Problem 4.1

Let $H$ be a Hilbert space. Let $\{a_n\}$ be a sequence in $H$ such that for each $h \in H$ the sum $\sum_{n=1}^{\infty} |(h, a_n)|^2$ is finite. (Such a sequence is said to be ‘weakly square summable’.)

(a) Prove that if $\sum_{n=1}^{\infty} \|a_n\|^2 < \infty$, then

$$\sum_{n=1}^{\infty} |(h, a_n)|^2 \leq C \|h\|^2 \quad \text{for all } h \in H,$$

where $C = \sum_{n=1}^{\infty} \|a_n\|^2$.

(b) Give an example of a sequence $\{a_n\}$ for which $\sum_{n=1}^{\infty} \|a_n\|^2 = \infty$.

(c) Prove that there exists a constant $C \geq 0$ such that

$$\sum_{n=1}^{\infty} |(h, a_n)|^2 \leq C \|h\|^2 \quad \text{for all } h \in H.$$

Hint: show that the operator $T : h \mapsto \{(h, a_n)\}$ from $H$ to $\ell^2$ has a closed graph.

(d) Prove that for each $\{\beta_n\} \in \ell^2$ the limit

$$\lim_{N \to \infty} \sum_{n=1}^{N} \beta_n a_n,$$

exists in the Hilbert space $H$.

Hint: consider the supremum $\sup_{\|h\| \leq 1} |(h, \sum_{n=M}^{N} \beta_n a_n)|$ for $N > M \geq 1$.

Problem 4.2

Let $(X, \| \cdot \|)$ be a normed space. A completion of $X$ is a Banach space $Y$ such that $X$ is isometrically isomorphic to a dense subspace of $Y$. 
(a) Show that $X$ has a completion and that every two completions of $X$ are isometrically isomorphic.

*Hint: use appropriate results of R&Y about the second dual of $X$.*

(b) Prove that, if $X$ is an inner product space, then its completion is a Hilbert space.

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**Problem 4.3**

As an example of a normed-space completion as in problem 4.2, note that for any $1 \leq p < \infty$, the space $L^p[a, b]$ is the completion of $C[a, b]$ with respect to the norm, $\|g\|_p = (\int_a^b |g(t)|^p \, dt)^{1/p}$. We specialize to the case $p = 1$, i.e. for $g \in C[a, b]$, define,

$$\|g\|_1 = \int_a^b |g(t)| \, dt.$$ 

Consider the linear functional $\ell : C[a, b] \to \mathbb{F}$ given by

$$\ell(g) = \int_a^b g(t) \, dt.$$

(a) Prove that the linear functional $\ell$ is bounded and compute $\|\ell\|$.

(b) Prove that there exists a unique bounded linear functional $\hat{\ell} : L^1[a, b] \to \mathbb{F}$ that extends $\ell$ to the $\|\cdot\|_1$-completion $L^1[a, b]$.

(The above constitutes an alternative way to define the space of *Lebesgue integrable* functions $g \in L^1[a, b]$ with *Lebesgue integral* $\hat{\ell}(g)$ (usually denoted $\int_a^b g(t) \, dt$).)

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**Problem 4.4**

(a) Let $X$ be a normed space, suppose $L \neq X$ is a closed linear subspace and that $a \in X$ is not in $L$. Prove that there exists $f \in X'$ such that $\|f\| = 1$, $f(x) = 0$ for all $x \in L$, and $f(a) = d(a, L)$.

*Hint: in the quotient space $X/L$ one has $\|[x]\| = d(x, L) := \inf\{\|x - z\| : z \in L\}$. Can you use a corollary of the Hahn-Banach theorem in this context?*

(b) As a corollary of part (a), prove the following theorem: Let $X$ be a normed space and let $L$ be a linear subspace of $X$. Then $L$ is dense in $X$ if and only if $\{f \in X' : f(x) = 0 \text{ for all } x \in L\} = \{0\}$.