

# Exam

## Asymptotic Statistics Mastermath stochastics

Re-sit exam

Date: Wednesday 17 January 2024

Time: 9.45–12.45h

Number of pages: 4 (including front page)

Number of questions: 3

Maximum number of points: 75

For each question is indicated how many points it is worth.

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### BEFORE YOU START

- Check if your version of the exam is complete.
- Write down **your name, student ID number**, and if applicable the **version number** on **each sheet** that you hand in. Also **number the pages**.
- Your **mobile phone** has to be switched off and be put in your coat or bag. Your **coat and bag** should be on the ground.
- Upper-right of first page **Write: Name + University + page-nr. on EVERY page**

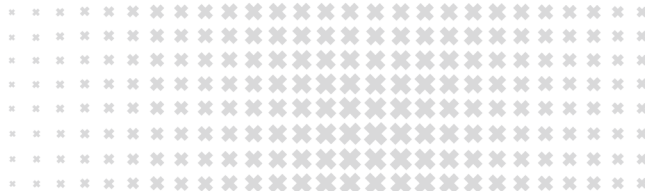
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### PRACTICAL MATTERS

- The first 30 minutes you are not allowed to leave the room, not even to visit the toilet.
- 15 minutes before the end, you will be warned that the time to hand in is approaching.
- If applicable, fill out the evaluation form at the end of the exam.
- You are obliged to identify yourself at the request of the examiner (or his representative) with a proof of your registration and a valid ID.
- During the examination it is not permitted to visit the toilet, unless the invigilator gives permission to do so.

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**Good luck!**

**Problem 1** (*Two estimators for the Poisson parameter*)

Let  $X_1, X_2, \dots$  be an *i.i.d.* sample from a Poisson distribution with parameter  $\lambda > 0$ . That means that,

$$P(X_i = k) = \frac{e^{-\lambda} \lambda^k}{k!},$$

for all  $i \geq 1, k \geq 0$ . Define two sequences of statistics  $(T_n)$  and  $(S_n)$  as follows:

$$S_n = \frac{1}{n} \sum_{i=1}^n 1\{X_i = 0\}, \quad T_n = \frac{1}{n} \sum_{i=1}^n 1\{X_i = 1\},$$

a. (5 points)

Show that  $\log(1/S_n)$  and  $T_n/S_n$  are consistent estimators for the parameter  $\lambda$ .

b. (5 points)

Show that  $\hat{\lambda}_1 = \log(1/S_n)$  is asymptotically normal and give the limit distribution.

c. (5 points)

Show that  $\hat{\lambda}_2 = T_n/S_n$  is asymptotically normal and give the limit distribution.

d. (5 points)

Describe what the relative efficiency of two estimator sequences signifies (e.g. through sample sizes). Based on your answers at parts *b.* and *c.*, calculate the relative efficiency of  $\hat{\lambda}_1$  and  $\hat{\lambda}_2$ .



**Problem 2** (A transformed exponential distribution)

Let  $X_1, X_2, \dots$  be *i.i.d.* non-negative real-valued random variables with single-observation distribution  $P_{\mu_0}$  and Lebesgue density  $p_\mu : \mathbb{R} \rightarrow [0, \infty)$  for some  $\mu_0 > 0$ , with  $p_\mu(x) = 0$  for  $x < 0$ , and

$$p_\mu(x) = 2\mu x e^{-\mu x^2},$$

for  $x \geq 0$  and  $\mu > 0$ . A change of variables  $Z = X^2$  leads to  $Z \sim \text{Exp}(\mu)$ .

*Hint: you may use the following integrals,*

$$\int_0^\infty x^2 e^{-x^2} dx = \frac{\sqrt{\pi}}{4}, \int x^3 e^{-x^2} dx = -\frac{e^{-x^2}(x^2 + 1)}{2} + C, \int x^5 e^{-x^2} dx = -\frac{e^{-x^2}(x^4 + 2x^2 + 2)}{2} + C,$$

a. (5 points)

Find the maximum-likelihood estimator  $\hat{\mu}_n$  for  $\mu_0$  based on the first  $n$  sample points  $X_1, \dots, X_n$ .

b. (5 points)

Calculate the expectation  $E_\mu X_1^2$  of the second moment for a single observation and show that  $\hat{\mu}_n$  is a consistent estimator sequence for  $\mu_0$ .

c. (5 points)

Calculate the Fisher information  $I_\mu$  for a single observation  $X_1$ , show that  $\mu \mapsto I_\mu$  is continuous and that  $I_\mu > 0$  for all  $\mu > 0$ .

d. (5 points)

Show that, for any  $x > 0$ , the map  $\mu \mapsto \log p_\mu(x)$  is Lipschitz in a neighbourhood of  $\mu_0$ . In other words, prove that for some  $\epsilon > 0$  and any  $\mu_1, \mu_2 > 0$  such that  $|\mu_1 - \mu_0| < \epsilon$  and  $|\mu_2 - \mu_0| < \epsilon$ ,

$$|\log p_{\mu_1}(x) - \log p_{\mu_2}(x)| \leq \dot{\ell}(x) |\mu_1 - \mu_2|,$$

for some measurable function  $\dot{\ell} : \mathbb{R} \rightarrow \mathbb{R}$  such that  $E_{\mu_0} \dot{\ell}^2 < \infty$ .

*Hint: for any  $\mu_1, \mu_2 \geq \mu > 0$ , we have  $|\log \mu_1 - \log \mu_2| \leq |\mu_1 - \mu_2|/\mu$ .*

e. (5 points)

State a theorem from the lecture notes and use parts a.–d. to prove that  $\sqrt{n}(\mu_n - \mu_0)$  is asymptotically normal under  $P_{\mu_0}$ . Give the variance of the limit distribution.

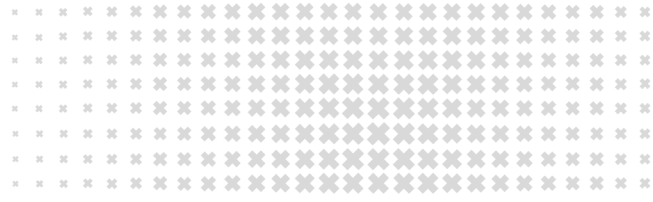
The moment estimator for  $\mu_0$  is,

$$\tilde{\mu}_n = \frac{\pi}{4} \left( \frac{1}{\bar{X}_n} \right)^2,$$

where  $\bar{X}_n$  denotes the sample average.

f. (5 points)

Use the delta rule to find the limit distribution for  $\sqrt{n}(\tilde{\mu}_n - \mu_0)$ .


**Problem 3** (*Domain boundary estimation*)

Let  $Y_1, Y_2, \dots$  be an *i.i.d.* sample from the uniform distribution  $P_\theta$  on  $[0, \theta]$ , for some  $\theta > 0$ . The distribution function for  $X \sim U[0, 1]$  is given by  $P(X \leq 0) = 0$ ,  $P(X \leq x) = x$ , ( $0 < x \leq 1$ ),  $P(X \leq 1) = 1$ . Denote the maximum of the first  $n$  observations by  $Y_{(n)} = \max\{Y_1, \dots, Y_n\}$ . (*Hint: in the problem, you may use that  $(1 + a/n)^n \rightarrow e^a$  as  $n \rightarrow \infty$ , for any  $a \in \mathbb{R}$ .*)

a. (5 points)

Given  $n \geq 1$ , find the maximum-likelihood estimator  $\hat{\theta}_n$  for  $\theta$ , based on the first  $n$  observations.

b. (5 points)

Show that, for given  $n \geq 1$  and  $\theta$ , the distribution function of satisfies,

$$P(Y_{(n)} \leq x) = \left(\frac{x}{\theta}\right)^n,$$

for all  $0 < x \leq \theta$ .

c. (5 points)

Show that  $\hat{\theta}_n$  is consistent for estimation of  $\theta$ .

d. (5 points)

Show that  $n(\theta - \hat{\theta}_n)$  converges weakly and give the limit distribution.

Given any estimators  $\tilde{\theta}_n$  for the parameter  $\theta$ , define the *bias*  $\Delta_n$  of  $\tilde{\theta}_n$  by the ( $\theta$ -dependent) expectation  $\Delta_n = P_\theta^n(\tilde{\theta}_n - \theta)$ .

e. (5 points)

For every  $n \geq 1$ , give the bias  $\Delta_n$  of  $\hat{\theta}_n$ . Find a real-valued sequence  $(a_n)$  such that the bias of the estimators  $a_n \hat{\theta}_n$  is exactly zero.