Exercise 10.1

Let $\mathcal{H}$ be the Hilbert space $\ell^2$ of all square-summable sequences in $\mathbb{C}$. For every $n \geq 1$, define the operator $S_n : \mathcal{H} \to \mathcal{H}$,

$$S_n(x_1, x_2, x_3, \ldots) = (0, 0, \ldots, 0, x_1, x_2, x_3, \ldots)$$

that shifts $x$ to the right $n$ times.

a. Show that for every $x \in \mathcal{H}$,

$$\lim_{n \to \infty} S_n^* x = 0.$$

Exercise 10.2

Let $\mathcal{H}$ be a complex Hilbert space with $T \in B(\mathcal{H})$ self-adjoint. Let $S \subset \mathcal{H}$ be a linear subspace and let $x_0$ be a point in $S$. Show that the following are equivalent:

a. the map $S \to \mathbb{R} : x \mapsto \|Lx\|$ assumes its maximum in $x_0$;

b. the map $S \to \mathbb{R} : x \mapsto |(Lx, x)|$ assumes its maximum in $x_0$;

c. $x_0$ is eigenvector of $L^2$ with eigenvalue $\|L^2\|$;

d. $x_0$ is eigenvector of $L$ with eigenvalue $\pm \|L\|$.

Exercise 10.3

Discuss the solutions of exercises 6.1, 6.5, 6.9–6.12, 6.14, 6.17, 6.21–6.23, 6.26 and 6.28 from Rynne and Youngson with your fellow students, until you understand all aspects in full detail.