

Exam

Asymptotic Statistics Mastermath stochastics

Mid-term exam

Date: Wednesday 25 October 2023

Time: 10.00–12.00

Number of pages: 3 (including front page)

Number of questions: 2

Maximum number of points: 60

For each question is indicated how many points it is worth.

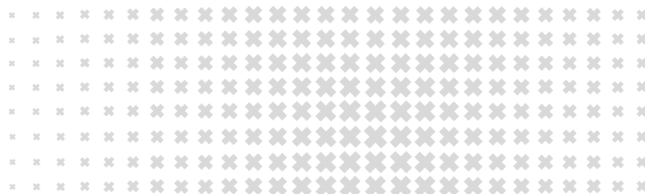
BEFORE YOU START

- Check if your version of the exam is complete.
- Write down **your name, student ID number**, and if applicable the **version number** on **each sheet** that you hand in. Also **number the pages**.
- Your **mobile phone** has to be switched off and be put in your coat or bag. Your **coat and bag** should be on the ground.
- Upper-right of first page **Write: Name + University + page-nr. on EVERY page**

PRACTICAL MATTERS

- The first 30 minutes you are not allowed to leave the room, not even to visit the toilet.
- 15 minutes before the end, you will be warned that the time to hand in is approaching.
- If applicable, fill out the evaluation form at the end of the exam.
- You are obliged to identify yourself at the request of the examiner (or his representative) with a proof of your registration and a valid ID.
- During the examination it is not permitted to visit the toilet, unless the invigilator gives permission to do so.

Good luck!

**Problem 1** (*Binomials weakly converging to Poisson*)

Let N denote the set of all non-negative integers $\{0, 1, 2, 3, \dots\}$ and let X, X_1, X_2, \dots denote random variables taking values in N .

a. (15 points)

Show that $X_n \rightsquigarrow X$, if and only if, $P(X_n = x) \rightarrow P(X = x)$ for each $x \in N$, as $n \rightarrow \infty$.

Recall that for $Y \sim \text{Bin}(m, p)$ and $Z \sim \text{Poisson}(\lambda)$,

$$P(Y = x) = \binom{m}{x} p^x (1-p)^{m-x}, \quad P(Z = x) = \frac{e^{-\lambda} \lambda^x}{x!},$$

give the densities with respect to the counting measure on N .

b. (15 points)

Assume that $X_n \sim \text{Bin}(n, p_n)$ with $p_n \in [0, 1]$ for all $n \geq 1$. Show that if, for some constant $\lambda > 0$, $n p_n \rightarrow \lambda$ as $n \rightarrow \infty$, then the sequence (X_n) converges weakly to $\text{Poisson}(\lambda)$.

Hint: In your calculation of $P(X_n = x)$, use Stirling's approximation for the factorials $n!$ and $(n-x)!$:

$$\frac{k!}{\sqrt{2\pi k}} \left(\frac{k}{e}\right)^{-k} \rightarrow 1,$$

as $k \rightarrow \infty$.


Problem 2 (Uniform integrability)

Let (X_n) be a sequence of random variables and let X be a random variable. Recall that X_n converges to X in probability (notation: $X_n \xrightarrow{P} X$), if for every $\epsilon > 0$, $P(|X_n - X| > \epsilon) \rightarrow 0$ as $n \rightarrow \infty$. For random variables (X_n) and X that are integrable, we define another form of stochastic convergence as follows: X_n converges to X in expectation (or in $L_1(P)$, notation: $X_n \xrightarrow{L_1} X$), if $E|X_n - X| \rightarrow 0$ as $n \rightarrow \infty$.

a. (10 punten)

Show that if $X_n \xrightarrow{L_1} X$, then also $X_n \xrightarrow{P} X$.

b. (5 punten)

Construct an example of a sequence (X_n) and a limiting random variable X such that $X_n \xrightarrow{P} X$, but *not* $X_n \xrightarrow{L_1} X$.

c. (5 punten)

Suppose that $X_n \xrightarrow{P} X$ and prove that if there exist constants $M > 0$ and $N \geq 1$ such that for all $n \geq N$, $P(|X_n| \leq M) = 1$, then $X_n \xrightarrow{L_1} X$.

From the above, it is clear that convergence in expectation implies convergence in probability but the converse is not true in general. The converse *does* hold under the sufficient condition of part c.. The question arises whether a sharp extra condition exists, *i.e.* a condition that is not just sufficient but also necessary for convergence in expectation (when combined with convergence in probability). We say that the sequence (X_n) is *uniformly integrable*, if

$$\lim_{M \rightarrow \infty} \sup_{n \geq 1} E|X_n|1_{\{|X_n| > M\}} = 0.$$

d. (5 punten)

Show that if (X_n) is uniformly integrable and $X_n \xrightarrow{P} X$, then $X_n \xrightarrow{L_1} X$.

e. (5 punten)

Show that if $X_n \xrightarrow{L_1} X$, then (X_n) is uniformly integrable.

Remark: the above constitutes a proof that $X_n \xrightarrow{L_1} X$, if and only if $X_n \xrightarrow{P} X$ and (X_n) is uniformly integrable.

