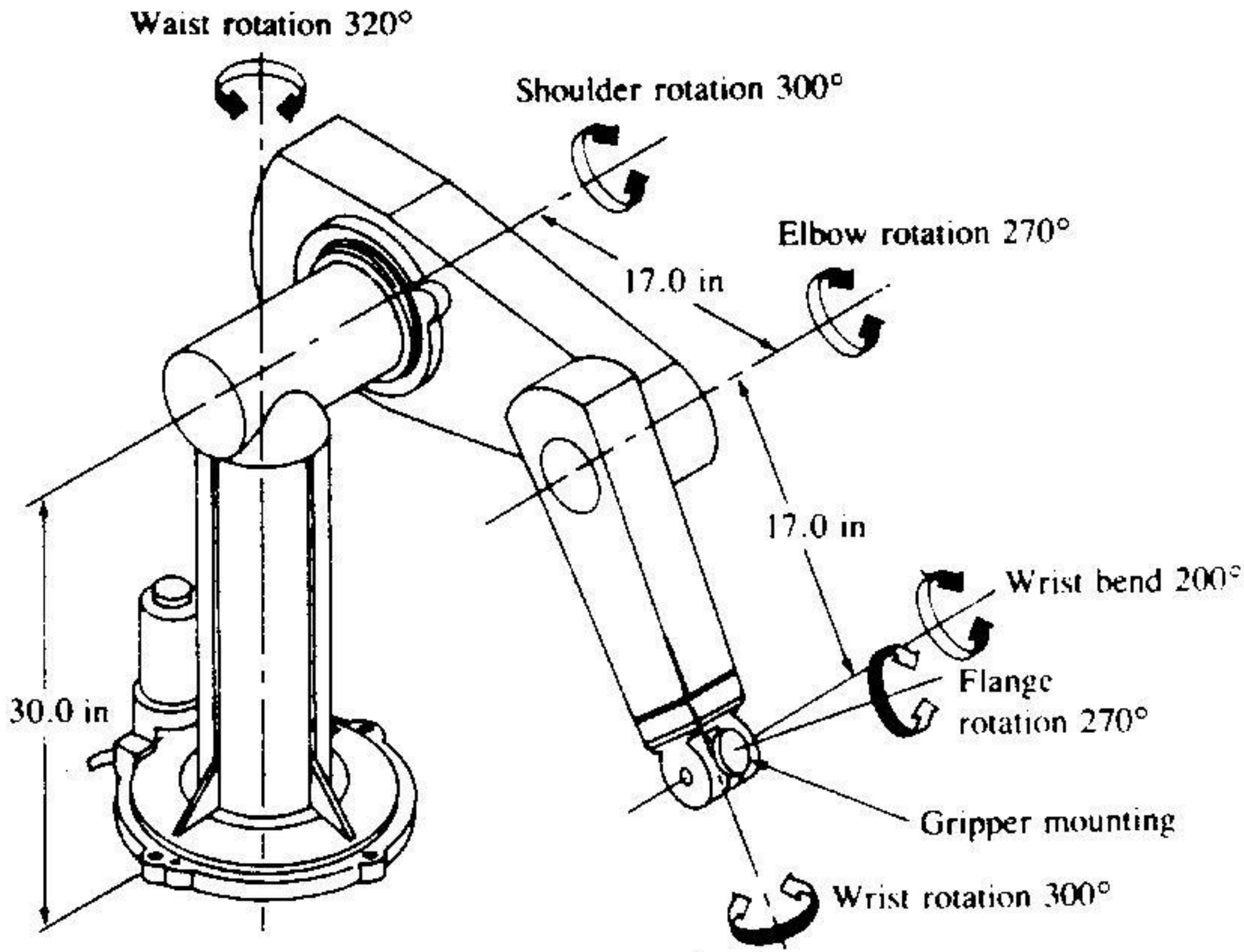


- θ_i – the joint angle from $\mathbf{x}_{i-1}$ to $\mathbf{x}_i$ about the $\mathbf{z}_{i-1}$ axis (use the right hand rule for the sign!)
- d_i – the distance from the origin of frame $i-1$ to the intersection of $\mathbf{z}_{i-1}$ and $\mathbf{x}_i$ (measured along $\mathbf{z}_{i-1}$).
- a_i – the shortest distance from the $\mathbf{z}_{i-1}$ to $\mathbf{z}_i$ axes (remember that this is measured along the perpendicular to *both* axes, so it is the amount of translation along the positive $\mathbf{x}_i$ axis!)
- α_i – the offset from $\mathbf{z}_{i-1}$ to $\mathbf{z}_i$, measured as an angle around the $\mathbf{x}_i$ axis (remember the right-hand rule)

Denavit-Hartenberg convention for robot joint parameters



$$[\,{}^{i-1}\mathbf{A}_i\,] = \begin{bmatrix} \cos\theta_i & -\cos\alpha_i\sin\theta_i & \sin\alpha_i\sin\theta_i & a_i\cos\theta_i \\ \sin\theta_i & \cos\alpha_i\cos\theta_i & -\sin\alpha_i\cos\theta_i & a_i\sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$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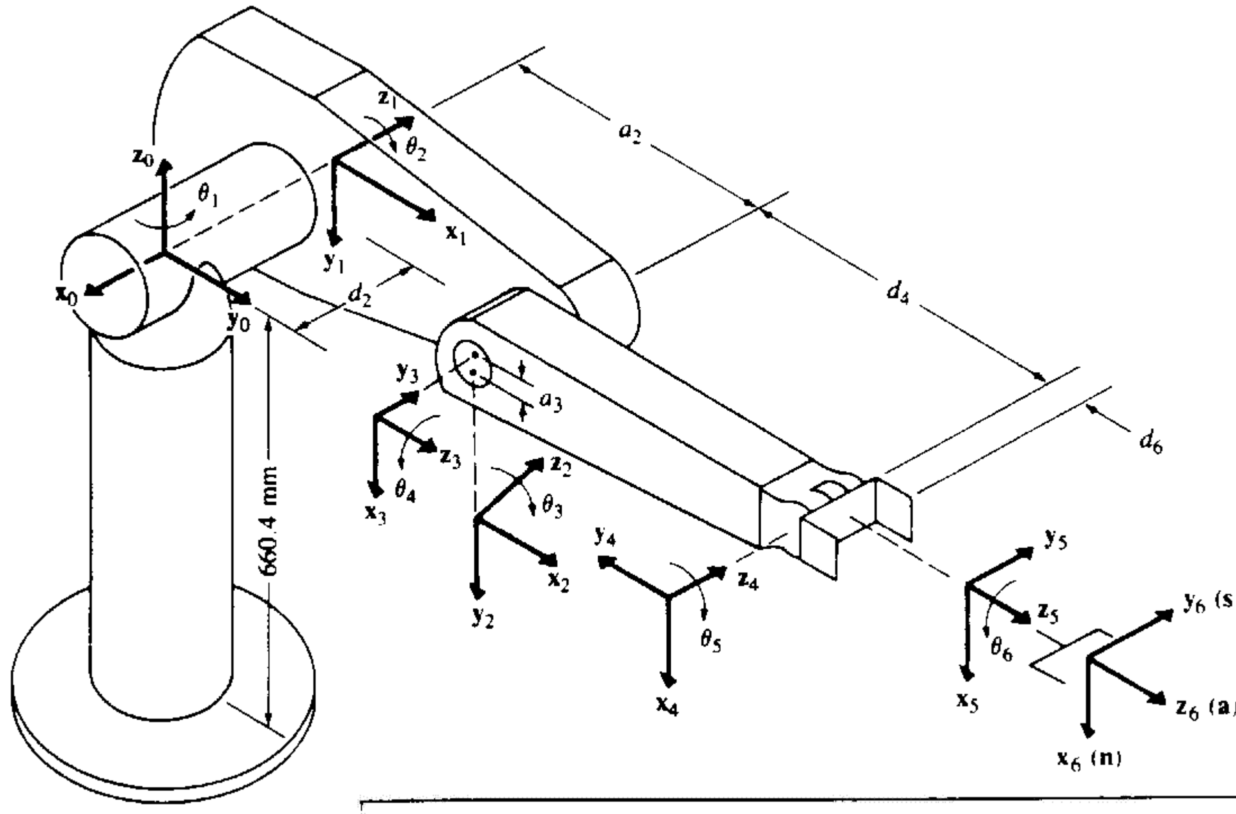
DH for PUMA

PUMA robot arm link coordinate parameters					
Joint i	θ_i	α_i	a_i	d_i	Joint range
1	90	−90	0	0	−160 to +160
2	0	0	431.8 mm	149.09 mm	−225 to 45
3	90	90	−20.32 mm	0	−45 to 225
4	0	−90	0	433.07 mm	−110 to 170
5	0	90	0	0	−100 to 100
6	0	0	0	56.25 mm	−266 to 266



PUMA robot arm link coordinate parameters					
Joint i	θ_i	α_i	a_i	d_i	Joint range
1	90	−90	0	0	−160 to +160
2	0	0	431.8 mm	149.09 mm	−225 to 45
3	90	90	−20.32 mm	0	−45 to 225
4	0	−90	0	433.07 mm	−110 to 170
5	0	90	0	0	−100 to 100
6	0	0	0	56.25 mm	−266 to 266

DH for 2-DoF planar arm

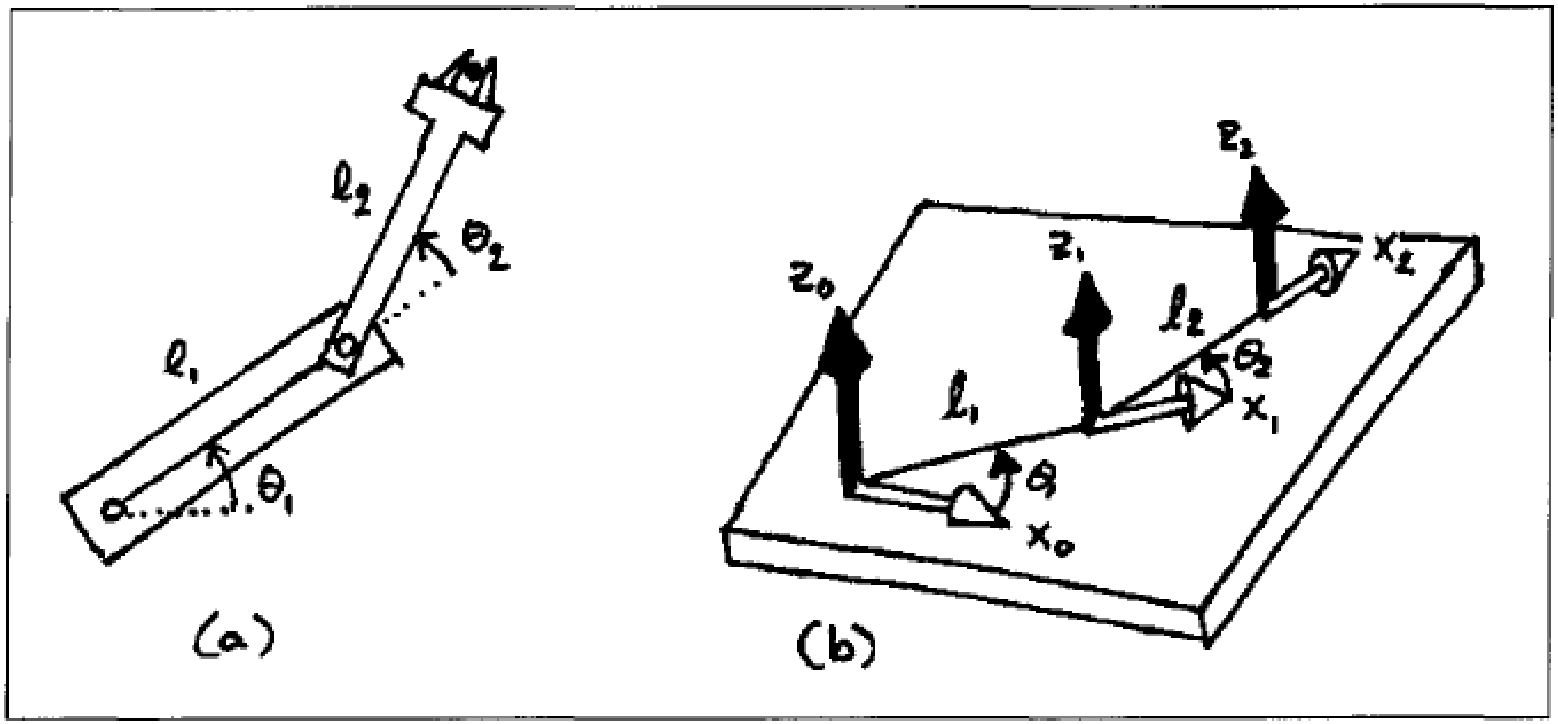


Figure 5.4: *A 2-dimensional 2-dof RR robot*

i	θ_i	a_i	a_i	d_i
1	θ_1	0	l_1	0
2	θ_2	0	l_2	0

Figure 5.5: *The Denavit Hartenberg parameters for the 2-D RR manipulator.*

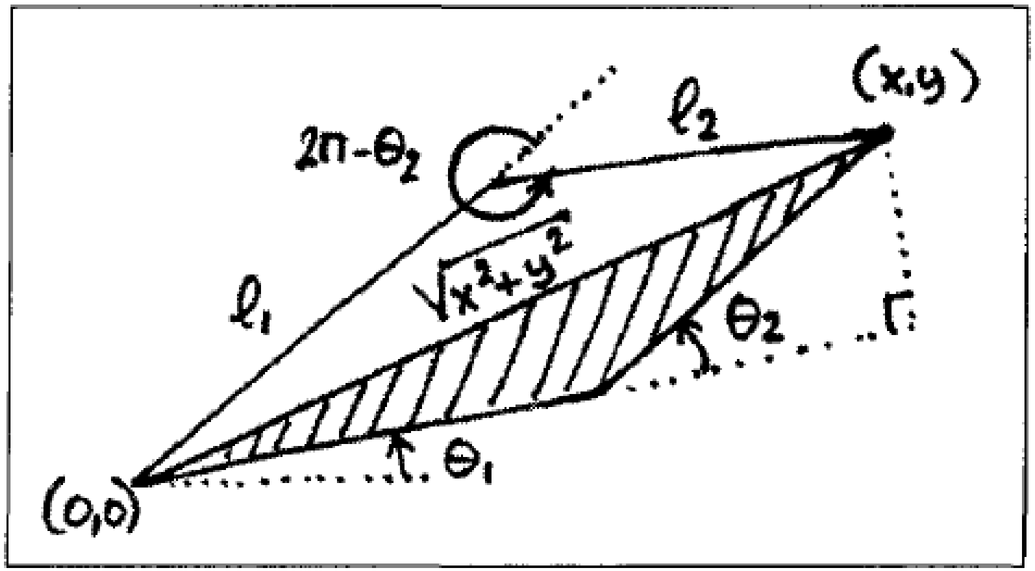
Planar robot arm DH kinematics

$$
\begin{aligned}
[{}^0\mathbf{A}_2] &= [{}^0\mathbf{A}_1][{}^1\mathbf{A}_2] \\
&= \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & l_1\cos\theta_1 \\ \sin\theta_1 & \cos\theta_1 & 0 & l_1\sin\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & l_2\cos\theta_2 \\ \sin\theta_2 & \cos\theta_2 & 0 & l_2\sin\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) & 0 & l_1\cos\theta_1 + l_2\cos(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & 0 & l_1\sin\theta_1 + l_2\sin(\theta_1+\theta_2) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c_{12} & -s_{12} & 0 & l_1c_1 + l_2c_{12} \\ s_{12} & c_{12} & 0 & l_1s_1 + l_2s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (5.4)
\end{aligned}
$$

Inverse kinematics
2D arm

$$\begin{bmatrix} c_{12} & -s_{12} & 0 & l_1c_1 + l_2c_{12} \\ s_{12} & c_{12} & 0 & l_1s_1 + l_2s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & x \\ r_{21} & r_{22} & r_{23} & y \\ r_{31} & r_{32} & r_{33} & z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$x = l_1c_1 + l_2c_{12}, \quad y = l_1s_1 + l_2s_{12}$$



$$x^2 + y^2 = l_1^2 + l_2^2 + 2l_1l_2c_2$$

$$c_2 = \frac{x^2 + y^2 - l_1^2 - l_2^2}{2l_1l_2}$$

$$s_2 = \pm\sqrt{1 - c_2^2}.$$

$$\theta_2 = \mathrm{atan2}(s_2, c_2)$$

$$\theta_1 = \mathrm{atan2}(y, x) - \mathrm{atan2}(l_2s_2, l_1 + l_2c_2).$$

Differential kinematics of 2D arm: Jacobian

$$
\begin{aligned}
x &= l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) \\
y &= l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2) \\
\phi &= \theta_1 + \theta_2
\end{aligned}
$$

$$
\begin{aligned}
\frac{dx}{dt} &= -(l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2))\frac{d\theta_1}{dt} - l_2 \sin(\theta_1 + \theta_2))\frac{d\theta_2}{dt} \\
\frac{dy}{dt} &= -(l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2))\frac{d\theta_1}{dt} + l_2 \cos(\theta_1 + \theta_2))\frac{d\theta_2}{dt} \\
\frac{d\phi}{dt} &= \frac{d\theta_1}{dt} + \frac{d\theta_2}{dt}
\end{aligned}
$$

$$
\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{d\phi}{dt} \end{bmatrix} = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{d\theta_1}{dt} \\ \frac{d\theta_2}{dt} \end{bmatrix}
$$

Jacobian matrix **J**: $[\frac{d\mathbf{x}}{dt}] = [\mathbf{J}][\frac{d\theta}{dt}]$

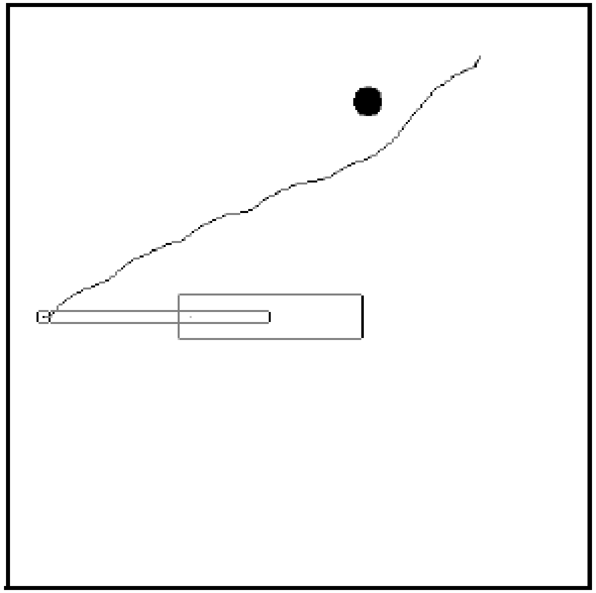
Position of 2 DoF robot arm

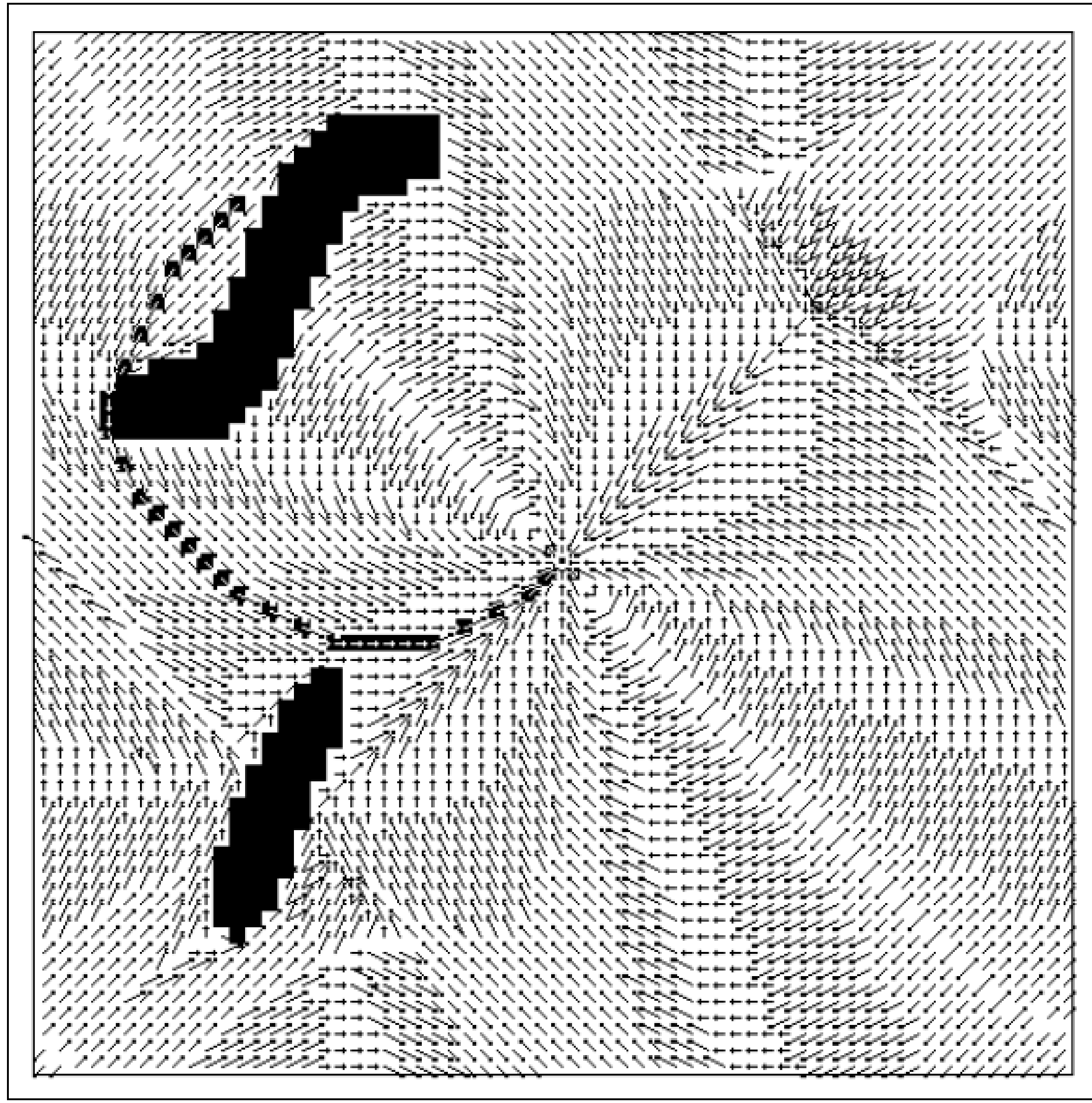
Position only, Jacobian:

$$\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{bmatrix} \begin{bmatrix} \Delta\theta_1 \\ \Delta\theta_2 \end{bmatrix}$$

Singularity detection by determinant:

$$\begin{aligned} \Delta &= (-l_1 s_1 - l_2 s_{12}) l_2 c_{12} + (l_1 c_2 + l_2 c_{12}) l_2 s_{12} = l_1 l_2 (c_1 s_{12} - s_1 c_{12}) \\ &= l_1 l_2 (c_1 (s_1 c_2 + c_1 s_2) - s_1 (c_1 c_2 - s_1 s_2)) = l_1 l_2 s_2 (s_1^2 + c_1^2) = l_1 l_2 s_2 \end{aligned}$$





Above: Task Space path corresponding to Config. Space solution.

Right: Resulting field of arrows from A* in Config. Space using minimum distance (straightest end effector path) criterion.

$$C(\theta_1,\theta_2,\delta\theta_1,\delta\theta_2) = \sqrt{(L_1\delta\theta_1)^2 + (L_2\delta\theta_2)^2 + 2L_1\delta\theta_1 L_2\delta\theta_2\cos(\theta_1-\theta_2)}$$

Minimum-distance-metric by differential kinematics

$$
\begin{aligned}
(\Delta s)^2 &= (\Delta x)^2 + (\Delta y)^2 = [\Delta x \;\; \Delta y] \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}^T \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} \\
&= \left([\mathbf{J}] \begin{bmatrix} \Delta\theta_1 \\ \Delta\theta_2 \end{bmatrix}\right)^T \left([\mathbf{J}] \begin{bmatrix} \Delta\theta_1 \\ \Delta\theta_2 \end{bmatrix}\right) = [\Delta\theta_1 \;\; \Delta\theta_2]\, [\mathbf{J}]^T [\mathbf{J}] \begin{bmatrix} \Delta\theta_1 \\ \Delta\theta_2 \end{bmatrix} \\
&= [\Delta\theta_1 \;\; \Delta\theta_2] \begin{bmatrix} l_1^2 + l_2^2 + 2 l_1 l_2 c_2 & l_2^2 + l_1 l_2 c_2 \\ l_2^2 + l_1 l_2 c_2 & l_2^2 \end{bmatrix} \begin{bmatrix} \Delta\theta_1 \\ \Delta\theta_2 \end{bmatrix} \\
&= ((-l_1 s_1 - l_2 s_{12})\Delta\theta_1 - l_2 s_{12} \Delta\theta_2)^2 + ((l_1 c_1 + l_2 c_{12})\Delta\theta_1 + l_2 c_{12} \Delta\theta_2)^2 \\
&= (l_1^2 + l_2^2 + 2 l_1 l_2 \cos\theta_2)(\Delta\theta_1)^2 + 2(l_2^2 + l_1 l_2 \cos\theta_2)\Delta\theta_1 \,\Delta\theta_2 + l_2^2 (\Delta\theta_2)^2
\end{aligned}
$$

(previous path planning slide used non-DH coordinates $\theta'_1 = \theta_1$ and $\theta'_2 = \theta_1 + \theta_2$, therefore slightly different formula)