Functionaal Analyse

Homework Assignment 5

6 Apr 2017

Problem 5.1

Consider the Banach space,

$$\ell^\infty = \{x : \mathbb{N} \to \mathbb{F} : \sup_{n\in\mathbb{N}} |x(n)| < \infty\}$$

with its norm,

$$\|x\|_\infty = \sup_{n\in\mathbb{N}} |x(n)|, \qquad x \in \ell^\infty.$$

Prove or disprove each of the following statements:

(a) There exists an $f \in (\ell^\infty)'$ such that $f(x) = \lim_{n\to\infty} x(n)$ for every $x \in \ell^\infty$ for which $\lim_{n\to\infty} x(n)$ exists.

(b) There exists an $f \in (\ell^\infty)'$ such that $f(x) = \sum_{n=1}^\infty x(n)$ for every $x \in \ell^\infty$ for which $\sum_{n=1}^\infty x(n)$ exists.

(c) There exist two distinct functionals $f, g \in (\ell^\infty)'$ such that $f(x) = g(x) = \lim_{n\to\infty} x(n)$ for every $x \in \ell^\infty$ for which $\lim_{n\to\infty} x(n)$ exists.

(d) There exists an $f \in (\ell^\infty)' \setminus \{0\}$ such that $f(e_n) = 0$ for all $n \in \mathbb{N}$. *(Here $e_n \in \ell^\infty$ is defined by $e_n(k) = \delta_{nk}$, for all $n, k \in \mathbb{N}$.)*

Problem 5.2

(a) Let C be a non-empty convex subset of a real normed space $(X, \|\cdot\|)$. Denote $H(f,\gamma) = \{x \in X\colon\ f(x) \le \gamma\}$ for $f \in X'$ and $\gamma \in \mathbb{R}$. Show that the closure $\overline{C}$ of C satisfies

$$\overline{C} = \bigcap_{f\in X',\, \gamma\in\mathbb{R}\colon C\subseteq H(f,\gamma)} H(f,\gamma).$$

(b) Give an example of a real normed space $(X, \|\cdot\|)$ and a non-convex set C for which the equality in (a) does not hold.

PROBLEM 5.3

Let $(X, \|\cdot\|)$ be a reflexive Banach space. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of bounded linear operators from X into X such that $\lim_{n\to\infty} f(T_n x)$ exists for all $f \in X'$ and all $x \in X$. Show that there exists a bounded linear operator T from X into X such that,

$$f(Tx) = \lim_{n\to\infty} f(T_n x) \quad \text{for all } f \in X' \text{ and all } x \in X.$$

(Hint: Use the Uniform Boundedness Principle (twice!) to show that $\sup_{n\in\mathbb{N}} \|T_n'\| < \infty$*. Show that the map* S *defined by* $(Sf)(x) := \lim_{n\to\infty}(T_n' f)(x)$ *is a bounded linear operator from* X' *into* X'*. Use* S' *and reflexivity to find* T*.)*