Functionaal Analyse

Exercises 9

Thursday 13 April 2017

Exercise 9.1

Let X be reflexive normed space. We say that $f \in X'$ is *norm-attaining*, if there exists an $x \in X$ such that $f(x) = \|f\|$.

a. Show that X is a Banach space.

b. Show that every $f \in X'$ is norm-attaining.

NB *James's theorem* says that if X is a Banach space and every $f \in X'$ is norm-attaining, then X is reflexive.

Exercise 9.2

Let X and Y be Banach spaces.

a. Show that X is isomorphic to Y, if and only if, X' is isomorphic to Y'.

b. Show that X is isometrically isomorphic to Y, if and only if, X' is isometrically isomorphic to Y'.

Recall that c_0 denotes the subspace of all $\{x_n\} \in \ell^\infty$ such that $x_n \neq 0$ only for a finite number of values for n. In exercise 7.1, we have seen that $(c_0)'$ is isometrically isomorphic to ℓ^1. Let $c \subset \ell^\infty$ denote the set of all sequences $\{x_n\}$ that converge.

c. Show that c' is also isometrically isomorphic to ℓ^1, but that c is not isometrically isomorphic to c_0. Does this contradict part *b.*?

Exercise 9.3

Let $(\Omega, \mathscr{F})$ be a measurable space with measure μ.

a. Find a condition on Ω that is equivalent to the statement that $L^1(\Omega, \mathscr{F}, \mu)$ is reflexive.
