Functionaal Analyse

Exercises 8

Thursday 6 April 2017

Exercise 8.1

Let X be a normed space. A subset $S \subset X$ is said to be *weakly bounded* if for all $f \in X'$, $\sup\{|f(x)| : x \in S\} < \infty$. A subset S is said to be *strongly bounded* if $\sup\{\|x\| : s \in S\} < \infty$.

a. Use the uniform boundedness principle to show that, if X is a Banach space, any subset S is weakly bounded, if and only if, S is strongly bounded.

Exercise 8.2

Let X be a Banach space and let $\{x_n\}$ be a sequence in X such that for all $f \in X'$, $\sup\{|f(x_n)| : n \geq 1\} < \infty$.

a. Use the uniform boundedness principle to show that $\sup\{\|x_n\| : n \geq 1\} < \infty$.

Exercise 8.3

Let X be a real normed space and let C be a convex subset of X that contains 0. Assume that,

$$X = \bigcup_{t>0} t\,C, \tag{1}$$

i.e. for every $x \in X$ there exists a $t > 0$ such that $t^{-1}x \in C$. For every $x \in X$, define,

$$p_C(x) = \inf\{t > 0 : t^{-1}x \in C\}.$$

a. Show that $p_C : X \to \mathbb{R}$ is a sublinear functional.

b. Show that if C is convex and open and $0 \in C$, then equation (1) holds.

c. Use parts *a.*, *b.* and *c.* to show that if C is convex and open and $0 \in C$, and $x_0 \in X \setminus C$, then there exists a linear functional ϕ_0 on X such that $\phi_0(x_0) = 1$ and $\phi_0(x) < 1$ for all $x \in C$.

d. Show that ϕ_0 is a bounded linear functional.
