# Describing the Characteristics of Circular and Elliptical Motion using Qualitative Representations

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## Abstract

Circular and elliptical motion are fundamental topics in physics education, yet learners often struggle to grasp them. We investigate how interactive qualitative representations can be used to describe the characteristic behavior of circular and elliptical motion. We use the vocabulary and algorithms known as qualitative reasoning, which make it possible to represent the distinct features of these systems in a conceptual way. Leveraging the close alignment between qualitative reasoning and human reasoning about dynamic systems, these representations have the potential to enhance understanding in this domain.

## 1 Introduction

Circular motion is a fundamental concept in physics that describes the motion of an object moving in a circular path. The direction of velocity (but not the speed) of an object in circular motion changes due to the centripetal force which causes centripetal acceleration. Centripetal force is directed towards the center of the circle. In the case that an object is orbiting another object (e.g., a planet orbiting a star) centripetal force is equal to gravitational force.

Celestial bodies generally follow elliptical orbits, although circular orbits are often used as a simplified approximation for easier understanding. Additionally, certain celestial bodies, like moons, exhibit nearly circular orbits around their parent planets. The elliptical motion of celestial bodies is governed by Kepler's laws of planetary motion, which can be explained by the gravitational forces exerted between celestial bodies. The strength of gravity depends on the distance between the bodies. As the distance changes within an elliptical orbit, gravity varies, resulting in different acceleration at different points along the orbit. The elliptical shape of the orbits arises from the balance between the gravitational force and the momentum (the product of the mass and velocity) of the object in motion.

In physics education, circular and elliptical motion is often explained on the basis of mathematical formulas. Learners then work through exercises involving calculations using these formulas to process and learn this knowledge. The use of supporting software is limited. Particularly, the conceptual knowledge that explains the working of the mechanisms is not available in an interactive format. This issue poses a challenge in physics education, as there have been numerous reported difficulties associated with understanding circular and elliptical motion [e.g., Alonzo & Steedle, 2009; Barniol & Zavala, 2014; Canlas, 2016; Liu & Fang, 2016].

In this paper we focus on describing circular and elliptical motion using interactive qualitative representations [Bredeweg *et al.*, 2023a]. For the work presented in this contribution we use the software Dynalearn [Bredeweg *et al.*, 2013]. This software is implemented as a server-based architecture deploying the Garp3 reasoning engine [Bredeweg *et al.*, 2009]. The front-end is web-based and provides a diagrammatic approach for users to construct and articulate their thoughts. Learning through the construction of qualitative representations has proven to be a successful approach [Bredeweg *et al.*, 2023a; Kragten & Bredeweg, 2023], highlighting the potential of the representations described in this contribution to enhance understanding.

## 2 Circular motion

To represent circular motion qualitatively, the following notions have to be addressed: entities, quantities, possible values and direction of change, causal dependences, correspondences, and finally simulation consisting of qualitatively distinct states and transitions between them.

### 2.1 Direction of change and values of quantities

**Entities** represent the physical objects that constitute the system. Let's assume we model a moon orbiting a planet. In that case, the qualitative representation will have two entities: Moon and Planet. **Quantities** represent the measurable properties of entities, as such, the entity Moon has a position, a velocity, etc.

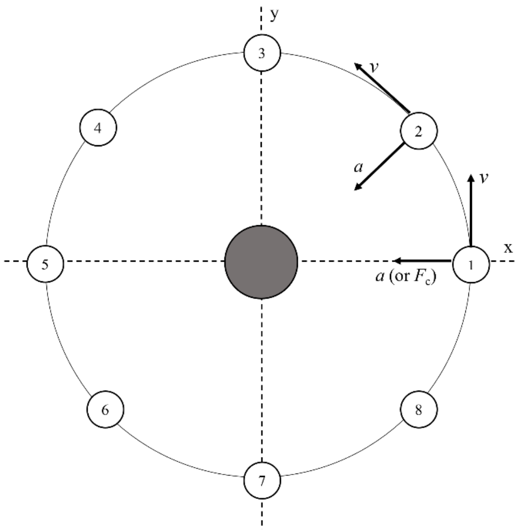


**Fig. 1.** Circular motion of a moon orbiting a planet. A system manifesting circular motion has eight qualitatively distinct states.

In a qualitative representation, each quantity has a value and a **direction of change**, represented as a tuple <v, ∂>. The possible values are represented in a quantity space, also for ∂. For instance, the direction of change can be captured by {−, 0, +}, referring to decreasing, steady, and increasing, respectively. However, the exact meaning of this depends on the context. To represent the dynamics of circular motion, we project the system on an x- and y-coordinate plane (Fig. 1). With regard to *Position*, ∂=+ is used to refer to 'increasing' on the x−axis (moving to the right) or on the y-axis (moving upward), while ∂=− refers to decreasing on these axes, and ∂=0 refers to remaining steady (no movement).

A similar quantity space can be used for the **possible values**, namely {min, −, 0, +, max}. If we consider *Position*, then 'min' refers to most-negative point on the x-axis (or y-axis), '−' refers to a negative interval between 'min' and '0', '0' refers to the origin of the plane, '+' refers to a positive interval between '0' and 'max', and 'max' refers to the most-positive point on the x-axis (or y-axis). Notice that, 'min', '0' and 'max' are points, while '−' and '+' are intervals. It turns out that the extreme values 'min' and 'max' are not needed for representing all the possible behaviors. This is because if the direction of change is zero within the negative and positive intervals, i.e., <−, 0> and <+, 0>, they also represent the minimum or maximum. Hence, we leave them out and work with the quantity space {−, 0, +}. Also note that, the planet is located at the origin of the coordinate plane.

## 2.2 Expected qualitative states

In a qualitative representation, each qualitatively distinct behavior of the system is represented as a **state**. Consequently, each state has a unique set of tuples <v, ∂> for the quantities describing the system. Given that the system is projected on a coordinate plane, the horizontal and vertical position, centripetal force, acceleration and velocity are the characteristic quantities. Together they describe the system using eight qualitatively distinct states (Fig. 1).

Table 1 shows the values and directions of change for each of the quantities in the eight states. Consider the position of the moon in state 1, in which case $x$=<+, 0> and $y$=<0, +>. The moon is at its most-right position (somewhere in the positive interval, hence '+') and there is no further change in the horizontal direction, hence ∂$x$=0. The y-coordinate is '0', but the moon is in an upward motion so there is a positive change in the vertical direction, hence ∂$y$=+.

In state 1, the centripetal force ($F_c$) and thereby the acceleration ($a$) is directed to the left. To describe the change of velocity we decompose the vectors of acceleration (and velocity) into a horizontal ($a_x$) and vertical component ($a_y$). For the horizontal acceleration holds $a_y$=<−, 0>, which represents that $a_y$ is at its most-negative value (the vector is directed to the left at its maximum value) and momentarily steady (for an infinite small moment). There is no vertical acceleration but there is a negative direction of change, hence $a_y$=<0, −>. There is no horizontal velocity and the change is negative, thus $v_x$=<0, −>. The vertical acceleration is at its maximum, thus $v_y$=<+, 0>.

**Table 1.** Eight qualitative states of circular motion. Quantities are position: x-axis ($x$) and y-axis ($y$), acceleration: horizontal ($a_x$) and vertical ($a_y$), and velocity: horizontal ($v_x$) and vertical ($v_y$). Each quantity has a value and a direction of change, shown as <v, ∂>. Force corresponds to acceleration. Force is not shown in this table.

| | State | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Quantity | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| $x$ | <+, 0> | <+, −> | <0, −> | <−, −> | <−, 0> | <−, +> | <0, +> | <+, +> |
| $y$ | <0, +> | <+, +> | <+, 0> | <+, −> | <0, −> | <−, −> | <−, 0> | <−, +> |
| $a_x$ | <−, 0> | <−, +> | <0, +> | <+, +> | <+, 0> | <+, −> | <0, −> | <−, −> |
| $a_y$ | <0, −> | <−, −> | <−, 0> | <−, +> | <0, +> | <+, +> | <+, 0> | <+, −> |
| $v_x$ | <0, −> | <−, −> | <−, 0> | <−, +> | <0, +> | <+, +> | <+, 0> | <+, −> |
| $v_y$ | <+, 0> | <+, −> | <0, −> | <−, −> | <−, 0> | <−, +> | <0, +> | <+, +> |

Note that state 1 is a point. The quantities only have these values and directions of change at this specific x- and y-coordinate in the system. In fact, state 1 has an infinite small duration. The system instantaneously moves into state 2, which has a duration. The values and directions of change in state 2 are true for the interval between state 1 and 3. State 3, 5 and 7 are also points. State 2, 4, 6 and 8 are intervals (with duration).

## 2.3 Adding dynamics to the representation

The next challenge is to add dynamics to the qualitative representation so that the latter can be simulated and successive states calculated from the information in the preceding states. Let us focus on the motion in horizontal direction. The implementation of this part is shown in Fig. 2. As discussed before, the entity Moon has four quantities to represent this part of the behavior: $x$, $F_x$, $a_x$ and $v_x$. All quantities have the quantity space {−, 0, +}. The direction of change is donated with $\partial$.

Two types of **causal dependencies** are distinguished: *proportionality* and *influence* [Bredeweg *et al.*, 2013]. When two quantities have a proportional relationship (P), a change in one quantity (the cause) results in a change in the other quantity. A proportional relationship can be positive (P+), where both quantities change in the same direction, or negative (P−), where the quantities change in the opposite direction. The relationship between the quantities $x$ and $F_x$ is negative proportional (P−). Note that $F_x$ is the horizontal component of the centripetal force which in this system is equal to the gravitational force, i.e., if the moon moves towards the origin of the coordinate plane (the location of the planet) the gravitational pull in the horizontal direction decreases (but increases in the vertical direction). The relationship between the quantities $F_x$ and $a_x$ is positive proportional (P+). This denotes that acceleration changes when the force applies changes.

Causal dependencies of type influence (I+, I−) can be added to represent the relationship between a process (also represented as a quantity) and another quantity. A process adds or removes something to the system per time unit. If an influence is positive (I+), a positive value of the process results in a change in the positive direction of the affected quantity, a negative value results in a change in the negative direction. The relationship between $a_x$ and $v_x$ is of the type positive influence (I+) (if $a_x$=− then $\delta v_x$=−, if $a_x$=0 then $\delta v_x$=0 and if $a_x$=+ then $\delta v_x$=+). For instance, if the acceleration in the horizontal direction is ‘0’ than there is no change in velocity. The relationship between $v_x$ and $x$ is also a positive influence (I+) (if $v_x$=− then $\delta x$=−, if $v_x$=0 then $\delta x$=0 and if $v_x$=+ then $\delta x$=+). For instance, if the velocity in the horizontal direction is negative ‘−‘, than the moon moves towards the negative side of the x-axis in the coordinate plane.

To determine the potential states of the system, **correspondences** (C) can be incorporated to describe the relationship between co-occurring values. In the present system, the values of $x$ and $F_x$ are dependent, they correspond inversely (if $x$=− then $F_x$=+, if $x$=0 then $F_x$=0 and if $x$=+ then $F_x$=−). The values of $F_x$ and $a_x$ are also dependent, they correspond regularly (if $F_x$=− then $a_x$=−, if $F_x$=0 then $a_x$=0 and if $F_x$=+ then $a_x$=+). The correspondences between $x$ and $F_x$, as well as $F_x$ and $a_x$, are directed, suggesting one-way dependencies between the values. To represent these directed correspondences, an arrow pointing in one direction is used (Fig. 2).

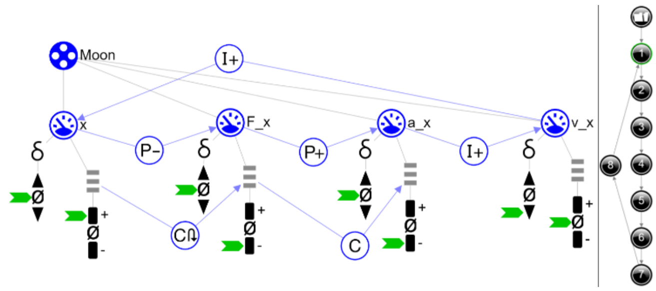


**Fig. 2.** Qualitative representation of the motion of a moon in horizontal direction. Quantities are position (x), force (F_x), acceleration (a_x) and velocity (v_x) (in the text we use $F_x$, $a_x$ and $v_x$). The representation is simulated with initial settings: <+, ?> and velocity <0,?> (not shown in the figure; ? refers to undefined). The simulation generates 8 states, as show on the RHS in the figure. The simulation result of state 1 is shown. From the representation it can be inferred that: x=<+, 0>, $F_x$=<−, 0>, $a_x$=<−, 0> and $v_x$=<0, −> (show in green). Correspondences are represented by the symbol C.

## 2.4 Simulation of horizontal motion

Fig. 2 shows the simulation for the horizontal motion, as it can be computed from the details discussed so far. The initial settings for this simulation are: $x$=<+, ?> and $v_x$=<0, ?> (? refers to undefined). All the other information can be inferred from this. The state graph (Fig. 2, RHS) shows that the system has eight states. The simulation result of state 1 is shown.

In state 1, the moon has no horizontal velocity ($v_x$=0), as determined by the initial settings. The causal dependency between $v_x$ and $x$ is of type positive influence (I+) and therefore the horizontal position of the moon does not change (if $v_x$=0 then $\delta x$=0). This results in $x$=<+, 0>, indicating that $x$ is at its maximum. There is an inversed correspondence between $x$ and $F_x$, indicating that the horizontal gravitational force on the moon is to the left (if $x$=+ then $F_x$=−). The correspondence between $F_x$ and $a_x$ indicates that the moon its horizontal acceleration is also to the left (if $F_x$=− then $a_x$=−). There is a negative proportional relationship (P-) between $x$ and $F_x$ and a positive proportional relationship (P+) between $F_x$ and $a_x$. The horizontal position of the moon does not change and as a result gravitational force in the horizontal direction does not change (if $\delta x$=0 then $\delta F_x$=0). Consequently, acceleration in the horizontal direction does not change (if $\delta F_x$=0 then $\delta a_x$=0). Therefore, in state 1,

$F_x$=<−, 0> and $a_x$ <−, 0>. Both quantities are maximal in the negative interval, i.e., both vectors ($F_x$ and $a_x$) have their maximal value (or magnitude) and are directed to the left. The causal dependency between $a_x$ and $v_x$ is of type positive influence (I+). The horizontal acceleration is to the left and as a result the direction of change of the horizontal velocity is to the left (if $a_x$=− then $\delta v_x$=−), i.e., $v_x$=<0, −>.

In state 2 (Table 2), the moon its velocity in the horizontal direction is to the left and increasing, i.e., $v_x$=<−, −>. As a result, the moon is on the right of the y-axis and moving towards the left, i.e., $x$=<+, −>. As the moon moves closer to the x-origin of the coordinate plane, the gravitational pull, and consequently, the acceleration in the horizontal direction towards the left, decreases, i.e., $F_x$=<−, +> and $a_x$ <−, +>.

The changes from state 2 propagate onwards, continuing until state 8. Upon reaching state 8, the values resemble those of the simulation's initial settings, initiating the repetition of circular motion.

## 2.5 Completing the model

Thus far we have managed to represent the movement of the celestial body in the horizontal direction. For this, it is important to see that the causal dependencies between quantities that describe vertical motion are similar to those of the horizontal direction. But how to represent the pendulum movement of the moon between its most-negative and most-positive position in the horizontal *and* vertical direction? Both pendulum movements have 8 possible states and without further information this results in 64 (8 x 8) possible states. For instance, the motion in the horizontal direction can go through all its 8 states while the motion in the vertical direction is still in its first state. An important insight is to realize that the pendulum movements in both directions are dependent.

Table 2 shows the correspondences between the values of the quantities in both directions when describing circular motion. All correspondences are bi-directional and apply to the entire quantity space. It is important to note that due to the bi-directional nature of correspondences, they also apply in the opposite direction. Table 2 includes six correspondences, namely between:

- $x$ and $a_x$. When the moon is positioned on the left side of the y-axis, its acceleration in the horizontal direction is towards the right (if $x$=− then $a_x$=+). If the moon crosses the y-axis, there is no acceleration in the horizontal direction (if $x$=0 then $a_x$=0). When the moon is located on the right side of the y-axis, its horizontal acceleration is towards the left (if $x$=+ then $a_x$=−).
- $x$ and $v_y$. When the moon is positioned on the left side of the y-axis, its vertical velocity is downward (if $x$=− then $v_y$=−). If the moon crosses the y-axis, there is no vertical velocity (if $x$=0 then $v_y$=0). When the moon is located on the right side of the y-axis, its vertical velocity is upward (if $x$=+ then $v_y$=+).
- $a_x$ and $v_y$. When the moon its acceleration in the horizontal is directed towards the left, its vertical velocity is upward (if $a_x$=− then $v_y$=+). If the moon has no acceleration in the horizontal direction, there is no vertical velocity (if $a_x$=0 then $v_y$=0). When the moon its acceleration in the horizontal direction is toward the right, its vertical velocity is downward ($a_x$ =+ then $v_y$=−).
- $y$ and $a_y$. When the moon is positioned below the x-axis, its acceleration in the vertical direction is upward (if $y$=− then $a_y$=+). If the moon crosses the x-axis, there is no acceleration in the vertical direction (if $y$=0 then $a_y$=0). When the moon is located above x-axis, its vertical acceleration is downward (if $y$=+ then $a_y$=−).
- $y$ and $v_x$. When the moon is positioned below the x-axis, its horizontal velocity is towards the right (if $y$=− then $v_x$=+). If the moon crosses the x-axis, there is no horizontal velocity (if $y$=0 then $v_x$=0). When the moon is located above the x-axis, its horizontal velocity is towards the left (if $y$=+ then $v_y$=−).
- $a_y$ and $v_x$. When the moon its acceleration in the vertical direction is downward, its horizontal velocity is to the left (if $a_y$=− then $v_x$=−). If the moon has no acceleration in the vertical direction, there is no horizontal velocity (if $a_y$=0 then $v_x$=0). When the moon its acceleration in the vertical direction is upward, its horizontal velocity is to the right ($a_y$ =+ then $v_x$=+).

**Table** 2. Correspondences between quantity spaces in circular motion. The correspondences establish the co-occurrence of values of quantities of the horizontal and vertical direction of circular motion.

| | *value* | $x$ | $y$ | $a_x$ | $a_y$ | $v_x$ | $v_y$ |
|---|---|---|---|---|---|---|---|
| $x$ | − | | | $+^*$ | | | $-^*$ |
| | 0 | | | $0^*$ | | | $0^*$ |
| | + | | | $-^*$ | | | $+^*$ |
| $y$ | − | | | | $+^*$ | $+^*$ | |
| | 0 | | | | $0^*$ | $0^*$ | |
| | + | | | | $-^*$ | $-^*$ | |
| $a_x$ | − | $+^*$ | | | | | $+^*$ |
| | 0 | $0^*$ | | | | | $0^*$ |
| | + | $-^*$ | | | | | $-^*$ |
| $a_y$ | − | | $+^*$ | | | $-^*$ | |
| | 0 | | $0^*$ | | | $0^*$ | |
| | + | | $-^*$ | | | $+^*$ | |
| $v_x$ | − | | $+^*$ | | $-^*$ | | |
| | 0 | | $0^*$ | | $0^*$ | | |
| | + | | $-^*$ | | $+^*$ | | |
| $v_y$ | − | $-^*$ | | $+^*$ | | | |
| | 0 | $0^*$ | | $0^*$ | | | |
| | + | $+^*$ | | $-^*$ | | | |

* bi-directional correspondence

We can now add correspondences between quantity spaces of the horizontal and vertical motion. We only add four bi-directional correspondences (indicated by an arrow point on both sides) to the qualitative representation (Fig. 3) because by adding the correspondence between $x$ and $v_y$ and $v_y$ and $a_x$, the correspondence between $x$ and $a_x$ becomes redundant. The same logic applies to the correspondence between $y$ and $y_x$ after adding the correspondences between $y$ and $v_x$ and $v_x$. and $a_y$. Note that we could have discarded other correspondences (or added them all). We made the decision to include correspondences between quantities of both directions, as they explicitly communicate the interdependence of the pendulum movements.

### 2.5 Simulation of the complete model

The representation is now ready and can be simulated. The starting condition for simulating the full representation is x=<+,?> and y=<0,?> which corresponds to state 1 in Figure 1 and Table 1. The state graph (Fig. 3, RHS) shows that the system has eight states.

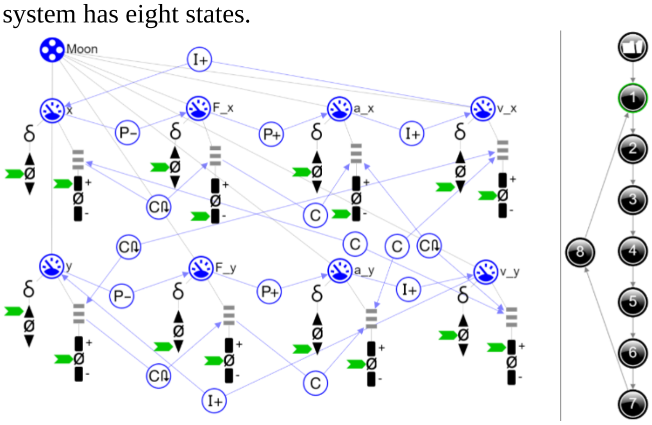


**Fig. 3.** Qualitative representation of circular motion. The vertical motion (with quantities $y$, $F_y$, $a_y$ and $v_y$) is comparable to the horizontal motion (with quantities $x$, $F_x$, $a_x$ and $v_x$). The simulation generates 8 states, as show in Fig. 3 on the RHS. The simulation result of state 1 is shown. An important insight concerns the four correspondences between the two motions.

Fig 4. Shows the value history of quantities $x$, $a_x$, $v_x$, $y$, $a_y$ and $v_y$ in the eight states. The value history shows the quantities, their possible values, their actual value, and their direction of change in each state. For instance, the quantity $x$ in state 1 is positive and its change of direction is zero. By adding the correspondences, the motion in the vertical direction is now half a period out of phase with the horizontal motion. The sinusoidal patterns define the typical behavior observed in simple harmonic motion.

The relationship between position, velocity, and acceleration in simple harmonic motion can be summarized as follows: when an object is at its equilibrium position, the velocity is maximum and the acceleration is zero. For example, in state 3, $x$ is at its equilibrium point on the x-axis and its direction of change is negative <0, −> and acceleration in the horizontal direction ($a_x$) is zero and its direction of change is positive <0, +>, i.e., the moon is in its equilibrium point on the x-axis and there is only gravitational pull in the vertical direction. The velocity in the horizontal direction is maximum in the negative direction <−, 0> , i.e., the moon is moving towards the left.

As the object moves away from the equilibrium position, the velocity decreases, and the acceleration increases in the opposite direction. When the object reaches its maximum displacement, the velocity becomes zero, and the acceleration is at its maximum (in the opposite direction). The cycle repeats as the object returns to the equilibrium position and continues oscillating.

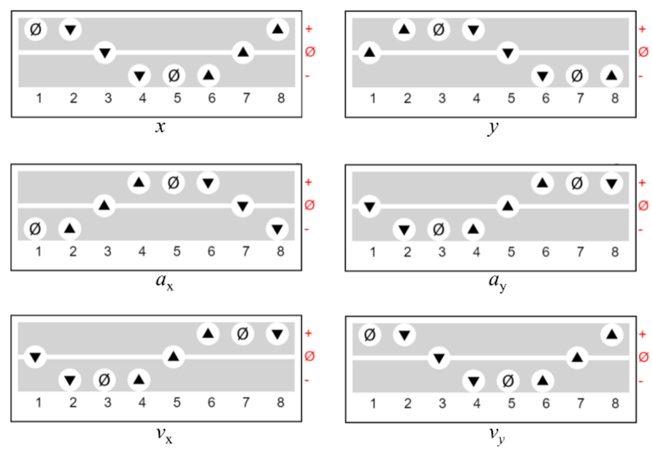


**Fig. 4.** Values history of $x$, $a_x$, $v_x$, $y$, $a_y$ and $v_y$ with regard to the eight states of circular motion.

# 3 Elliptical motion

Elliptical motion can be described by twelve distinct qualitative states (Fig. 5).

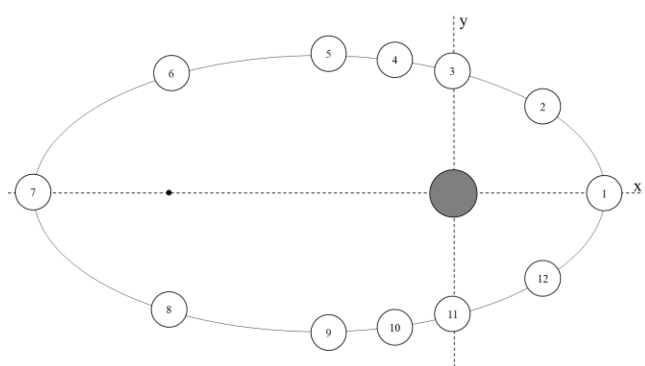


**Fig. 5.** Elliptical motion of a star orbiting a black hole. A system manifesting elliptical motion has twelve qualitatively distinct states.

A concrete example is a star orbiting a black hole, where the black hole is in one of the focal points of the ellipse. Within an elliptical orbit, as the distance from the black hole changes, the gravitational force exerted on the star varies, leading to corresponding alterations in acceleration. The equilibrium between gravitational force and the star's

**Table 3.** Twelve qualitative states of elliptical motion. Quantities are position: x-axis ($x$) and y-axis ($y$), acceleration: horizontal ($a_x$) and vertical ($a_y$), and velocity: horizontal ($v_x$) and vertical ($v_y$). Each quantity has a value and a direction of change, shown as <v,∂>. Force corresponds to acceleration. Force is not shown in this table.

| | State | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Quantity | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| $x$ | <+, 0> | <+, –> | <0, –> | <–, –> | <–, –> | <–, –> | <–, 0> | <–, +> | <–, +> | <–, +> | <0, +> | <+, +> |
| $y$ | <0, +> | <+, +> | <+, +> | <+, +> | <+, 0> | <+, –> | <0, –> | <–, –> | <–, 0> | <–, +> | <–, +> | <–, +> |
| $a_x$ | <–, 0> | <–, +> | <0, +> | <+, +> | <+, +> | <+, +> | <+, 0> | <+, –> | <+, –> | <+, –> | <0, –> | <–, –> |
| $a_y$ | <0, –> | <–, –> | <–, –> | <–, –> | <–, 0> | <–, +> | <0, +> | <+, +> | <+, 0> | <+, –> | <+, –> | <+, –> |
| $v_x$ | <0, –> | <–, –> | <–, 0> | <–, +> | <–, +> | <–, +> | <0, +> | <+, +> | <+, +> | <+, +> | <0, +> | <+, –> |
| $v_y$ | <+, 0> | <+, –> | <+, –> | <+, –> | <0, –> | <–, –> | <–, 0> | <–, +> | <0, +> | <+, +> | <+, +> | <+, +> |

momentum gives rise to the elliptical shape of the orbit. The specific shape of the ellipse depends on the starting situation of the object's motion, such as its distance, velocity, and direction relative to the central body. However, regardless of the specific shape, the presence of twelve states remains constant.

## 3.1 Expected qualitative states

Table 3 shows the values and directions of change for each of the quantities in the twelve states. States 1, 2, 6, 7, 8 and 12 in elliptical motion are similar to states 1, 2, 4, 5, 6, 8 in circular motion, respectively. In elliptical motion, there are six distinct states (3, 4, 5, 9, 10, and 11) that do not exist in circular motion, whereas states 3 and 7 in circular motion do not exist in elliptical motion. Although the relationships between position, force, acceleration, and velocity still govern the movements in both the horizontal and vertical directions, they are interdependent in a distinct manner compared to circular motion.

## 3.2 Completing the model

Table 4 shows the correspondences of elliptical motion and marks the differences with circular motion. Four bi-directional correspondences are the same as in circular motion: between $x$ and $a_x$, $a_x$ and $v_x$, $y$ and $a_x$, and $y$ and $v_x$. The other two correspondences (between $x$ and $v_y$, and $a_x$ and $v_y$) are different compared to circular motion: the values that correspond may differ, the correspondence can change from bi-directional to directed, or there may be no correspondence at all. Because in circular motion all correspondences are bi-directional, we will describe the specific changes for each pair of corresponding values in the context of elliptical motion below:

- $x$ and $v_y$:
  - *(i)* In circular motion: if $x$=− then $v_y$=−. In elliptical motion there is no correspondence between $x$=− and values of $v_y$. That is, when the star is positioned on the left side of the y-axis ($x$=−), its vertical velocity is either downward ($v_y$=− in states 6, 7 and 8), it has no vertical velocity ($v_y$=0 in states 4 and 10), or vertical velocity is upward ($v_y$=+ in states 3 and 11). In elliptical motion the correspondence in the other direction (if $v_y$=− then $x$=−) is directed. When the star its vertical velocity is downward $v_y$=−, its position is on the left side of the y-axis ($x$=− in states 6, 7 and 8). This correspondence is directed because when the star is on the left side of the y-axis ($x$=−), it can also have no velocity in the vertical direction ($v_y$=0 in states 5 and 9) or its vertical velocity is upward ($v_y$=+ in states 4 and 10).
  - *(ii)* In circular motion: if $x$=0 then $v_y$=0. In elliptical motion, when the star crosses the y-axis its vertical velocity is upward (if $x$=0 then $v_y$=+ in state 3 and 11). So the value of this correspondence changed and it is now directed because the star its vertical velocity is also upward ($v_y$=+) when it is on the on the left ($x$=− in states 4 and 10) or on the right side of the y-axis ($x$=+ in states 1, 2, and 12). The correspondence in the other direction (if $v_y$=0 then $x$=0) changed its value and is now directed. When the star has no

Table 4. Correspondences in elliptical motion. The correspondences establish the co-occurrence of values of quantities of the horizontal and vertical direction of elliptical motion.

| | *value* | $x$ | $y$ | $a_x$ | $a_y$ | $v_x$ | $v_y$ |
|---|---|---|---|---|---|---|---|
| $x$ | – | | | $+^{*}$ | | | $^{3}$ |
| | 0 | | | $0^{*}$ | | | $+^{1,2}$ |
| | + | | | $-^{*}$ | | | $+^{2}$ |
| $y$ | – | | | | $+^{*}$ | $+^{*}$ | |
| | 0 | | | | $0^{*}$ | $0^{*}$ | |
| | + | | | | $-^{*}$ | $-^{*}$ | |
| $a_x$ | – | $+^{*}$ | | | | | $+^{2}$ |
| | 0 | $0^{*}$ | | | | | $+^{1,2}$ |
| | + | $-^{*}$ | | | | | $^{3}$ |
| $a_y$ | – | | $+^{*}$ | | | $-^{*}$ | |
| | 0 | | $0^{*}$ | | | $0^{*}$ | |
| | + | | $-^{*}$ | | | $+^{*}$ | |
| $v_x$ | – | | $+^{*}$ | | $-^{*}$ | | |
| | 0 | | $0^{*}$ | | $0^{*}$ | | |
| | + | | $-^{*}$ | | $+^{*}$ | | |
| $v_y$ | – | $-^{2}$ | | $+^{2}$ | | | |
| | 0 | $-^{1,2}$ | | $+^{1,2}$ | | | |
| | + | $^{3}$ | | $^{3}$ | | | |

$^{*}$ bi-directional correspondence; differences compared to circular motion: $^{1}$value differs, $^{2}$ correspondence changed from bi-directional to directed, $^{3}$ no correspondence anymore.

vertical velocity, its position is on the left side of y-axis (if $v_y$=0 then $x$=− in states 5 and 9). This correspondence is directed because when the star its position is on the left side of the y-axis ($x$=−), its vertical velocity can be upward ($v_y$=+ in states 4 and 10) or downward ($v_y$=− in states 6, 7 and 8).

*(iii)* In circular motion: if $x$=+ then $v_y$=+. In elliptical motion, when the position of the star is on the right side of the y-axis, its vertical velocity is also upward (if $x$=+ then $v_y$=+ in states 1, 2, and 12), but the correspondence is now directed. It is directed because the star its vertical velocity is also upward ($v_y$=+) in states 3 and 11 ($x$=0) and states 4 and 10 ($x$=−). The correspondence in the other direction (if $v_y$=+ then $x$=+) does not exist in elliptical motion because the star its vertical velocity is upward ($v_y$=+) through the full quantity space of $x$ {−, 0, +}.

- $a_x$ and $v_y$:

*(i)* In circular motion: if $a_x$=− then $v_y$=+. In elliptical motion, when the star its acceleration in the horizontal direction is towards the left, its vertical velocity is also upward (if $a_x$=− then $v_y$=+ in states 1, 2, and 12). However, this correspondence is directed in elliptical motion because the star its vertical velocity is also upward ($v_y$=+) when horizontal acceleration is to the left ($a_x$=− states 4 and 10) or when there is no horizontal acceleration ($a_x$=0 in states 3 and 11). Therefore, the correspondence in the other direction (if $v_y$=+ then $a_x$=−) does not exist in elliptical motion.

*(ii)* In circular motion: if $a_x$=0 then $v_y$=0. In elliptical motion, when the star has no acceleration in the horizontal direction, its vertical velocity is upward (if $a_x$=0 then $v_y$=+ in states 3 and 11). This correspondence is directed because the star its vertical velocity is also upward ($v_y$=+) when acceleration in the horizontal direction is to the left ($a_x$=− in states 2 and 12) and to the right ($a_x$=+ in states 4 and 10). The value of the correspondence in the other direction (if $v_y$=0 then $a_x$=0) has changed. When the star is has no velocity in the vertical direction, the gravitational pull and thereby the acceleration in the horizontal direction is toward the right (if $v_y$=0 then $a_x$=+ in states 5 and 9). The correspondence is directed, because when the star its acceleration in the horizontal direction is to the right ($a_x$=+), velocity in the vertical direction can be downward ($v_y$=− in states 6, 7 and 8) or upward ($v_y$=+ in states 4 and 10).

*(iii)* In circular motion: if $a_x$=+ then $v_y$=−. In elliptical motion there is no correspondence between $a_x$=+ and values of $v_y$. That is, when the star its horizontal acceleration is to the right ($a_x$=+), its vertical velocity is downward ($v_y$=− in states 6, 7 and 8), ), it has no vertical velocity ($v_y$=0 in state 3 and 11), or its vertical motion is velocity ($v_y$=+ in state 4 and 10). Therefore, the correspondence in the other direction (if $v_y$=− then $a_x$=+) is directed.

We can now add the correspondences between quantity spaces of both directions (vertical and horizontal) to describe elliptical motion (Fig. 6). As mentioned before, we do not need to add all correspondences from Table 4 because adding one correspondence can make another redundant. We add the bi-directional correspondences that are the same as in circular motion. We also add the directed correspondences that define states 3 and 11 (if $x$=0 then $v_y$=+) and states 5 and 9 (if $v_y$=0 then x=−).

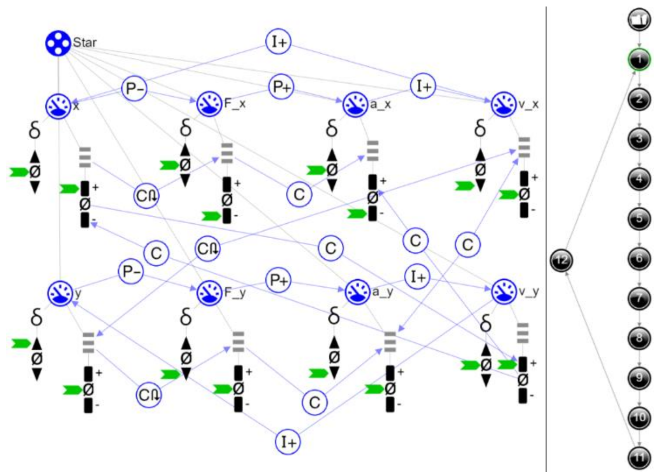


**Fig. 6.** Qualitative representation of elliptical motion. The simulation generates 12 states. The simulation result of state 1 is shown.

## 3.3 Simulation of the complete model

The representation can be simulated with initial conditions that correspond to state 1 in Fig. 5: x=<+,?> , y=<0,?> and $v_x$ = <+,?>. The latter initial condition is needed because there is no correspondence that automatically sets the value of $v_x$ in state 1.

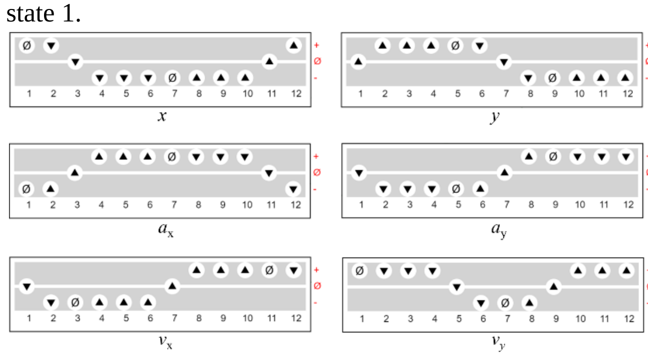


**Fig. 7.** Values history of $x$, $a_x$, $v_x$, $y$, $a_y$ and $v_y$ with regard to the twelve states of elliptical motion.

Fig 7. Shows the value history of quantities $x$, $a_x$, $v_x$, $y$, $a_y$ and $v_y$ in the twelve states of elliptical motion. While motions in both directions still exhibit sinusoidal patterns, it is important to note that in the case of elliptical motion, the system no longer strictly adheres to simple harmonic motion. The varying changes in gravitational force introduce complexities that deviate from the characteristics of circular motion in both directions.

## 4 Conclusion and discussion

In this paper, we present qualitative representations of circular and elliptical motion. The motions are depicted on a x- and y-coordinate plane. This allows for the decomposition of motion into a horizontal and vertical direction. To describe the dynamics of circular and elliptical motion, the representations include the quantities: position ($x$, $y$), force ($F_x$, $F_y$), acceleration ($a_x$, $a_y$), and velocity ($v_x$, $v_y$). The quantities have a quantity space that encompasses negative, zero, and positive values, hence $\{-, 0, +\}$. Note that force, acceleration and velocity are vectors and their qualitative value indicate both value *and* direction.

We describe the dependencies between quantities and the correspondences that exist in both circular and elliptical motion. Specifically, we focus on the correspondences between horizontal and vertical motion and highlight the differences between circular and elliptical motion.

Circular motion can be described by eight qualitatively distinct states, featuring six bi-directional correspondences between the quantities in the horizontal and vertical direction. When these correspondences are added to the representation, the system's behavior follows a pattern of two simple harmonic motions that are half a period out of phase.

Elliptical motion consists of twelve distinct qualitative phases. The dependencies between the quantities in both directions are similar to circular motion. However, compared to circular motion, there are changes in two correspondences: (*i*) between the horizontal position ($x$) and velocity in the vertical direction ($v_y$), and (*ii*) between acceleration in the vertical direction ($a_x$) and velocity in the vertical direction ($v_y$). These changes manifest in different ways: the values that correspond may differ, the correspondence itself may transition from being bi-directional to directed, or in some cases, there is no correspondence at all between certain values. These variations in correspondences highlight the distinct nature of elliptical motion compared to circular motion.

In conclusion, qualitative representations, such as the ones presented in this paper, offer an alternative approach to describing and understanding circular and elliptical motion, bypassing the traditional mathematical methods. By constructing qualitative representations, learners can gain valuable insights into the behavior of these systems, fostering a deeper comprehension of the concepts involved [Kragten & Bredeweg, 2023]. Future research aimed at continuous improvement of the pedagogical approach should examine how students learn optimally by constructing such representations and identify the essential support they need during the learning process.

## 5 References

[Alonzo & Steedle, 2009] Alicia Alonzo & Jeffrey Steedle. Developing and assessing a force and motion learning progression. *Science Education*, *93*(3), 389–421, 2009.

[Barniol & Zavala, 2014] Pablo Barniol & Genaro Zavala. Test of understanding of vectors: A reliable multiple-choice vector concept test. *Physical Review Special Topics-Physics Education Research*, *10*(1), 010121, 2014.

[Bredeweg et al., 2009] Bert Bredeweg, Floris Linnebank, Anders Bouwer, & Jochem Liem (2009). Garp3 — Workbench for qualitative modelling and simulation. *Ecological Informatics*, *4*(5-6), 263-281.

[Bredeweg et al., 2013] Bert Bredeweg, Jochem Liem, Wouter Beek, Floris Linnebank, Jorge Gracia, Esther Lozano,… David Mioduser. DynaLearn - An Intelligent Learning Environment for Learning Conceptual Knowledge. *AI Magazine*, *34*(4), 46-65, 2013.

[Bredeweg et al., 2023a] Bert Bredeweg, Marco Kragten, Joanna Holt, Patricia Kruit, Tom van Eijck, Monique Pijls,… Muriel de Boer. Learning with Interactive Knowledge Representations. *Applied Sciences*, *13*(9), 5256, 2023.

[Bredeweg et al., 2023b] Bert Bredeweg, Marco Kragten, Joanna Holt, Dennis Vaendel, Joris Hanse, & Stef Bloemen. Stargazing Live! Inspiring with Real Data in a Mobile Planetarium and Learning Through Conceptual Modelling. In *ITS: 19th International Conference*, LNCS 13891, 257-269, 2023.

[Canlas, 2016] Ian Canlas. University students' alternative conceptions on circular motion. *International Journal of Scientific & Technology Research*, *5*(3), 2277-8816, 2016.

[Kragten et al., 2022] Marco Kragten, Emile Jaspar, & Bert Bredeweg. Learning Domain Knowledge and Systems Thinking using Qualitative Representations in Secondary Education (grade 10-12). Paper presented at 35th *International Workshop on Qualitative Reasoning*, Vienna, Austria, 2022.

[Kragten & Bredeweg, 2023] Marco Kragten & Bert Bredeweg. Learning about circular motion of celestial bodies with interactive qualitative representations. *Artificial Intelligence in Education*: 24nd International Conference, AIED, 2023.

[Liu & Fang, 2016] Gang Liu & Ning Fang. Student misconceptions about force and acceleration in physics and engineering mechanics education. *International Journal of Engineering Education, 32*(1), 19–29, 2016.