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Existence and phase structure of random inverse limit measures

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Part I

The Bourbaki-Prokhorov-Schwartz theorem

Daniell-Kolmogorov existence theorem (I)

Setting

Let $\mathscr{X}$ be a Polish space. To define a random function $f : \mathscr{X} \to \mathbb{R}$, consider all *finite* subsets $S = \{s_1, \ldots, s_n\}$ of $\mathscr{X}$, and probability distributions Π_S such that,

$$f_S = \big(f(s_1), f(s_2), \ldots, f(s_n)\big) \sim \Pi_S.$$

Consistency

for any $S' \subset S$, $\Pi_{S'}$ is marginal to Π_S;

for any permutation π of S, $\Pi_{\pi(S)} = \Pi_S \circ \pi^{-1}$.

Theorem 3.1 *(Daniell, 1922; Kolmogorov, 1933)*
For any consistent collection $(\Pi_S : S \subset \mathscr{X})$, there exists a probability space $(\Omega, \mathscr{F}, \Pi)$ that permits $(f(x) : x \in \mathscr{X})$ as a stochastic process.

Daniell-Kolmogorov existence theorem (II)

Advantages

THE tool to prove *existence of stochastic processes*

Π_S are easy to work with

Properties of Π_S induce properties of Π

Example *(Kolmogorov's continuity theorem)*

If there exist $\alpha, \beta > 0$ such that, for any S and any $s, t \in S$,

$$\mathbb{E}_{\Pi_S}\left|f(s) - f(t)\right|^{\alpha} \leq K|s-t|^{1+\beta},$$

then f is γ-Hölder continuous for any $0 < \gamma < \beta/\alpha$.

Disadvantage

$\Omega = \mathbb{R}^{\mathscr{X}}$ and $\mathscr{F}$ is Borel σ-algebra for pointwise convergence

Random histograms

Specify

Let $\mathscr{X}$ be a Hausdorff space with Borel σ-algebra $\mathscr{B}$. To define a random measure $\mu : \mathscr{B} \to \mathbb{R}$, consider *finite* partitions $\alpha = \{A_1, \ldots, A_n\}$ of $\mathscr{X}$, $(A \in \mathscr{B}, A \neq \varnothing)$, and probability distributions Π_α such that,

$$\mu_\alpha = \Big(\mu(A_1), \mu(A_2), \ldots, \mu(A_n)\Big) \sim \Pi_\alpha.$$

Coherence

For any $\beta \geq \alpha$, with $\mu_\beta \sim \Pi_\beta$,

$$\left(\sum_{B \subset A_1} \mu_\beta(B), \ldots, \sum_{B \subset A_n} \mu_\beta(B)\right) \sim \Pi_\alpha.$$

Goal

Under which conditions does a coherent system of random histograms define a probability distribution Π on the space $M(\mathscr{X})$ where the μ live?

The Bourbaki-Prokhorov-Schwartz theorem (I)

Theorem 6.1 *(Bourbaki (1969), Integration II, Ch. 9)*
Let $(\mathscr{Y}_\alpha, \psi_{\alpha\beta})$ be an inverse system of Hausdorff spaces, T a Hausdorff space and $\psi_\alpha : T \to \mathscr{Y}_\alpha$ a coherent and separating family of continuous mappings.
Let $(\mu_\alpha, \psi_{\alpha\beta})$ be a coherent inverse system of positive measures on $(\mathscr{Y}_\alpha, \psi_{\alpha\beta})$. There exists a bounded positive Radon measure μ on T projecting to μ_α for all α, if and only if,

for every $\epsilon > 0$, there is a compact $H \subset T$ s.t. for all α,

$$\mu_\alpha\Big(\mathscr{Y}_\alpha \setminus \psi_\alpha(H)\Big) \leq \epsilon.$$

The Bourbaki-Prokhorov-Schwartz theorem (II)

Setting

Let $\mathscr{X}$ be Hausdorff with Borel σ-algebra $\mathscr{B}$. Choose $T = M^1(\mathscr{X})$, with a Hausdorff topology that we focus on later.

Projections

For all $\alpha = \{A_1, \ldots, A_n\}$, define histogram projections,

$$\varphi_{*\,\alpha} : M^1(\mathscr{X}) \to M^1(\mathscr{X}_\alpha) : P \mapsto P_\alpha = \big(P(A_1), P(A_2), \ldots, P(A_n)\big),$$

and maps to coarsen histograms, for $\beta \geq \alpha$,

$$\varphi_{*\,\alpha\beta} : M^1(\mathscr{X}_\beta) \to M^1(\mathscr{X}_\alpha) : P_\beta \mapsto \left(\sum_{B \subset A_1} P_\beta(B), \ldots, \sum_{B \subset A_n} P_\beta(B) \right).$$

($\varphi_{*\,\alpha} = \varphi_{*\,\alpha\beta} \circ \varphi_{*\,\beta}$, $(\alpha \leq \beta)$, and $\varphi_{*\,\alpha\gamma} = \varphi_{*\,\alpha\beta} \circ \varphi_{*\,\beta\gamma}$, $(\alpha \leq \beta \leq \gamma)$.)

The Bourbaki-Prokhorov-Schwartz theorem (III)

Coherence and random histograms

For any α, choose a probability distribution $\Pi_\alpha \in M^1(\mathscr{X}_\alpha)$ s.t., for all $\beta \geq \alpha$,

$$\Pi_\beta \circ \varphi_{*\,\alpha\beta}^{-1} = \Pi_\alpha.$$

Bourbaki-Prokhorov-Schwartz

Assume that the histogram projections $\varphi_{*,\alpha}$ are separating and continuous. Choose Π_α that form a coherent system of probability measures. There exists a Radon probability measure Π on $M^1(\mathscr{X})$, projecting to Π_α for all α, if and only if:

for any $\epsilon > 0$, there is a compact $H \subset M^1(\mathscr{X})$ s.t. for all α,

$$\Pi_\alpha\Big(M^1(\mathscr{X}_\alpha) \setminus \varphi_{*\,\alpha}(H)\Big) < \epsilon. \qquad \text{(P)}$$

Part II

Phases of random histogram limits

Histogram limits with the weak topology (I)

Weak topology

Consider $M^1(\mathscr{X})$ with the coarsest topology $\mathscr{T}_W$ s.t.,

$$M^1(\mathscr{X}) \to \mathbb{R} : P \mapsto \int f \, dP,$$

is continuous for every bounded, measurable $f : \mathscr{X} \to \mathbb{R}$.

Dunford-Pettis-Grothendieck

$H \subset M^1(\mathscr{X})$ is weakly compact, if and only if, there exists a $Q \in M^1(\mathscr{X})$ s.t.,

$$\lim_{L \to \infty} \sup_{P \in H} \left\| P - P \wedge LQ \right\| = 0.$$

Histogram limits with the weak topology (II)

Support of a $\mathscr{T}_W$-Radon probability measure Π

With $G = \int P \, d\Pi \in M^1(\mathscr{X})$, (the *mean measure* of Π),

$$\mathrm{supp}_W(\Pi) \subset \left\{ P \in M^1(\mathscr{X}) : P \ll G \right\}.$$

Such Π describe random Radon-Nikodym densities $dP/dG \in L^1(G)$.

Theorem 11.1 *(Existence of weak histogram limits)*
Let Π_α be coherent probability measures. There is a $\mathscr{T}_W$-Radon probability measure Π on $M^1(\mathscr{X})$ projecting to Π_α for all α, if and only if:

there is a $Q \in M^1(\mathscr{X})$ s.t., for every $\epsilon, \delta > 0$ there is a $L > 0$ s.t.,

$$\Pi_\alpha\Big(\{ P_\alpha \in M^1(\mathscr{X}_\alpha) : \| P_\alpha - P_\alpha \wedge L Q_\alpha \|_{1,\mathscr{X}_\alpha} > \delta \} \Big) < \epsilon, \qquad \text{(PW)}$$

for all $\alpha \in \mathscr{A}$.

Random histogram limits with the TV topology

Total variational topology

Consider $M^1(\mathscr{X})$ with the total-variational metric,

$$d_{TV}(P,Q) = \sup_{B \in \mathscr{B}} |P(B) - Q(B)|,$$

and call the metric topology $\mathscr{T}_{TV}$.

Borel σ-algebras are the same!

If $\mathscr{X}$ is separable and $\mathscr{P}$ is dominated, $\mathscr{B}_W = \mathscr{B}_{TV}$.

Theorem 12.1 *(Existence of total-variational histogram limits)* *Let Π_α be coherent probability measures. There is a $\mathscr{T}_{TV}$-Radon probability measure Π on $M^1(\mathscr{X})$ projecting to Π_α for all α, if and only if, condition (PW) holds.*

Random histogram limits with the tight topology (I)

Tight topology

Consider $M^1(\mathscr{X})$ with the coarsest topology $\mathscr{T}_T$ s.t.,

$$M^1(\mathscr{X}) \to \mathbb{R} : P \mapsto \int f \, dP,$$

is continuous for every bounded, *continuous* $f : \mathscr{X} \to \mathbb{R}$.

Prokhorov

Let $\mathscr{X}$ be Polish. $H \subset M^1(\mathscr{X})$ is tightly compact, if and only if, for all $\epsilon > 0$, there is a compact $K \subset \mathscr{X}$ s.t.,

$$\sup_{P \in H} P(\mathscr{X} \setminus K) < \epsilon,$$

On H *inner regularity* holds uniformly.

Continuity of projections

The mappings $P \mapsto P(A)$ are *not continuous*! So the histogram projections $\varphi_{*\alpha}$ are not continuous...

Random histogram limits with the tight topology (I)

Continuity of projections

To make $P \mapsto P(A)$ continuous for all A in all α, we consider a zero-dimensional refinement $\mathscr{Y}$ of $\mathscr{X}$.

Tight topology

Consider $M^1(\mathscr{Y})$ with the coarsest topology $\mathscr{T}_T$ s.t.,

$$M^1(\mathscr{Y}) \to \mathbb{R} : P \mapsto \int f\,dP,$$

is continuous for every bounded, *continuous* $f : \mathscr{Y} \to \mathbb{R}$.

Prokhorov

Let $\mathscr{Y}$ be Polish. $H \subset M^1(\mathscr{Y})$ is tightly compact, if and only if, for all $\epsilon > 0$, there is a compact $\hat{K} \subset \mathscr{Y}$ s.t.,

$$\sup_{P \in H} P(\mathscr{Y} \setminus \hat{K}) < \epsilon,$$

On H *inner regularity* holds uniformly.

Random histogram limits with the tight topology (II)

Support of a $\mathscr{T}_T$-Radon probability measure Π

With G again the mean measure of Π,

$$\text{supp}_T(\Pi) \subset \Big\{ P \in M^1(\mathscr{X}) : \text{supp}(P) \subset \text{supp}(G) \Big\}.$$

Such Π are *not limited* to Radon-Nikodym densities in $L^1(G)$.

Theorem 15.1 *(Existence of tight histogram limits)*
Let Π_α be coherent probability measures. There is a $\mathscr{T}_T$-Radon probability measure Π on $M^1(\mathscr{X})$ projecting to Π_α for all α, if and only if:

for all $\epsilon, \delta > 0$ there is a compact $\hat{K}$ in $\mathscr{Y}$ s.t.,

$$\Pi_\alpha\Big(\{P_\alpha \in M^1(\mathscr{X}_\alpha) : P_\alpha(\mathscr{X}_\alpha \setminus \hat{K}_\alpha) > \delta\Big) < \epsilon, \qquad \text{(PT)}$$

for all $\alpha \in \mathscr{A}$.

Random histogram limits on compactifications

Compactification

If (PT) does not hold, $\mathscr{Y}$ needs more points

Stone-Čech compactification

compact Hausdorff $\check{\mathscr{Y}}$, $i : \mathscr{Y} \to \check{\mathscr{Y}}$ continuous, injective, dense

Partitions α of $\check{\mathscr{Y}}$ have histograms of $i^{-1}(\alpha)$

Condition (PT) and theorem 15.1 hold ($\widehat{K} = \check{\mathscr{Y}}$)

Theorem 16.1 *(Existence of histogram limits on compactifications) Let* Π_α *be coherent probability measures. There is a* $\mathscr{T}_T$*-Radon probability measure* Π *on* $M^1(\check{\mathscr{Y}})$ *projecting to* Π_α *for all* α*.*

Kingman's completely random measures

Completely random histograms

If $A_i \cap A_j = \varnothing$, then $\nu(A_i), \nu(A_j)$ are independent

Cumulants

The positive measures $\lambda_t : \mathscr{B} \to [0, \infty]$ defined by,

$$\lambda_t(B) = \log \int e^{t\nu(B)} \, d\Pi(\nu).$$

Theorem 17.1 *(Kingman, 1967)*
If all histograms are completely random and cumulants σ-finite,

$$\nu = \nu_n + \nu_f + \nu_r, \tag{1}$$

where,

ν_n *is non-random, non-atomic*

ν_f *is random purely atomic on a* fixed $\mathscr{X}' \subset \mathscr{X}$

ν_r *is random purely atomic, independent of* ν_r

Phases of random histogram limits (I)

Theorem 18.1 *(Phases of random histogram limits)*
Let Π_α *be a system of histogram distributions with a limit* Π*.*

(i.) (absolutely-continuous)
Under condition (PW), the random P *lies in* $L^1(G)$*:*

$$\Pi\Big(\{P \in M^1(\mathscr{X}) : P \ll G\}\Big) = 1.$$

(ii.) (fixed-atomic)
if, in addition, the Π_α *are (normalized) completely random,*

$$P(A) = Z^{-1}(\nu_n(A) + \nu_f(A)), \quad Z = \nu_n(\mathscr{X}) + \nu_f(\mathscr{X}).$$

with $\nu_n \ll G$ *non-random, non-atomic and* ν_f *random atomic, supported on a fixed set.*

Phases of random histogram limits (II)

Theorem 18.1 *(continued)*
If $\mathscr{X}$ is Polish,

(iii.) *(continuous-singular)*
Under condition (PT), random P has support in support of G,

$$\Pi\Big(\{P \in M^1(\mathscr{X}) : \mathrm{supp}(P) \subset \mathrm{supp}(G)\}\Big) = 1.$$

(iv.) *(random-atomic)*
if, in addition, histograms are (normalized) completely random,

$$P(A) = Z^{-1}(\nu_n(A) + \nu_f(A) + \nu_r(A)).$$

with ν_r atomic, supported on a random set.

Phases of random histogram limits (III)

Theorem 18.1 *(continued)*
If $\mathscr{X}$ is Polish, $\check{\mathscr{Y}}$ is compact Hausdorff, $\check{G} \in M^1(\check{\mathscr{Y}})$

(iii.) *(compact-singular)*
Random P has support in support of $\check{G}$,

$$\Pi\Big(\{P \in M^1(\mathscr{X}) : \mathrm{supp}(P) \subset \mathrm{supp}(\check{G})\}\Big) = 1.$$

(iv.) *(compact-atomic)*
if, in addition, histograms are (normalized) completely random,

$$P(A) = Z^{-1}(\nu_n(A) + \nu_f(A) + \nu_r(A)).$$

with ν_r atomic, supported on a random set.

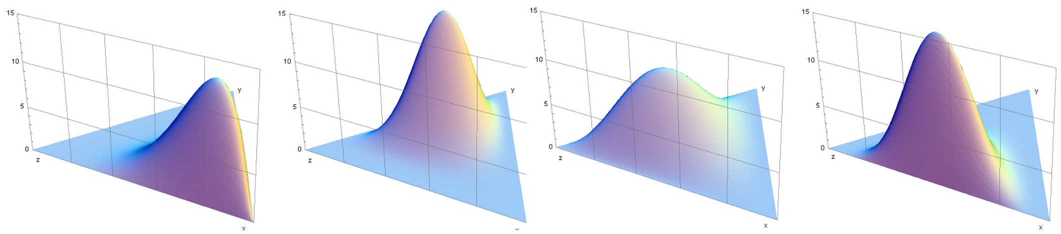
Part III

Applications

Dirichlet histogram systems

Definition 22.1 *(Dirichlet distribution)*
A $p = (p_1, \ldots, p_k)$ $p_l \geq 0$ *and* $\sum_l p_l = 1$ *is Dirichlet distributed with parameter* $\nu = (\nu_1, \ldots, \nu_k)$, $p \sim D_\nu$, *if it has density*

$$f_\nu(p) = C(\nu) \prod_{l=1}^{k} p_l^{\nu_l - 1}$$



Definition 22.2 *(Dirichlet process, Ferguson 1973,1974)*
Let $\mathscr{X}$ *be Polish and let* ν *be a bounded positive Borel measure on* $(\mathscr{X}, \mathscr{B})$. *The Dirichlet histogram system is defined by,*

$$\Big(P(A_1), \ldots, P(A_N)\Big) \sim \mathrm{Dir}_{\nu,\alpha} = D_{(\nu(A_1),\ldots,\nu(A_N))},$$

(for all finite Borel-measurable partitions $\alpha = \{A_1, \ldots, A_N\}$ *of* $\mathscr{X}$*).*

Existence and phases of Dirichlet histogram limits (I)

Dirichlet histograms normalized completely random

For independent, Gamma-distributed $Z_i \sim \Gamma(\nu_i, 1)$, $(1 \leq i \leq N)$,

$$\left(\frac{Z_1}{S}, \ldots, \frac{Z_N}{S}\right) \sim D_{(\nu_1,\ldots,\nu_N)}$$

where $S = Z_1 + \ldots + Z_N$.

Mean measure condition (Orbanz, 2011)

if the histogram means $G_\alpha : \sigma(\alpha) \to [0,1]$,

$$G_\alpha(A) = \int_{M^1(\mathscr{X}_\alpha)} P_\alpha(A) \, d\mathrm{Dir}_{\nu,\alpha}(P_\alpha),$$

together form a measure (on the ring $\mathscr{R} = \cup\{\sigma(\alpha) : \alpha \in \mathscr{A}\}$), then there is a tight histogram limit Dir_ν.

Existence and phases of Dirichlet histogram limits (II)

Theorem 24.1 *Let $\mathscr{X}$ be a Polish space and let ν be a bounded, positive Borel measure on $\mathscr{X}$. Then there is a $\mathscr{T}_T$-Radon probability measure* Dir_ν *on* $M^1(\mathscr{X})$ *projecting to* $\mathrm{Dir}_{\nu,\alpha}$ *for all α, describing a random probability measure in the random atomic phase.*

Theorem 24.2 *Let $\mathscr{X}$ be a Polish space and let ν be a bounded, positive, purely atomic Borel measure on $\mathscr{X}$. Then there is a $\mathscr{T}_W$-Radon probability measure* Dir_ν *on* $M^1(\mathscr{X})$*, projecting to* $\mathrm{Dir}_{\nu,\alpha}$ *for all α, describing a random probability measure in the fixed-atomic phase.*

Pólya tree histogram systems (I)

Infinite splitting

$\mathscr{A} = \{\alpha_m : m \geq 1\}$.

Make α_{m+1} from α_m, by splitting every $A \in \alpha_m$ into two subsets

$\alpha_0 = \{\mathscr{X}\}$, $\alpha_1 = \{A_0, A_1\}$, $\alpha_2 = \{A_{00}, A_{01}, A_{10}, A_{11}\}$, $\ldots$,

Binary sequence labels $\alpha_m = \{A_\varepsilon : \varepsilon = e_1 \ldots e_m \in \mathscr{E}_m\}$.

Splitting variables

Parameters $\beta_{\varepsilon 0}, \beta_{\varepsilon 1} \geq 0$,

Define random $V_{\varepsilon 0} \sim \text{Beta}(\beta_{\varepsilon 0}, \beta_{\varepsilon 1})$ (and $V_{\varepsilon 1} = 1 - V_{\varepsilon 0}$),

If $\varepsilon \neq \varepsilon'$, $V_{\varepsilon 0}$ and $V_{\varepsilon' 0}$ are independent,

$P(A_{\varepsilon 0} | A_\varepsilon) = V_{\varepsilon 0}$, (and $P(A_{\varepsilon 1} | A_\varepsilon) = V_{\varepsilon 1}$)

Pólya tree histogram systems (II)

Random histograms

Random $P_{\alpha_m} = (P_\alpha(A_\varepsilon) : \varepsilon \in \mathscr{E}_m)$,

$$P(A_\varepsilon) = V_{e_1} V_{e_1 e_2} \ldots V_{e_1 \ldots e_m} = \prod_{l=1}^{m} V_{e_1 \ldots e_l}$$

Pólya tree histogram distributions

$$\Big(P(A_\varepsilon) : \varepsilon \in \mathscr{E}_m\Big) \sim \Pi_{\alpha_m}$$

Homogeneous Pólya tree histogram systems

Choose $\beta_m > 0$ for all $m \geq 1$, set $\beta_\varepsilon = \beta_m$, for all $\varepsilon \in \mathscr{E}_m$.

Existence of tight Pólya tree histogram limits

Pólya tree histogram systems on $(0,1]$

Let $\mathscr{X} = (0,1]$ and α_m such that $A_\varepsilon = (l,u]$ for all $\varepsilon \in \mathscr{E}_m$, $(m \geq 1)$. For all $m \geq 0$, $o_m = 0\ldots0 \in \mathscr{E}_m$ and $\iota_m = 1\ldots1 \in \mathscr{E}_m$

Theorem 27.1 *There is a* $\mathscr{T}_T$*-Radon prob msr* Pol_β *on* $M^1((0,1])$ *projecting to* Π_{α_m} *for all* $m \geq 1$*, if and only if,*

$$\prod_{m\geq 0} \frac{\beta_{\varepsilon o_m 0}}{\beta_{\varepsilon o_m 0} + \beta_{\varepsilon o_m 1}} = 0, \qquad (2)$$

for every $\varepsilon \in \mathscr{E}$. Pol_β *describes a random probability measure in the continuous-singular phase.*

Dirichlet-Pólya tree histogram systems

if $\beta_\varepsilon = \beta_{\varepsilon 0} + \beta_{\varepsilon 1}$ for every ε, then $\Pi_{\alpha_m} = \mathrm{Dir}_{\nu,\alpha_m}$. Such Π_{α_m} describe random probability measures in the random atomic phase.

Existence of weak Pólya tree histogram limits

Theorem 28.1 *Let $\mathscr{X}$ be Polish. For given β's, there is a $\mathscr{T}_W$-Radon probability measure Pol_β on $M^1(\mathscr{X})$ projecting to Π_{α_m} for all $m \geq 1$, if,*

$$\sup_{m\geq 1} \sum_{\varepsilon\in\mathscr{E}_m} \prod_{l=1}^{m} \frac{1}{\beta_{\varepsilon_{l-1}0} + \beta_{\varepsilon_{l-1}1}} \left(\frac{\beta_{\hat{\varepsilon}}}{\beta_{\varepsilon_{l-1}0} + \beta_{\varepsilon_{l-1}1} + 1} + \beta_{\varepsilon} \right) < \infty. \qquad (3)$$

where $\hat{\varepsilon}$ denotes ε with the last digit flipped: $\hat{\varepsilon} = \varepsilon_{m-1}(\neg e_m)$. Pol_β describes a random probability measure in the absolutely continuous phase.

Limits of homogeneous Pólya tree systems

If $\beta_m^{-1} = O(m^{-1})$, the homogeneous Pólya tree histogram system has a weak limit. (Compare with $\beta_m^{-1} = O(m^{-2})$ (Kraft, 1964; Ghosal, van der Vaart, 2017)).

Gaussian histogram systems

Definition

(a) Let $\mathscr{X}$ be Polish; let $\mathscr{A}$ be resolving and generated by a basis

(b) Let λ be a bounded signed Radon measure on $\mathscr{X}$

(c) Let Σ be a bounded signed Radon measure on $\mathscr{X} \times \mathscr{X}$, s.t.

  (i) symmetry $\Sigma(A \times B) = \Sigma(B \times A)$ and,

  (ii) For all $\alpha = \{A_1, \ldots, A_k\}$, the $k \times k$ matrix Σ_α

$$\Sigma_{\alpha,ij} = \Sigma(A_i \times A_j),$$

  is positive semi-definite

(d) Define random elements of $M(\mathscr{X}_\alpha)$ by,

$$\Big(\Phi_\alpha(A_1), \ldots, \Phi_\alpha(A_k)\Big) \sim N_k\Big(\lambda_\alpha, \Sigma_\alpha\Big) = \Pi_{\lambda,\Sigma,\alpha},$$

$(N_k(\lambda, \Sigma)$ denotes the multivariate normal distribution on $\mathbb{R}^k)$.

The mean positive measure

Zero-mean Gaussian histograms

If $\Pi_{0,\Sigma,\alpha}$ has a limit Π_{Σ}, then $\Pi_{\lambda,\Sigma,\alpha}$ has a limit $\Pi_{\lambda,\Sigma}$:

$$\Pi_{\lambda,\Sigma}(B) = \Pi_{\Sigma}(B - \lambda).$$

Mean positive G

Given zero-mean Gaussian $\Phi_\alpha \sim \Pi_{0,\Sigma,\alpha}$, the random positive measure $|\Phi_\alpha| : \sigma(\alpha) \to [0,\infty)$ has a mean,

$$\sigma_\alpha(A) = \int_{M(\mathscr{X}_\alpha)} |\Phi_\alpha|(A)\, d\Pi_{0,\Sigma,\alpha}(\Phi_\alpha) = \sqrt{\frac{2}{\pi}} \sum_{A_i \subset A} \sqrt{\Sigma_{\alpha,ii}}$$

Define the set-function $\sigma : \mathscr{R} \to [0,\infty)$,

$$\sigma(A) = \sup\Big\{\sigma_\alpha(A) : \alpha \in \mathscr{A}\Big\}$$

Tight and weak Gaussian histogram limits

Theorem 31.1 *(Weak and tight Gaussian histogram limits)* *Consider* $\Phi_\alpha \sim \Pi_{0,\Sigma,\alpha}$ *based on* $\Sigma \in M(\mathscr{X} \times \mathscr{X})$. *Assume that* σ *is a positive, locally bounded measure.*

If σ *is bounded (e.g. when* $\mathscr{X}$ *is compact) then there is a* $\mathscr{T}_W$*-Radon limiting prob msr* Π_Σ *on* $M(\mathscr{X})$, *describing a random signed measure* Φ *in the absolutely continuous phase.*

If σ *is unbounded then there is a* $\mathscr{T}_T$*-Radon limiting prob msr* Π_Σ *on* $M(\mathscr{X})$, *describing a random signed measure* Φ *in the continuous-singular phase.*

Theorem 31.2 *(Compactified Gaussian histogram limits)* *Consider* $\Phi_\alpha \sim \Pi_{0,\Sigma,\alpha}$ *based on* $\Sigma \in M(\mathscr{X} \times \mathscr{X})$.

There is a $\mathscr{T}_T$*-Radon limiting prob msr* Π_Σ *on* $M(\check{\mathscr{Y}})$, *describing a random signed measure* Φ *in the compact-singular phase.*

The Gaussian Free field in d dimensions

Gaussian free field in d dimensions

Compact $\mathscr{X} = K \subset \mathbb{R}^d$, $\mu = 0$ and $\Sigma_{\Delta,d}(A \times B) = \int_{A\times B} G_d(x-y)\,dx\,dy$

$d = 1$ **GFF is random function (Wiener sample path)**

Theorem 31.1 works

The GFF is in the absolutely-continuous phase and we can write,

$$\Phi(A) = \int_A W(t)\,\mathrm{d}t.$$

The random RN density functions are Wiener sample paths

$d \geq 2$ **GFF is a random generalized function**

Theorem 31.2 works (and theorem 31.1 does not).

The GFF is in the compact-singular phase and we can write,

$$\Phi(A) = \int_{i^{-1}(A)} \phi(x)\,\mathrm{d}^d x$$

where ϕ is a random generalized function (on $\check{\mathscr{Y}}$)