UNIVERSITY OF AMSTERDAM

Faculty of Science

Exam

Asymptotic Statistics

Mastermath stochastics

Mid-term exam
Date: Wednesday 25 October 2023
Time: 10.00–12.00

Number of pages: 3 (including front page)
Number of questions: 2
Maximum number of points: 60
For each question is indicated how many points it is worth.

BEFORE YOU START

- Check if your version of the exam is complete.
- Write down **your name, student ID number**, and if applicable the **version number** on **each sheet** that you hand in. Also **number the pages**.
- Your **mobile phone** has to be switched off and be put in your coat or bag. Your **coat and bag** should be on the ground.
- Upper-right of first page **Write: Name + University + page-nr. on EVERY page**

PRACTICAL MATTERS

- The first 30 minutes you are not allowed to leave the room, not even to visit the toilet.
- 15 minutes before the end, you will be warned that the time to hand in is approaching.
- If applicable, fill out the evaluation form at the end of the exam.
- You are obliged to identify yourself at the request of the examiner (or his representative) with a proof of your registration and a valid ID.
- During the examination it is not permitted to visit the toilet, unless the invigilator gives permission to do so.

Good luck!

Problem 1 *(Binomials weakly converging to Poisson)*

Let N denote the set of all non-negative integers $\{0, 1, 2, 3, \ldots\}$ and let $X, X_1, X_2, \ldots$ denote random variables taking values in N.

a. *(15 points)*
Show that $X_n \rightsquigarrow X$, if and only if, $P(X_n = x) \to P(X = x)$ for each $x \in N$, as $n \to \infty$.

Recall that for $Y \sim \mathrm{Bin}(m, p)$ and $Z \sim \mathrm{Poisson}(\lambda)$,

$$P(Y = x) = \binom{m}{x} p^x (1-p)^{m-x}, \qquad P(Z = x) = \frac{e^{-\lambda}\,\lambda^x}{x!},$$

give the densities with respect to the counting measure on N.

b. *(15 points)*
Assume that $X_n \sim \mathrm{Bin}(n, p_n)$ with $p_n \in [0, 1]$ for all $n \geq 1$. Show that if, for some constant $\lambda > 0$, $n\,p_n \to \lambda$ as $n \to \infty$, then the sequence (X_n) converges weakly to $\mathrm{Poisson}(\lambda)$.

Hint: In your calculation of $P(X_n = x)$, use Stirling's approximation for the factorials $n!$ and $(n - x)!$:

$$\frac{k!}{\sqrt{2\pi\,k}} \left(\frac{k}{e}\right)^{-k} \to 1,$$

as $k \to \infty$.

Problem 2 *(Uniform integrability)*

Let (X_n) be a sequence of random variables and let X be a random variable. Recall that X_n converges to X *in probability* (notation: $X_n \xrightarrow{P} X$), if for every $\epsilon > 0$, $P(|X_n - X| > \epsilon) \to 0$ as $n \to \infty$. For random variables (X_n) and X that are integrable, we define another form of stochastic convergence as follows: X_n converges to X *in expectation* (or *in* $L_1(P)$, notation: $X_n \xrightarrow{L_1} X$), if $E|X_n - X| \to 0$ as $n \to \infty$.

a. *(10 punten)*
   Show that if $X_n \xrightarrow{L_1} X$, then also $X_n \xrightarrow{P} X$.

b. *(5 punten)*
   Construct an example of a sequence (X_n) and a limiting random variable X such that $X_n \xrightarrow{P} X$, but *not* $X_n \xrightarrow{L_1} X$.

c. *(5 punten)*
   Suppose that $X_n \xrightarrow{P} X$ and prove that if there exist constants $M > 0$ and $N \geq 1$ such that for all $n \geq N$, $P(|X_n| \leq M) = 1$, then $X_n \xrightarrow{L_1} X$.

From the above, it is clear that convergence in expectation implies convergence in probability but the converse is not true in general. The converse *does* hold under the sufficient condition of part *c.*. The question arises whether a sharp extra condition exists, *i.e.* a condition that is not just sufficient but also necessary for convergence in expectation (when combined with convergence in probability). We say that the sequence (X_n) is *uniformly integrable*, if

$$\lim_{M \to \infty} \sup_{n \geq 1} E|X_n| 1\{|X_n| > M\} = 0.$$

d. *(5 punten)*
   Show that if (X_n) is uniformly integrable and $X_n \xrightarrow{P} X$, then $X_n \xrightarrow{L_1} X$.

e. *(5 punten)*
   Show that if $X_n \xrightarrow{L_1} X$, then (X_n) is uniformly integrable.

Remark: the above constitutes a proof that $X_n \xrightarrow{L_1} X$*, if and only if* $X_n \xrightarrow{P} X$ *and* (X_n) *is uniformly integrable.*

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