1. Let $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$ be a dominated model of distributions on the interval $[0, 1]$. Assume that the parameter space $\Theta$ consists of continuous functions $\theta : [0, 1] \to \mathbb{R}$ and that there exists a constant $C > 0$ such that Hellinger metric distances (see definition (8.2) of the lecture notes) satisfy,

$$h(P_{\theta_1}, P_{\theta_2}) \leq C \|\theta_1 - \theta_2\|_\infty,$$

for all $\theta_1, \theta_2 \in \Theta$. In addition, assume that $\Theta$ is uniformly bounded (i.e., there exists a constant $M > 0$ such that $\|\theta\|_\infty < M$ for all $\theta \in \Theta$) and that the family $\Theta$ is equicontinuous.

a. Show that $N(\delta, \mathcal{P}, h) < \infty$ for all $\delta > 0$.

Let $P_0 \in \mathcal{P}$ be given. As a consequence of the Minimax theorem (see theorem 8.3), there exists a constant $L > 0$ such that, for every Hellinger ball $W$ there is a test sequence $(\phi_n)$ such that,

$$P_0^n \phi_n + \sup_{P \in W} P^n (1 - \phi_n) \leq e^{-nLh(P_0, W)^2},$$

for all $n \geq 1$. (Here, $h(P_0, W) = \inf_{P \in \mathcal{P}} d(P_0, P)$.) Fix some $\epsilon > 0$ and consider the complement $V = \{P \in \mathcal{P} : h(P, P_0) \geq \epsilon\}$ of the Hellinger ball of radius $\epsilon$ around $P_0$. For parts b. and c. below, assume that $\Theta$ is uniformly bounded and equicontinuous.

b. Show that there exists a test sequence $(\psi_n)$ and a constant $L' > 0$ such that,

$$P_0^n \psi_n + \sup_{P \in V} P^n (1 - \psi_n) \leq e^{-nL'\epsilon^2}.$$

c. Vary on the proof of theorem 7.8 to show that the posterior associated with a KL-prior on $\Theta$ is consistent.

2. Approximation in measure from within by compact subsets has a deep background in analysis. Central is the following notion: for a given Hausdorff topological space $\Theta$, a Radon measure $\Pi$ is a Borel measure that is locally finite (meaning that any $\theta \in \Theta$ has a neighbourhood $U$ such that $\Pi(U) < \infty$) and inner regular (meaning that for any measurable subset $S \subset \Theta$ and any $\epsilon > 0$, there exists a compact $K \subset S$ such that $\mu(S \setminus K) < \epsilon$).
a. Let $\Theta$ be a Hausdorff topological space with a finite Borel measure $\Pi$. Denote by $\mathcal{R}$ the collection of all Borel sets $S$ for which,

$$
\Pi(S) = \sup\{\Pi(K) : K \subset S, \text{ } K \text{ compact}\},
$$

$$
\Pi(\Theta \setminus S) = \sup\{\Pi(K) : K \subset \Theta \setminus S, \text{ } K \text{ compact}\}.
$$

Show that $\mathcal{R}$ is a $\sigma$-algebra iff $\Theta \in \mathcal{R}$.

b. Show that any probability measure on a Polish space is Radon.

c. Keeping in mind that most (almost all, in fact) models in non-parametric statistics are parametrized by Polish spaces, comment on the nature of conditions like the first in the displayed inequalities of theorem 7.8, (i) of Theorem 7.9 and inequality (9.2).

3. Prove metric entropy bound (8.3) of lemma 8.6.