Functionaal Analyse

Homework Assignment 3

9 Mar 2017

Problem 3.1

(a) Use Fourier series to show that,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

(Hint: Consider the function $f \in L^2_{\mathbb{C}}[-\pi,\pi]$ given by $f(x) = x$ (for $x \in [-\pi,\pi]$) and use Parseval's theorem (i.e. R&Y Theorem 3.47(c)). You may use that $L^2_{\mathbb{C}}[-\pi,\pi]$ is a Hilbert space, that it contains all continuous functions from $[-\pi,\pi]$ to $\mathbb{C}$, that the inner product of two continuous functions f and g is given by $(f,g) = \int_{\pi}^{\pi} f(x)\overline{g(x)}\,\mathrm{d}x$, and Corollary 3.57 of R&Y.)

Problem 3.2

Recall that for a given linear space X and subspace Y, the *quotient space* X/Y is defined as the collection of all equivalence classes of the equivalence relation $u \sim v$ whenever $u - v \in Y$. For $x \in X$, denote the equivalence class of x by $[x]$.

(a) Let X be a Banach space and suppose Y is a closed linear subspace of X. Show that the quotient X/Y is again a Banach space, with its norm given by,

$$\|[x]\| = \inf_{u\in[x]} \|u\|,$$

where $[x] = \{u \in X\colon\ u - x \in Y\}$.
(Hint: the easiest way to show completeness is probably to apply the result of exercise 2.2 in Homework Assignment 2 to X/Y.)

(b) Let $\mathscr{H}$ be a Hilbert space and suppose Y is a closed linear subspace of $\mathscr{H}$. From the previous exercise we know that $\mathscr{H}/Y$ is a Banach space. Prove that it is even a Hilbert space and that it is isomorphic to $Y^{\perp}$ as a Hilbert space. (Two Hilbert spaces are isomorphic if there exists a linear bijection between them which preserves the inner product.)

PROBLEM 3.3

Let $(X, \|\cdot\|)$ be a Banach space.

(a) For each $k \in \mathbb{N}$, let $A_k \subseteq X$ be compact and $r_k \in \mathbb{R}$, $r_k > 0$, such that

$$A_{k+1} \subseteq \{x + u \colon\ x \in A_k \text{ and } u \in X \text{ with } \|u\| \leq r_k\}$$

for every $k \in \mathbb{N}$ and,

$$\sum_{k=1}^{\infty} r_k < \infty.$$

Show that the closure of $\bigcup_{k=1}^{\infty} A_k$ is compact.

(b) Let $p \geq 1$ and let $\{r_k\}$ be a sequence in $\mathbb{R}$ such that $r_k > 0$ for all $k \in \mathbb{N}$ and $\sum_{k=1}^{\infty} r_k < \infty$. Show that,

$$K = \big\{x = \{x_k\} \in \ell^p \colon\ |x_k| \leq r_k \text{ for all } k \in \mathbb{N}\big\},$$

is compact in ℓ^p.