Problem 3.1

(a) Use Fourier series to show that,

\[ \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \]

(Hint: Consider the function \( f \in L^2[-\pi, \pi] \) given by \( f(x) = x \) (for \( x \in [-\pi, \pi] \)) and use Parseval’s theorem (i.e. R&Y Theorem 3.47(c)). You may use that \( L^2[-\pi, \pi] \) is a Hilbert space, that it contains all continuous functions from \([-\pi, \pi]\) to \( \mathbb{C} \), that the inner product of two continuous functions \( f \) and \( g \) is given by \( (f, g) = \int_{-\pi}^{\pi} f(x)\overline{g(x)} \, dx \), and Corollary 3.57 of R&Y.)

Problem 3.2

Recall that for a given linear space \( X \) and subspace \( Y \), the quotient space \( X/Y \) is defined as the collection of all equivalence classes of the equivalence relation \( u \sim v \) whenever \( u - v \in Y \). For \( x \in X \), denote the equivalence class of \( x \) by \([x]\).

(a) Let \( X \) be a Banach space and suppose \( Y \) is a closed linear subspace of \( X \). Show that the quotient \( X/Y \) is again a Banach space, with its norm given by,

\[ \|[x]\| = \inf_{u \in [x]} \|u\|, \]

where \([x] = \{u \in X : u - x \in Y\}\).

(Hint: the easiest way to show completeness is probably to apply the result of exercise 2.2 in Homework Assignment 2 to \( X/Y \).)

(b) Let \( \mathcal{H} \) be a Hilbert space and suppose \( Y \) is a closed linear subspace of \( \mathcal{H} \). From the previous exercise we know that \( \mathcal{H}/Y \) is a Banach space. Prove that it is even a Hilbert space and that it is isomorphic to \( Y^\perp \) as a Hilbert space. (Two Hilbert spaces are isomorphic if there exists a linear bijection between them which preserves the inner product.)
Problem 3.3

Let \((X, \| \cdot \|)\) be a Banach space.

(a) For each \(k \in \mathbb{N}\), let \(A_k \subseteq X\) be compact and \(r_k \in \mathbb{R}, r_k > 0\), such that

\[ A_{k+1} \subseteq \{ x + u : x \in A_k \text{ and } u \in X \text{ with } \|u\| \leq r_k \} \]

for every \(k \in \mathbb{N}\) and,

\[ \sum_{k=1}^{\infty} r_k < \infty. \]

Show that the closure of \(\bigcup_{k=1}^{\infty} A_k\) is compact.

(b) Let \(p \geq 1\) and let \(\{r_k\}\) be a sequence in \(\mathbb{R}\) such that \(r_k > 0\) for all \(k \in \mathbb{N}\) and \(\sum_{k=1}^{\infty} r_k < \infty\). Show that,

\[ K = \{ x = \{x_k\} \in \ell^p : |x_k| \leq r_k \text{ for all } k \in \mathbb{N} \}, \]

is compact in \(\ell^p\).