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Random histogram limits and quantum field theory

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Part I

The Bourbaki-Prokhorov-Schwartz theorem

Daniell-Kolmogorov existence theorem (I)

Setting

Let $\mathscr{X}$ be a Polish space. To define a random function $f : \mathscr{X} \to \mathbb{R}$, consider all *finite* subsets $S = \{s_1, \ldots, s_n\}$ of $\mathscr{X}$, and probability distributions Π_S such that,

$$f_S = \big(f(s_1), f(s_2), \ldots, f(s_n)\big) \sim \Pi_S.$$

Consistency

for any $S' \subset S$, $\Pi_{S'}$ is marginal to Π_S;

for any permutation π of S, $\Pi_{\pi(S)} = \Pi_S \circ \pi^{-1}$.

Theorem 3.1 *(Daniell, 1922; Kolmogorov, 1933)*

For any consistent collection $(\Pi_S : S \subset \mathscr{X})$, there exists a probability space $(\Omega, \mathscr{F}, \Pi)$ that permits $(f(x) : x \in \mathscr{X})$ as a stochastic process.

Daniell-Kolmogorov existence theorem (II)

Advantages

THE tool to prove *existence of stochastic processes*

Π_S are easy to work with

Properties of Π_S induce properties of Π

Example *(Kolmogorov's continuity theorem)*

If there exist $\alpha, \beta > 0$ such that, for any S and any $s, t \in S$,

$$\mathbb{E}_{\Pi_S}\Big|X_s - X_t\Big|^\alpha \le K|s-t|^{1+\beta},$$

then f is γ-Hölder continuous for any $0 < \gamma < \beta/\alpha$.

Disadvantage

$\Omega = \mathbb{R}^{\mathscr{X}}$ and $\mathscr{F}$ is Borel σ-algebra for pointwise convergence

Random histograms

Specify

Let $\mathscr{X}$ be a Hausdorff space with Borel σ-algebra $\mathscr{B}$. To define a random measure $\mu : \mathscr{B} \to \mathbb{R}$, consider *finite* partitions $\alpha = \{A_1, \dots, A_n\}$ of $\mathscr{X}$, $(A \in \mathscr{B}, A \neq \varnothing)$, and probability distributions Π_α such that,

$$\mu_\alpha = \Big(\mu(A_1), \mu(A_2), \dots, \mu(A_n)\Big) \sim \Pi_\alpha.$$

Coherence

For any $\beta \geq \alpha$, with $\mu_\beta \sim \Pi_\beta$,

$$\left(\sum_{B \subset A_1} \mu_\beta(B), \dots, \sum_{B \subset A_n} \mu_\beta(B)\right) \sim \Pi_\alpha.$$

Goal

Under which conditions does a coherent system of random histograms define a probability distribution Π on the space $M(\mathscr{X})$ where the μ live?

The Bourbaki-Prokhorov-Schwartz theorem (I)

Theorem 6.1 *(Bourbaki (1969), Integration II, Ch. 9)*
Let $(\mathscr{Y}_\alpha, \psi_{\alpha\beta})$ be an inverse system of Hausdorff spaces, T a Hausdorff space and $\psi_\alpha : T \to \mathscr{Y}_\alpha$ a coherent and separating family of continuous mappings.
Let $(\mu_\alpha, \psi_{\alpha\beta})$ be a coherent inverse system of positive measures on $(\mathscr{Y}_\alpha, \psi_{\alpha\beta})$. There exists a bounded positive Radon measure μ on T projecting to μ_α for all α, if and only if,

for every $\epsilon > 0$, there is a compact $H \subset T$ s.t. for all α,

$$\mu_\alpha\Big(\mathscr{Y}_\alpha \setminus \psi_\alpha(H)\Big) \leq \epsilon.$$

The Bourbaki-Prokhorov-Schwartz theorem (II)

Setting

Let $\mathscr{X}$ be Hausdorff with Borel σ-algebra $\mathscr{B}$. Choose $T = M^1(\mathscr{X})$, with a Hausdorff topology that we focus on later.

Projections

For all $\alpha = \{A_1, \ldots, A_n\}$, define histogram projections,

$$\varphi_{*\,\alpha} : M^1(\mathscr{X}) \to M^1(\mathscr{X}_\alpha) : P \mapsto P_\alpha = \Big(P(A_1), P(A_2), \ldots, P(A_n)\Big),$$

and maps to coarsen histograms, for $\beta \geq \alpha$,

$$\varphi_{*\,\alpha\beta} : M^1(\mathscr{X}_\beta) \to M^1(\mathscr{X}_\alpha) : P_\beta \mapsto \left(\sum_{B \subset A_1} P_\beta(B), \ldots, \sum_{B \subset A_n} P_\beta(B)\right).$$

($\varphi_{*\,\alpha} = \varphi_{*\,\alpha\beta} \circ \varphi_{*\,\beta}$, ($\alpha \leq \beta$), and $\varphi_{*\,\alpha\gamma} = \varphi_{*\,\alpha\beta} \circ \varphi_{*\,\beta\gamma}$, ($\alpha \leq \beta \leq \gamma$).)

The Bourbaki-Prokhorov-Schwartz theorem (III)

Coherence and random histograms

For any α, choose a probability distribution $\Pi_\alpha \in M^1(\mathscr{X}_\alpha)$ s.t., for all $\beta \geq \alpha$,

$$\Pi_\beta \circ \varphi_{*\,\alpha\beta}^{-1} = \Pi_\alpha.$$

Bourbaki-Prokhorov-Schwartz

Assume that the histogram projections $\varphi_{*,\alpha}$ are separating and continuous. Choose Π_α that form a coherent system of probability measures. There exists a Radon probability measure Π on $M^1(\mathscr{X})$, projecting to Π_α for all α, if and only if:

for any $\epsilon > 0$, there is a compact $H \subset M^1(\mathscr{X})$ s.t. for all α,

$$\Pi_\alpha\Big(M^1(\mathscr{X}_\alpha) \setminus \varphi_{*\,\alpha}(H)\Big) < \epsilon. \tag{P}$$

Part II

Phases of random histogram limits

Histogram limits with the weak-star topology (I)

Weak-star topology

Consider $M^1(\mathscr{X})$ with the coarsest topology $\mathscr{T}_W$ s.t.,

$$M^1(\mathscr{X}) \to \mathbb{R} : P \mapsto \int f \, dP,$$

is continuous for every bounded, measurable $f : \mathscr{X} \to \mathbb{R}$.

Dunford-Pettis-Grothendieck

$H \subset M^1(\mathscr{X})$ is weak-star compact, if and only if, there exists a $Q \in M^1(\mathscr{X})$ s.t.,

$$\lim_{L \to \infty} \sup_{P \in H} \left\| P - P \wedge LQ \right\| = 0.$$

Histogram limits with the weak-star topology (II)

Support of a $\mathscr{T}_W$-Radon probability measure Π

With $G = \int P \, d\Pi \in M^1(\mathscr{X})$, (the *mean measure* of Π),

$$\mathrm{supp}_W(\Pi) \subset \left\{ P \in M^1(\mathscr{X}) : P \ll G \right\}.$$

Such Π describe random Radon-Nikodym densities $dP/dG \in L^1(G)$.

Theorem 11.1 *(Existence of weak-star histogram limits)*
Let Π_α be coherent probability measures. There is a $\mathscr{T}_W$-Radon probability measure Π on $M^1(\mathscr{X})$ projecting to $\Pi_\alpha \in M^1(\mathscr{X}_\alpha)$ for all α, if and only if:

there is a $Q \in M^1(\mathscr{X})$ s.t., for every $\epsilon, \delta > 0$ there is a $L > 0$ s.t.,

$$\Pi_\alpha\Big(\{ P_\alpha \in M^1(\mathscr{X}_\alpha) : \| P_\alpha - P_\alpha \wedge L Q_\alpha \|_{1, \mathscr{X}_\alpha} > \delta \} \Big) < \epsilon, \qquad \text{(PW)}$$

for all $\alpha \in \mathscr{A}$.

Random histogram limits with the TV topology

Total variational topology

Consider $M^1(\mathscr{X})$ with the total-variational metric,

$$d_{TV}(P,Q) = \sup_{B \in \mathscr{B}} |P(B) - Q(B)|,$$

and call the metric topology $\mathscr{T}_{TV}$.

Borel σ-algebras are the same!

If $\mathscr{X}$ is separable and $\mathscr{P}$ is dominated, $\mathscr{B}_W = \mathscr{B}_{TV}$.

Theorem 12.1 *(Existence of total-variational histogram limits)*
Let Π_α be coherent probability measures. There is a $\mathscr{T}_{TV}$-Radon probability measure Π on $M^1(\mathscr{X})$ projecting to $\Pi_\alpha \in M^1(\mathscr{X}_\alpha)$ for all α, if and only if, condition (PW) holds.

Random histogram limits with the tight topology (I)

Tight topology

Consider $M^1(\mathscr{X})$ with the coarsest topology $\mathscr{T}_T$ s.t.,

$$M^1(\mathscr{X}) \to \mathbb{R} : P \mapsto \int f \, dP,$$

is continuous for every bounded, *continuous* $f : \mathscr{X} \to \mathbb{R}$.

Prokhorov

Let $\mathscr{X}$ be Polish. $H \subset M^1(\mathscr{X})$ is tightly compact, if and only if, for all $\epsilon > 0$, there is a compact $K \subset \mathscr{X}$ s.t.,

$$\sup_{P \in H} P(\mathscr{X} \setminus K) < \epsilon,$$

On H *inner regularity* holds uniformly.

Continuity of projections

The mappings $P \mapsto P(A)$ are *not continuous*! So the histogram projections $\varphi_{*\,\alpha}$ are not continuous...

Random histogram limits with the tight topology (I)

Continuity of projections

To make $P \mapsto P(A)$ continuous for all A in all α, we consider a zero-dimensional refinement $\mathscr{Y}$ of $\mathscr{X}$.

Tight topology

Consider $M^1(\mathscr{Y})$ with the coarsest topology $\mathscr{T}_T$ s.t.,

$$M^1(\mathscr{Y}) \to \mathbb{R} : P \mapsto \int f \, dP,$$

is continuous for every bounded, *continuous* $f : \mathscr{Y} \to \mathbb{R}$.

Prokhorov

Let $\mathscr{Y}$ be Polish. $H \subset M^1(\mathscr{Y})$ is tightly compact, if and only if, for all $\epsilon > 0$, there is a compact $\hat{K} \subset \mathscr{Y}$ s.t.,

$$\sup_{P \in H} P(\mathscr{Y} \setminus \hat{K}) < \epsilon,$$

On H *inner regularity* holds uniformly.

Random histogram limits with the tight topology (II)

Support of a $\mathscr{T}_T$-Radon probability measure Π

With G again the mean measure of Π,

$$\mathrm{supp}_T(\Pi) \subset \Big\{ P \in M^1(\mathscr{X}) : \mathrm{supp}(P) \subset \mathrm{supp}(G) \Big\}.$$

Such Π are *not limited* to Radon-Nikodym densities in $L^1(G)$.

Theorem 15.1 *(Existence of tight histogram limits)*
Let Π_α be coherent probability measures. There is a $\mathscr{T}_T$-Radon probability measure Π on $M^1(\mathscr{X})$ projecting to $\Pi_\alpha \in M^1(\mathscr{X}_\alpha)$ for all α, if and only if:

for all $\epsilon, \delta > 0$ there is a compact $\hat{K}$ in $\mathscr{Y}$ s.t.,

$$\Pi_\alpha\Big(\{ P_\alpha \in M^1(\mathscr{X}_\alpha) : P_\alpha(\mathscr{X}_\alpha \setminus \hat{K}_\alpha) > \delta \Big) < \epsilon, \qquad \text{(PT)}$$

for all $\alpha \in \mathscr{A}$.

Kingman's completely random measures

Completely random histograms

If $A_i \cap A_j = \varnothing$, then $\nu(A_i), \nu(A_j)$ are independent

Cumulants

The positive measures $\lambda_t : \mathscr{B} \to [0, \infty]$ defined by,

$$\lambda_t(B) = \log \int e^{t\nu(B)} \, d\Pi(\nu).$$

Theorem 16.1 *(Kingman, 1967)*
If all histograms are completely random and cumulants σ-finite,

$$\nu = \nu_n + \nu_f + \nu_r, \tag{1}$$

where,

ν_n *is non-random, non-atomic*

ν_f *is random purely atomic on a* fixed $\mathscr{X}' \subset \mathscr{X}$

ν_r *is random purely atomic, independent of* ν_r

Phases of random histogram limits (I)

Theorem 17.1 *(Phases of random histogram limits)*
Let Π_α *be a system of histogram distributions with a limit* Π.

(i.) (absolutely-continuous)
Under condition (PW), the random P *lies in* $L^1(G)$:

$$\Pi\Big(\{P \in M^1(\mathscr{X}) : P \ll G\}\Big) = 1.$$

(ii.) (fixed-atomic)
if, in addition, the Π_α *are (normalized) completely random,*

$$P(A) = Z^{-1}(\nu_n(A) + \nu_f(A)), \quad Z = \nu_n(\mathscr{X}) + \nu_f(\mathscr{X}).$$

with $\nu_n \ll G$ *non-random, non-atomic and* ν_f *random atomic, supported on a fixed set.*

Phases of random histogram limits (II)

Theorem 17.1 *(continued)*
If $\mathscr{X}$ is Polish,

(iii.) *(continuous-singular)*
Under condition (PT), random P has support in support of G,

$$\Pi\Big(\{P \in M^1(\mathscr{X}) : \mathrm{supp}(P) \subset \mathrm{supp}(G)\}\Big) = 1.$$

(iv.) *(random-atomic)*
if, in addition, histograms are (normalized) completely random,

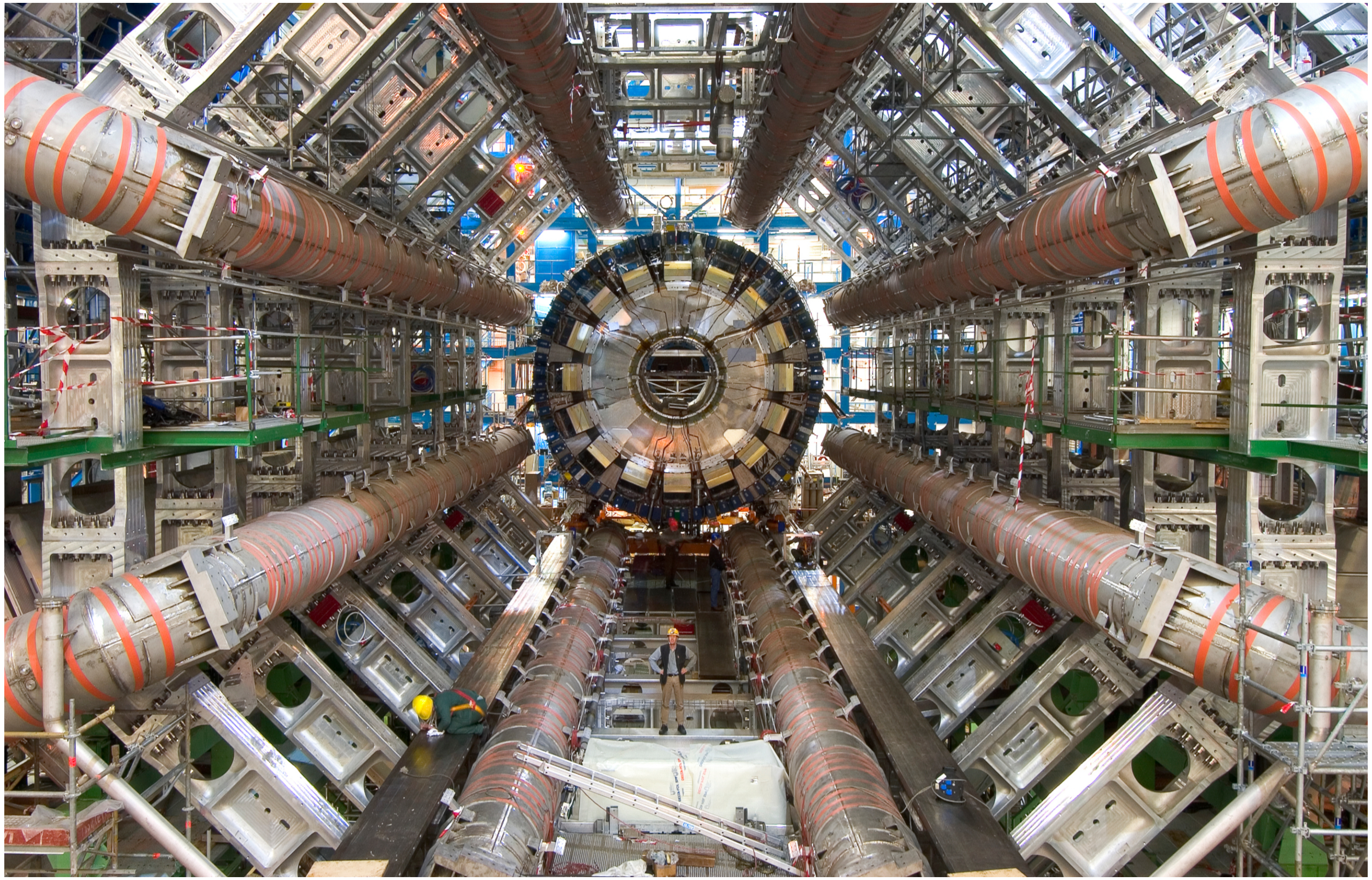
$$P(A) = Z^{-1}(\nu_n(A) + \nu_f(A) + \nu_r(A)).$$

with ν_r atomic, supported on a random set.

Part III

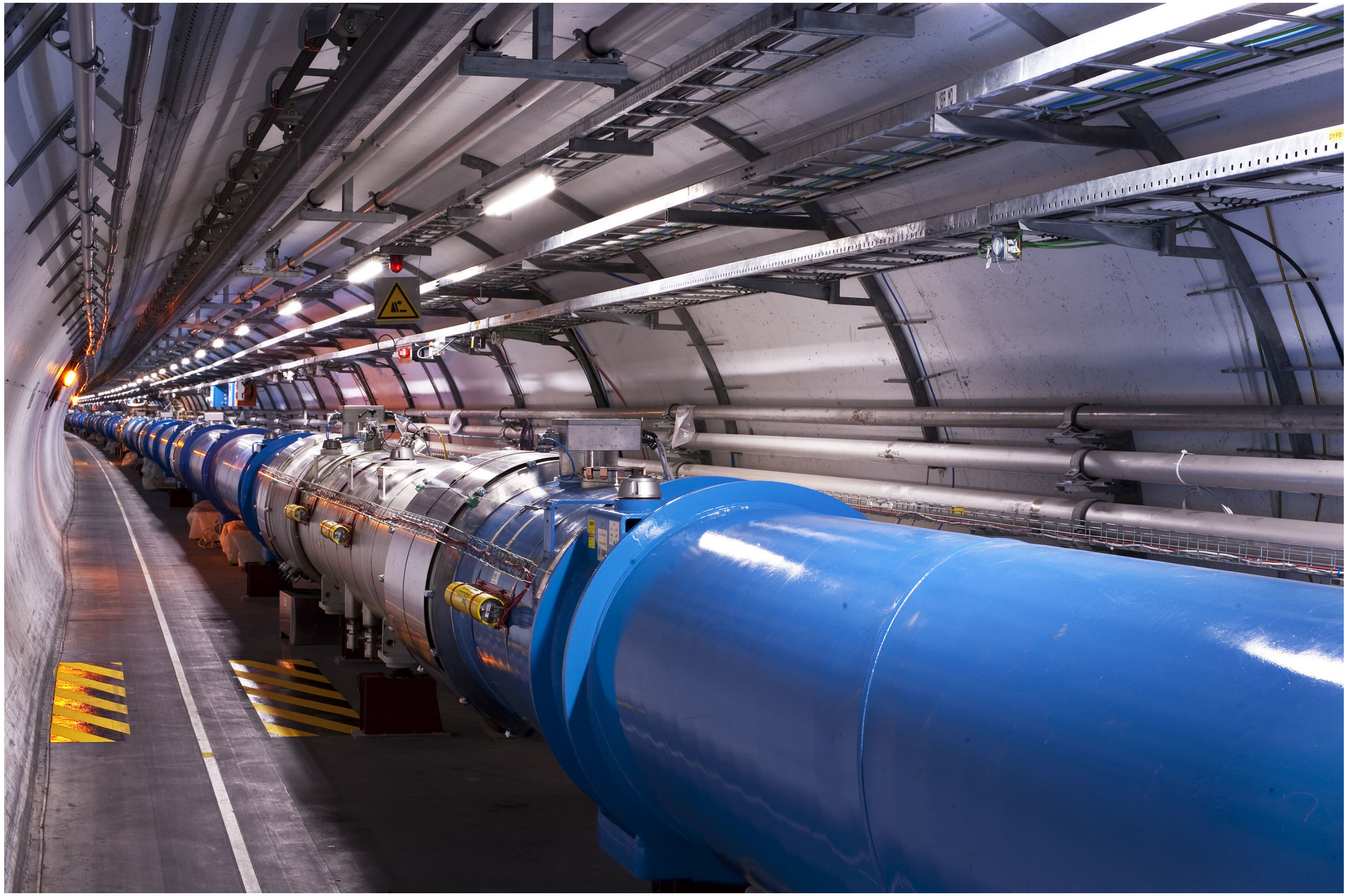
Quantum field theory
and the Gaussian free field

Particle collider experiments



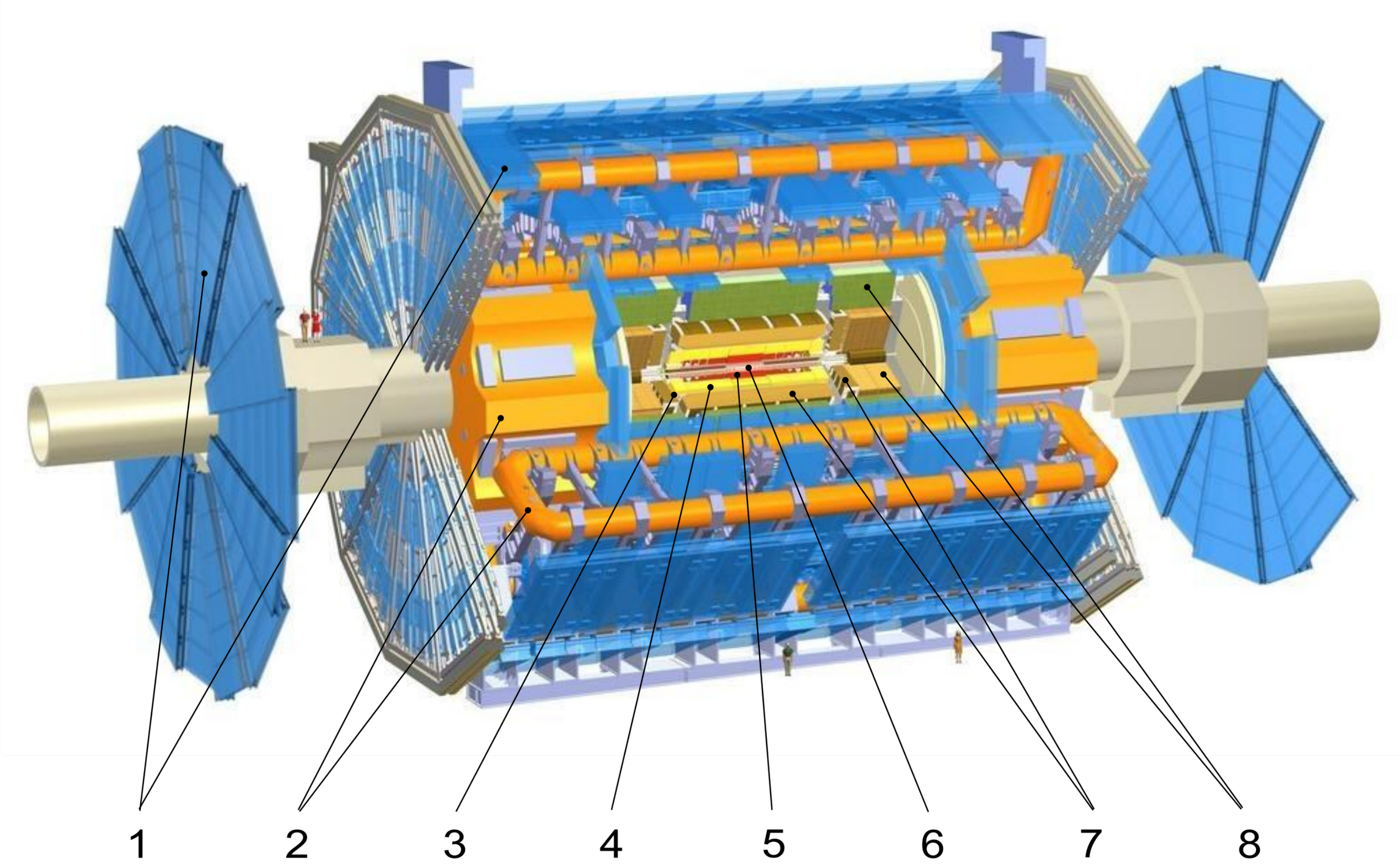
Interior of ATLAS detector (image from CERN, wikipedia)

Particle collider experiments



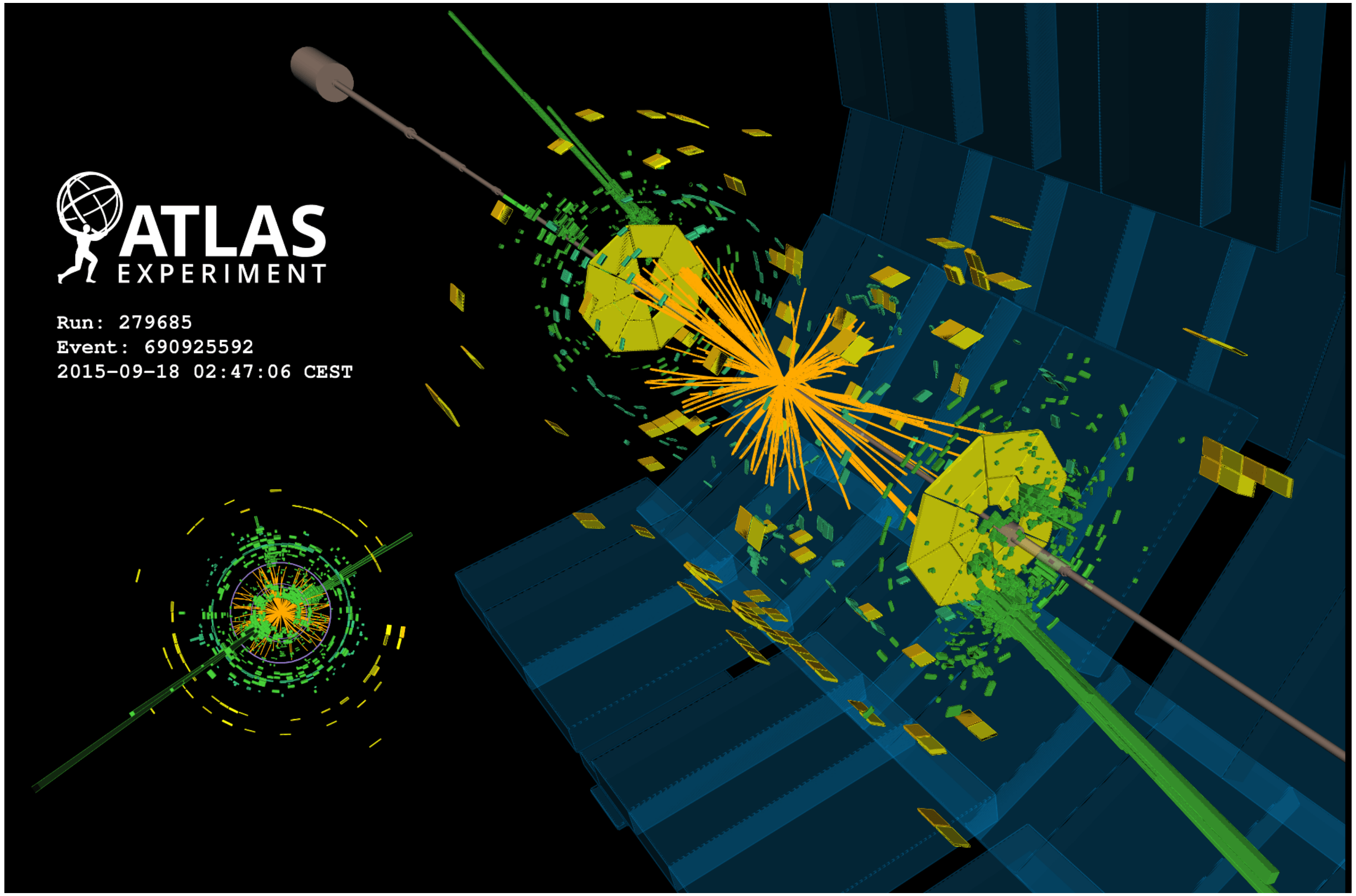
Tunnel with LHC ring (image from CERN, wikipedia)

Particle collider experiments



Schematic of ATLAS detector (image from CERN, wikipedia)

Particle collider experiments



Schematic of ATLAS detector (image from CERN, wikipedia)

Quantum field theory (I)

Hilbert space representation

In-state

$$|\text{in}\rangle = |\vec{k}_1, \vec{k}_2, \vec{k}_3, \ldots\rangle$$

Out-state

$$\langle\text{out}| = \langle\vec{k}_1, \vec{k}_2, \vec{k}_3, \ldots|$$

Amplitude

$$P(\text{in} \to \text{out}) = |\langle\text{out}|\text{in}\rangle|^2$$

Fock space

$$\mathscr{H}_\infty = \overline{\bigoplus_{n=0}^{\infty} S_n \mathscr{H}_1^{\otimes n}}, \qquad \mathscr{H}_1 = \overline{\bigoplus\{\psi_{\vec{k}} = e^{i\vec{k}\cdot\vec{x}} \, : \, \vec{k} \in \mathbb{R}^3\}}$$

Quantum field theory (II)

Wightman functions and path integrals

Sources and sinks

$$\langle \mathsf{out}|\mathsf{in}\rangle = \Big\langle \int \cdots \int j_1(x_1)\phi(x_1)\ldots j_m(x_m)\phi(x_m)\, dx_1 \ldots dx_m \Big\rangle_W$$

Wightman functions (satisfying Wightman axioms)

$$W_m(x_1, x_2, \ldots, x_m) = \Big\langle \phi(x_1)\ldots\phi(x_m)\Big\rangle_W$$

Feynman's *path integral*

$$\Big\langle f(\phi)\Big\rangle_W = Z^{-1}\int_{\mathscr{H}_\infty} f(\phi)\, e^{iS(\phi)}\, \mathscr{D}\phi,$$

Action

$$S(\phi) = \int_{\mathbb{R}^d}\Big(\phi\Delta\phi + m^2\phi^2(x) + \lambda\phi^4(x) + j(x)\phi(x)\Big)d^dx$$

Part IV

The Gaussian Free Field
and emergence of the particle

Gaussian random histogram limits (I)

Signed, non-normalized random histograms

- $\mathscr{X}$ is a Polish space
- μ is a bounded, signed Borel measure on $\mathscr{X}$
- Σ is a bounded, signed Borel measure on $\mathscr{X} \times \mathscr{X}$
  - $\Sigma(A \times B) = \Sigma(B \times A)$,
  - for every α, the matrix Σ_α,

$$\Sigma_{\alpha,ij} = \Sigma(A_i \times A_j)$$

is *positive definite*.

Gaussian random histograms

$$\Phi_\alpha = \Big(\Phi(A_1), \ldots, \Phi(A_n)\Big) \sim N_\alpha = N(\mu_\alpha, \Sigma_\alpha).$$

Gaussian free field in d dimensions

$\mathscr{X} = K \subset \mathbb{R}^d$, $\mu = 0$ and $\Sigma_{\Delta,d}(A \times B) = \int_{A\times B} G_d(x-y)\, dx\, dy$

Phases of Gaussian histogram limits

Theorem 28.1 *(Tight Gaussian histogram limits)*
By σ-additivity of λ and Σ, any Gaussian histogram system is coherent and has a tight limit Π on $M^1(\mathscr{X})$.

Theorem 28.2 *(Weak-star Gaussian histogram limits)*
If,

$$\limsup_{\alpha} \sum_i \sqrt{\Sigma_{\alpha,ii}} < \infty,$$

then the histogram system has a weak-star limit Π.

The Gaussian Free field in d dimensions

$d = 1$ **GFF is random function (Brownian motion)**

Theorem 28.2 works

The GFF is in the absolutely-continuous phase and we can write,

$$\Phi(A) = \int_A B(t)\, dt.$$

The random RN density functions are Brownian paths

$d \geq 2$ **GFF is a random generalized function**

Theorem 28.1 works (and theorem 28.2 does not).

The GFF is in the continuous-singular phase and we can write,

$$\Phi(A) = \int_A \phi(t)\, d^d x$$

where ϕ is a random rank-0 generalized function

Diagonalization of Gaussian histogram limits

Diagonalized covariance

For every α, write $\Sigma_\alpha = O_\alpha^T \circ D_\alpha \circ O_\alpha$ and consider the coherent histogram system,

$$\Psi_\alpha = \Big(\Psi(A_1), \ldots, \Psi(A_n)\Big) \sim N_\alpha = N(0, D_\alpha),$$

with $\psi_{*\,\alpha\beta} = O_\alpha \circ \varphi_{*\,\alpha\beta} \circ O_\beta^T$, $(\beta \geq \alpha)$, and $\psi_{*\,\alpha} = O_\alpha \circ \varphi_{*\,\alpha}$.

Theorem 28.1 works but theorem 28.2 doesn't.

... And such random histograms are completely random,

$$\Psi(A) = \Psi_n(A) + \Psi_f(A) + \Psi_r(A).$$

with,

- $\nu_n \ll \mathbb{E}_{\prod}|\Psi|$ non-random, non-atomic,
- ν_f random atomic, supported on a fixed set,
- ν_r atomic, supported on a random set.

Particles emerge in Gaussian histogram limits

Diagonalization has Fourier transformation as limit
In the limit,

$$\mathbb{E}_{\sqcap}\Psi(A)\Psi(B) = \int_{A\times B} \frac{1}{p^2}\,\delta_d(p-q)\,d^dp\,d^dq$$

Decomposition By completely randomness,

$$\Psi(A) = \Psi_n(A) + \Psi_f(A) + \Psi_r(A).$$

with (in momentum space)

- Ψ_n non-random,
  classical sources, non-zero μ, boundary conditions, solitons
- Ψ_f random atomic, on a fixed set
  on-shell particles, on *mass spectrum*, "physical particles"
- Ψ_r random atomic, on a random set
  off-shell particles, quantum-only "virtual particles"

Wightman functions of the Gaussian free field?

Schwinger functions

For all α and any $A_1, \ldots, A_m \in \alpha$, define $\langle \phi_\alpha(x_1) \ldots \phi_\alpha(x_m) \rangle_S$ by

$$\int_{A_1 \times \cdots \times A_m} \langle \phi_\alpha(x_1) \ldots \phi_\alpha(x_m) \rangle_S \, dx_1 \cdots dx_m = \mathbb{E}_{\Pi_\alpha}(\Phi_\alpha(A_1) \cdots \Phi_\alpha(A_m))$$

Wick rotation *(Osterwalder-Schrader, 1973, 1975)*

If the Schwinger functions satisfy

(E0) Temperedness + linear growth
(E1) Euclidean covariance
(E2) Positivity
(E3) Symmetry
(E4) Cluster property

then they can be continued analytically to Wightman functions.

Part V

Interactions, exact renormalization and effective field theories

Interacting scalar fields in d dimensions

Theorem 34.1 *(Martingale convergence (Doob, 1948))*
For any functions $M_\alpha(\Phi_\alpha) \geq 0$ such that

$$\mathbb{E}_\beta[M_\beta(\Phi_\beta)|\mathscr{F}_\alpha] = M_\alpha(\Phi_\alpha)$$

there exists a Borel-measurable martingale limit $M(\Phi)$ such that,

$$\mathbb{E}_\Pi[M(\Phi)|\mathscr{F}_\alpha] = M_\alpha(\Phi_\alpha)$$

Call $M(\Phi)$ the bare interaction Lagrangian, and define Π_M,

$$p_M(\Phi) = Z_M^{-1}\, e^{-M(\Phi)}, \qquad \Pi_M(C) = \int_C p_M(\Phi)\, d\Pi(\Phi)$$

Corollary 34.2 *Since $\Pi_M \ll \Pi$,*

$$\Pi_M\Big(\Psi(A) = \Psi_n(A) + \Psi_f(A) + \Psi_r(A)\Big) = 1$$

Effective field theory

Define effective interaction Lagrangians,

$$L_\alpha(\Phi_\alpha) = -\log \mathbb{E}[\, p_M(\Phi) | \mathscr{F}_\alpha]$$

Theorem 35.1 *For any* α*,*

$$\begin{aligned}
&\mathbb{E}_{\Pi_M}(\Phi_\alpha(A_1)\cdots\Phi_\alpha(A_m)) \\
&\qquad = \mathbb{E}_{\Pi}\Big(p_M(\Phi)(\Phi_\alpha(A_1)\cdots\Phi_\alpha(A_m)\Big) \\
&\qquad = \mathbb{E}_{\Pi_\alpha}\Big(e^{-L_\alpha(\Phi_\alpha)}(\Phi_\alpha(A_1)\cdots\Phi_\alpha(A_m)\Big) \\
&\qquad = \mathbb{E}_{L,\alpha}\Big(\Phi_\alpha(A_1)\cdots\Phi_\alpha(A_m)),
\end{aligned}$$

Feynman diagrams

Suppose L_α is polynomial in $(\Phi_\alpha(A) : A \in \alpha)$,

$$L_\alpha(\Phi_\alpha) = \sum_{i \in I} \lambda_{\alpha,i}\, \Phi_\alpha(A_1)^{n_{i,1}} \ldots \Phi_\alpha(A_n)^{n_{i,n}}$$

for some finite set I of monomials, and, for the set J of all monomials,

$$\begin{aligned}
&\mathbb{E}_{\Pi_\alpha}\Big(e^{-L_\alpha(\Phi_\alpha)}\Phi_\alpha(A_1)\cdots\Phi_\alpha(A_l)\Big) \\
&\quad = \mathbb{E}_{\Pi_\alpha}\Big(\sum_{m=0}^{\infty} \frac{(-1)^m}{m!} L(\Phi_\alpha)^m \Phi_\alpha(A_1)\cdots\Phi_\alpha(A_l)\Big) \\
&\quad = \sum_{j \in J} \mu_j \mathbb{E}_{\Pi_\alpha}\Big(\Phi_\alpha(A_1)^{m_{j,1}} \ldots \Phi_\alpha(A_j)^{m_{j,n}}\Big)
\end{aligned}$$

Theorem 36.1 *(Isserlis theorem)*
For multivariate-normally distributed $(X_1, \ldots, X_n)$,

$$\mathbb{E}(X_1^{m_1} \ldots X_n^{m_n}) = \sum_p \prod_{(ij) \in p} \mathbb{E}(X_i X_j)$$