Functionaal Analyse

Exercises 6

Thursday 16 Mar 2017

Exercise 6.1

Let X, Y be Banach spaces and let $A : E \subset X \to Y$ be given. We say that A is a *closed operator* if the graph $\mathscr{G}(A)$ of A is a closed subset of $X \times Y$.

a. Show that A is a closed operator, if and only if, for any sequence $\{x_n\} \subset E$ such that $x_n \to x$ and $Ax_n \to y$, for some $x \in X$ and $y \in Y$, we have that $x \in E$ and $Ax = y$.

b. Show that a closed operator A defined on all of X is bounded. Also show that if $A : E \to Y$ is closed and injective, then $A^{-1} : A(E) \subset Y \to X$ is closed.

Given $a < b$ in $\mathbb{R}$, let $C[a, b]$ denote the space of all continuous, real-valued functions on $[a, b]$, with uniform norm $\| \cdot \|_\infty$. Consider the subspace $C^1[a, b]$ of continuously differentiable functions and the operator D that maps f to its derivative, $Df = f'$.

c. Show that $D : C^1[a, b] \to C[a, b]$ is closed but not bounded.

Let $B : F \subset X \to Y$ be a linear operator. such that the closure of its graph $\mathscr{G}(B)$ in $X \times Y$ happens to be the graph of a linear map $A : E \subset X \to Y$ for some E, then B is called *closable* and A is called the *closure* of B.

d. Show that a linear operator $B : F \subset X \to Y$ is closable, if and only if, for any $\{x_n\}, \{y_n\} \subset F$ such that $x_n \to x$, $y_n \to x$ for some x and $\{Bx_n\}$, $\{By_n\}$ converge, we have $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} By_n$.

e. Show that if we restrict the domain of $f \mapsto f'$ to $C^\infty[a, b]$, that is, we consider $\tilde{D} : C^\infty[a, b] \subset C[a, b] \to C[a, b] : f \mapsto f'$, then $\tilde{D}$ is not a closed operator. Also show that $\tilde{D}$ has a closure, namely $D : C^1[a, b] \to C[a, b]$.

Endow $C^1[a, b]$ with the norm $\|f\| := \|f\|_\infty + \|f'\|_\infty$.

f. Show that with this norm on the domain, $D : C^1[a, b] \to C[a, b]$ is bounded.

Exercise 6.2

a. Study the proof of the Uniform Boundedness Principle (theorem 4.52 in Rynne and Youngson) and discuss it with your fellow students, until you understand all aspects in full detail.
