Functionaal Analyse

Exercises 5

Thursday 9 Mar 2017

Exercise 5.1

Let X and Y be Banach spaces and $T : X \to Y$ a one-to-one, bounded linear operator.

a. Show that $T^{-1} : T(X) \subset Y \to X$ is bounded, if and only if $T(X)$ is closed in Y. *Hint: for this proof, it may be most convenient to use the Open Mapping theorem in the form of corollary 4.44, called the Closed Graph Theorem.*

Exercise 5.2

Let X and Y be Banach spaces and $T : X \to Y$ a bounded linear operator. Assume that T is bijective.

a. Show that there exist $a, b > 0$ such that,

$$a\|x\|_X \leq \|Tx\|_Y \leq b\|x\|_X.$$

Consider a vector space X with two norms, $\|\cdot\|_1$ and $\|\cdot\|_2$, and assume that for both norms, X is complete.

b. Show that, if there exists a constant $c > 0$ such that $\|x\|_1 \leq c\|x\|_2$ (for all $x \in X$), then there exists a constant $C > 0$ such that $\|x\|_2 \leq C\|x\|_1$ (for all $x \in X$).

Exercise 5.3

a. Study the proof of the Open Mapping theorem (the proof of theorem 4.43 in Rynne and Youngson), with particular attention for the application of the Baire Category theorem. Discuss it with your fellow students, until you understand all aspects in full detail.
