Problem 1.1

Let \((M, d)\) be a metric space. For \(x \in M\) and \(r > 0\) define the open ball \(B_x(r)\) with center \(x\) and radius \(r\) as,

\[ B_x(r) = \{ y \in M : d(x, y) < r \}, \]

and the closed ball \(C_x(r)\) with center \(x\) and radius \(r\) as,

\[ C_x(r) = \{ y \in M : d(x, y) \leq r \}. \]

a. Show that it is not true in general that \(\overline{B_x(r)} = C_x(r)\), by constructing a counterexample.

Problem 1.2

Consider the space \(\ell_1\) of absolutely convergent sequences \(x\) in \(\mathbb{F}\), that is, \(x : \mathbb{N} \to \mathbb{F} : k \mapsto x_k\) such that,

\[ \|x\|_1 = \sum_{k=1}^{\infty} |x_k| < \infty. \]

(Throughout this problem we do not use theorem 1.61 in R and Y.)

a. Show that \(\| \cdot \|_1 : \ell_1 \to \mathbb{R}\) is a norm on \(\ell_1\).

Let \(\{x_n\}\) be a Cauchy sequence in \(\ell_1\). (To avoid confusion, note that each individual \(x_n\) is a map \(\mathbb{N} \to \mathbb{F} : k \mapsto x_{n,k}\).)

b. Show that for each \(k \geq 1\), the sequence \(\{x_{n,k}\}\) is Cauchy in \(\mathbb{F}\).

c. Prove that \(\ell_1\) with the norm \(\| \cdot \|_1\) is complete.
Problem 1.3

Endow the upper halfplane \( M = \{ (x, y) \in \mathbb{R}^2 : y \geq 0 \} \) in \( \mathbb{R}^2 \) with a topological basis defined in the following way: we consider, for every point \((x, y) \in M \) with \( y > 0 \) and every \( 0 < \epsilon < y \), the neighbourhood \( \{(z_1, z_2) \in M : \|(x - z_1, y - z_2)\| < \epsilon\} \); furthermore, for every \((x, 0) \in M \) and every \( \epsilon > 0 \), we consider the neighbourhood \( \{x\} \cup \{(z_1, z_2) \in M : \|(x - z_1, \epsilon - z_2)\| < \epsilon \} \). (In these definitions \( \|\cdot\| \) denotes the usual, Euclidean norm on \( \mathbb{R}^2 \).) We denote the resulting topological space by \((M, T)\).

a. Prove that \( M \) is separable in the topology \( T \).

b. Prove that \( M \) is first-countable but not second-countable in the topology \( T \).

c. Show that \( M \) has a subspace that is not separable.

d. Argue that \((M, T)\) is not metrizable. Make the argument twice, showing that for a metric space: firstly a. contradicts b., and secondly a. contradicts c..

(Note: this space is also an example of a topological space that is completely regular but not normal. We do not prove this fact here.)

Problem 1.4

Consider \( \mathbb{N} \) with the discrete topology as a factor in the definition of the space \( \mathbb{N}^\mathbb{N} \) of all maps \( \mathbb{N} \to \mathbb{N} \), endowed with the product topology. This topological space is called the Baire space and often denoted by \( \mathcal{N} \).

a. Describe the usual, product-space basis \( \mathcal{B} \) for the topology of \( \mathcal{N} \).

The elements of \( \mathcal{N} \) can be viewed as maps \( f : \mathbb{N} \to \mathbb{N} \). With this in mind, consider the map \( d : \mathcal{N} \times \mathcal{N} \to \mathbb{R} \) where \( d(f, g) = 1/k \) if \( f(i) = g(i) \) for all \( 1 \leq i \leq k - 1 \) and \( f(k) \neq g(k) \).

b. Show that \( d \) is a metric on \( \mathcal{N} \).

c. Show that every element of \( \mathcal{B} \) contains an open ball with respect to the metric \( d \); vice versa, show that every open ball with respect to \( d \) contains an element from \( \mathcal{B} \).

From c. we conclude that the product-space topology on \( \mathcal{N} \) equals the metric topology on \( \mathcal{N} \) associated with \( d \). In other words, the topological space \( \mathcal{N} \) is metrizable with metric \( d \).
d. Show that $N$ is separable.

e. Show that $(N, d)$ is complete.

From c. and e. we conclude that $N$ satisfies the Baire category theorem. Topological spaces that are metrizable, complete and separable are called *Polish* spaces and play a central role in modern analysis.