Functionaal Analyse

Homework Assignment 1

9 Feb 2017

Problem 1.1

Let (M, d) be a metric space. For $x \in M$ and $r > 0$ define the *open ball* $B_x(r)$ with center x and radius r as,

$$B_x(r) = \{y \in M \,:\, d(x, y) < r\},$$

and the *closed ball* $C_x(r)$ with center x and radius r as,

$$C_x(r) = \{y \in M \,:\, d(x, y) \leq r\}.$$

a. Show that it is *not* true in general that $\overline{B_x(r)} = C_x(r)$, by constructing a counterexample.

Problem 1.2

Consider the space ℓ_1 of absolutely convergent sequences x in $\mathbb{F}$, that is, $x : \mathbb{N} \to \mathbb{F} : k \mapsto x_k$ such that,

$$\|x\|_1 = \sum_{k=1}^{\infty} |x_k| < \infty.$$

(Throughout this problem we do *not* use theorem 1.61 in R and Y.)

a. Show that $\|\cdot\|_1 : \ell_1 \to \mathbb{R}$ is a norm on ℓ_1.

Let $\{x_n\}$ be a Cauchy sequence in ℓ_1. (To avoid confusion, note that each individual x_n is a map $\mathbb{N} \to \mathbb{F} : k \mapsto x_{n,k}$.)

b. Show that for each $k \geq 1$, the sequence $\{x_{n,k}\}$ is Cauchy in $\mathbb{F}$.

c. Prove that ℓ_1 with the norm $\|\cdot\|_1$ is complete.

PROBLEM 1.3

Endow the upper halfplane $M = \{(x, y) \in \mathbb{R}^2 : y \geq 0\}$ in $\mathbb{R}^2$ with a topological basis defined in the following way: we consider, for every point $(x, y) \in M$ with $y > 0$ and every $0 < \epsilon < y$, the neighbourhood $\{(z_1, z_2) \in M : \|(x - z_1, y - z_2)\| < \epsilon\}$; furthermore, for every $(x, 0) \in M$ and every $\epsilon > 0$, we consider the neighbourhood $\{x\} \cup \{(z_1, z_2) \in M : \|(x - z_1, \epsilon - z_2)\| < \epsilon\}$. (In these definitions $\|\cdot\|$ denotes the usual, Euclidean norm on $\mathbb{R}^2$.) We denote the resulting topological space by $(M, \mathscr{T})$.

a. Prove that M is separable in the topology $\mathscr{T}$.

b. Prove that M is first-countable but *not* second-countable in the topology $\mathscr{T}$.

c. Show that M has a subspace that is *not* separable.

d. Argue that $(M, \mathscr{T})$ is not metrizable. Make the argument twice, showing that for a metric space: firstly *a.* contradicts *b.*, and secondly *a.* contradicts *c.*.

(Note: this space is also an example of a topological space that is *completely regular* but *not normal*. We do not prove this fact here.)

PROBLEM 1.4

Consider $\mathbb{N}$ with the discrete topology as a factor in the definition of the space $\mathbb{N}^{\mathbb{N}}$ of all maps $\mathbb{N} \to \mathbb{N}$, endowed with the product topology. This topological space is called the *Baire space* and often denoted by $\mathscr{N}$.

a. Describe the usual, product-space basis $\mathscr{B}$ for the topology of $\mathscr{N}$.

The elements of $\mathscr{N}$ can be viewed as maps $f : \mathbb{N} \to \mathbb{N}$. With this in mind, consider the map $d : \mathscr{N} \times \mathscr{N} \to \mathbb{R}$ where $d(f, g) = 1/k$ if $f(i) = g(i)$ for all $1 \leq i \leq k - 1$ and $f(k) \neq g(k)$.

b. Show that d is a metric on $\mathscr{N}$.

c. Show that every element of $\mathscr{B}$ contains an open ball with respect to the metric d; vice versa, show that every open ball with respect to d contains an element from $\mathscr{B}$.

From *c.* we conclude that the product-space topology on $\mathscr{N}$ equals the metric topology on $\mathscr{N}$ associated with d. In other words, the topological space $\mathscr{N}$ is *metrizable* with metric d.

d. Show that $\mathscr{N}$ is separable.

e. Show that $(\mathscr{N}, d)$ is complete.

From *c.* and *e.* we conclude that $\mathscr{N}$ satisfies the Baire category theorem. Topological spaces that are metrizable, complete and separable are called *Polish* spaces and play a central role in modern analysis.