Problem 5.1

Consider the Banach space,
\[ \ell^\infty = \{ x : \mathbb{N} \to \mathbb{F} : \sup_{n \in \mathbb{N}} |x(n)| < \infty \} \]
with its norm,
\[ \|x\|_\infty = \sup_{n \in \mathbb{N}} |x(n)|, \quad x \in \ell^\infty. \]

Prove or disprove each of the following statements:

(a) There exists an \( f \in (\ell^\infty)' \) such that \( f(x) = \lim_{n \to \infty} x(n) \) for every \( x \in \ell^\infty \) for which \( \lim_{n \to \infty} x(n) \) exists.

(b) There exists an \( f \in (\ell^\infty)' \) such that \( f(x) = \sum_{n=1}^\infty x(n) \) for every \( x \in \ell^\infty \) for which \( \sum_{n=1}^\infty x(n) \) exists.

(c) There exist two distinct functionals \( f, g \in (\ell^\infty)' \) such that \( f(x) = g(x) = \lim_{n \to \infty} x(n) \) for every \( x \in \ell^\infty \) for which \( \lim_{n \to \infty} x(n) \) exists.

(d) There exists an \( f \in (\ell^\infty)' \setminus \{0\} \) such that \( f(e_n) = 0 \) for all \( n \in \mathbb{N} \). (Here \( e_n \in \ell^\infty \) is defined by \( e_n(k) = \delta_{nk} \), for all \( n, k \in \mathbb{N} \).)

Problem 5.2

(a) Let \( C \) be a non-empty convex subset of a real normed space \( (X, \|\cdot\|) \). Denote \( H(f, \gamma) = \{ x \in X : f(x) \leq \gamma \} \) for \( f \in X' \) and \( \gamma \in \mathbb{R} \). Show that the closure \( \overline{C} \) of \( C \) satisfies
\[ \overline{C} = \bigcap_{f \in X', \gamma \in \mathbb{R} : \emptyset \subseteq H(f, \gamma)} H(f, \gamma). \]

(b) Give an example of a real normed space \( (X, \|\cdot\|) \) and a non-convex set \( C \) for which the equality in (a) does not hold.
Let $(X, \|\cdot\|)$ be a reflexive Banach space. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of bounded linear operators from $X$ into $X$ such that $\lim_{n \to \infty} f(T_n x)$ exists for all $f \in X'$ and all $x \in X$. Show that there exists a bounded linear operator $T$ from $X$ into $X$ such that,

$$f(T x) = \lim_{n \to \infty} f(T_n x) \quad \text{for all } f \in X' \text{ and all } x \in X.$$ 

(Hint: Use the Uniform Boundedness Principle (twice!) to show that $\sup_{n \in \mathbb{N}} \|T_n'\| < \infty$. Show that the map $S$ defined by $(Sf)(x) := \lim_{n \to \infty} (T_n' f)(x)$ is a bounded linear operator from $X'$ into $X'$. Use $S'$ and reflexivity to find $T$.)