Exercise 9.1
Let $X$ be reflexive normed space. We say that $f \in X'$ is norm-attaining, if there exists an $x \in X$ such that $f(x) = \|f\|$.

a. Show that $X$ is a Banach space.

b. Show that every $f \in X'$ is norm-attaining.

NB James’s theorem says that if $X$ is a Banach space and every $f \in X'$ is norm-attaining, then $X$ is reflexive.

Exercise 9.2
Let $X$ and $Y$ be Banach spaces.

a. Show that $X$ is isomorphic to $Y$, if and only if, $X'$ is isomorphic to $Y'$.

b. Show that $X$ is isometrically isomorphic to $Y$, if and only if, $X'$ is isometrically isomorphic to $Y'$.

Recall that $c_0$ denotes the subspace of all $\{x_n\} \in \ell^\infty$ such that $x_n \neq 0$ only for a finite number of values for $n$. In exercise 7.1, we have seen that $(c_0)'$ is isometrically isomorphic to $\ell^1$. Let $c \subset \ell^\infty$ denote the set of all sequences $\{x_n\}$ that converge.

c. Show that $c'$ is also isometrically isomorphic to $\ell^1$, but that $c$ is not isometrically isomorphic to $c_0$. Does this contradict part b.?

Exercise 9.3
Let $(\Omega, \mathcal{F})$ be a measurable space with measure $\mu$.

a. Find a condition on $\Omega$ that is equivalent to the statement that $L^1(\Omega, \mathcal{F}, \mu)$ is reflexive.