Functionaal Analyse

Homework Assignment 1

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Problem 4.1

Let H be a Hilbert space. Let $\{a_n\}$ be a sequence in H such that for each $h \in H$ the sum $\sum_{n=1}^{\infty} |(h, a_n)|^2$ is finite. (Such a sequence is said to be 'weakly square summable'.)

(a) Prove that *if* $\sum_{n=1}^{\infty} \|a_n\|^2 < \infty$, then

$$\sum_{n=1}^{\infty} |(h, a_n)|^2 \leq C\|h\|^2 \quad \text{for all } h \in H,$$

where $C = \sum_{n=1}^{\infty} \|a_n\|^2$.

(b) Give an example of a sequence $\{a_n\}$ for which $\sum_{n=1}^{\infty} \|a_n\|^2 = \infty$.

(c) Prove that there exists a constant $C \geq 0$ such that

$$\sum_{n=1}^{\infty} |(h, a_n)|^2 \leq C\|h\|^2 \quad \text{for all } h \in H.$$

Hint: show that the operator $T : h \mapsto \{(h, a_n)\}$ *from* H *to* ℓ^2 *has a closed graph.*

(d) Prove that for each $\{\beta_n\} \in \ell^2$ the limit

$$\lim_{N\to\infty} \sum_{n=1}^{N} \beta_n a_n,$$

exists in the Hilbert space H.
Hint: consider the supremum $\sup_{\|h\|\leq 1} |(h, \sum_{n=M}^{N} \beta_n a_n)|$ *for* $N > M \geq 1$.

Problem 4.2

Let $(X, \|\cdot\|)$ be a normed space. A *completion* of X is a Banach space Y such that X is isometrically isomorphic to a dense subspace of Y.

(a) Show that X has a completion and that every two completions of X are isometrically isomorphic.
Hint: use appropriate results of R&Y about the second dual of X.

(b) Prove that, if X is an inner product space, then its completion is a Hilbert space.

PROBLEM 4.3

As an example of a normed-space completion as in problem 4.2, note that for any $1 \leq p < \infty$, the space $L^p[a,b]$ is the completion of $C[a,b]$ with respect to the norm, $\|g\|_p = (\int_a^b |g(t)|^p \, dt)^{1/p}$. We specialize to the case $p = 1$, *i.e.* for $g \in C[a,b]$, define,

$$\|g\|_1 = \int_a^b |g(t)| \, dt.$$

Consider the linear functional $\ell : C[a,b] \to \mathbb{F}$ given by

$$\ell(g) = \int_a^b g(t) \, dt.$$

(a) Prove that the linear functional ℓ is bounded and compute $\|\ell\|$.

(b) Prove that there exists a unique bounded linear functional $\widehat{\ell} : L^1[a,b] \to \mathbb{F}$ that extends ℓ to the $\|\cdot\|_1$-completion $L^1[a,b]$.

(The above constitutes an alternative way to define the space of *Lebesgue integrable* functions $g \in L^1[a,b]$ with *Lebesgue integral* $\widehat{\ell}(g)$ (usually denoted $\int_a^b g(t) \, dt$).)

PROBLEM 4.4

(a) Let X be a normed space, suppose $L \neq X$ is a closed linear subspace and that $a \in X$ is not in L. Prove that there exists $f \in X'$ such that $\|f\| = 1$, $f(x) = 0$ for all $x \in L$, and $f(a) = d(a, L)$.
Hint: in the quotient space X/L one has $\|[x]\| = d(x, L) := \inf\{\|x - z\| : z \in L\}$. Can you use a corollary of the Hahn-Banach theorem in this context?

(b) As a corollary of part *(a)*, prove the following theorem: Let X be a normed space and let L be a linear subspace of X. Then L is dense in X if and only if $\{f \in X' : f(x) = 0 \text{ for all } x \in L\} = \{0\}$.
