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Nieuwe koppelingen in $N = 2$ supergravitatie

(met een samenvatting in het Nederlands)

PROEFSCHRIFT

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Chapter 1

Introduction

1.1 Supersymmetry: the Why

Over the last two decades it has become clear that the standard model provides a highly accurate picture of the physics at subnuclear scales. Confirmation of measurements made in collider experiments and the detection of new particles predicted by the standard model, have established it as one of the most successful theoretical descriptions in modern physics. At the moment, its most challenging experimental aspect is the discovery of the Higgs boson, the heaviest fundamental degree of freedom in the standard model.

However, at energy scales beyond those currently attainable in particle accelerators, the standard model is expected to be a low-energy manifestation of a new theory that describes a whole new range of physics. Renormalization group extrapolations of the strong, weak and electromagnetic coupling constants show a remarkable coincidence at an energy scale of the order 10^{15} GeV. It is generally believed that this signals the so-called Grand Unification of the standard model interactions, *i.e.* that they are consistently described by a single simple gauge group instead of the well-known factorization into SU(3), SU(2) and U(1). The grand unification theories, or GUT's, inherently pose a new problem. There is a gap of roughly thirteen orders of magnitude between the observed spectrum of the standard model and the GUT-scale. In a quantum field theoretical expansion such a gap is not stable, unless fine-tuning of the parameters is assumed to high accuracy. The above very briefly sketches what is usually referred to as the *hierarchy problem*. Furthermore, grand unified models make no attempt to incorporate gravitational interactions, even though their characteristic scale lies only four orders of magnitude below the Planck scale (as compared to the thirteen orders of magnitude mentioned earlier).

The 'Grand' in GUT may therefore have been a bit premature. A more radical departure from the standard model is called for. An alternative for the continuation of the standard model to higher energies is a supersymmetric extension. A supersymmetric field theory is an ordinary field theory in every respect, except that its action is invariant under a global transformation that relates bosonic and fermionic degrees of freedom. Coming back to the unification of the strong, weak and electromagnetic interactions, a so-called superGUT is claimed to overcome the hierarchy problem through a cancellation of bosonic and fermionic loops to a high order in perturbation theory. So the aforementioned fine-tuning is achieved by assuming that the group of global symmetries of the theory includes boson-fermion transformations. More on GUT's, the hierarchy problem and supersymmetry can be found in [1].

An even more intriguing feature of supersymmetry is the fact that theories with *local* supersymmetry naturally (and necessarily) incorporate general relativity. Correspondingly, these models are called supergravity models. It can even be shown that the non-renormalizability of the usual field theory of gravity is cured at low-loop level by Bose-Fermi cancellations. In fact, cancellations of divergent contributions to the perturbative expansion arise between all boson and fermion fields that are related by a supersymmetry transformation. The convergence of perturbative supergravity can therefore be improved if one considers supergravity models that incorporate more then one supersymmetry, so-called N-extended supergravity models, where N represents the number of supersymmetries. Initially this sparked the hope that the models with maximal supersymmetry, $N = 8$ supergravity models, were perturbatively finite.

Unfortunately, also extended supergravity theories turned out to have their problems (heralding the demise of yet another superlative). Even $N = 8$ supersymmetry seems not enough to ensure cancellation of bosonic and fermionic divergences at higher loop level. At the moment, the only description of near Planck-scale physics that stands a chance of being consistent is given by so-called string models [2]. Let us briefly review a few basic concepts in string theory: a string model employs conformal quantum field theories on a $(1 + 1)$-dimensional space (the 'worldsheet') to describe the dynamics of a string, which is embedded in space-time. Based on the topology of the spatial worldsheet dimension, we distinguish open and closed strings, the former having end-points and the latter being loops. The characteristic scale for strings, the so-called string tension, is of the order of the Planck scale. The simplest string theory describes only free bosonic degrees of freedom on the worldsheet and is called (open or closed) bosonic string theory. Even though the worldsheet theory is free, the embedded strings can split and join, representing interactions of strings in space-time. If the conformal field theory on the worldsheet is supersymmetric, we speak of a *superstring model.* Quantum mechanical consistency requires that the string model is embedded in a space-time of so-called critical dimension: for the closed bosonic string this dimension is $D = 26$, whereas for a superstring model it is $D = 10$.

In many string models, the spectrum of string states and the associated spins and masses can be calculated exactly. In closed bosonic string theory and superstring theory, one of the massless states has spin 2 and interacts with other states in a way that is appropriate for a graviton. For that reason, string theory is a candidate for a model of quantum gravity. However, the spectrum of the bosonic string contains a tachyonic state, which leads to infrared divergences in string perturbation theory and instability of the perturbative vacuum. In superstring models, the tachyonic state can be eliminated by a so-called GSO-projection, which amounts to a choice for the possible boundary conditions of the worldsheet superconformal field theory. The absence of the tachyon selects GSO-projected superstring models as the only string models that could lead to a *consistent* model of Planck scale physics and quantum gravity.

Based on the superstring action, its massless spectrum and the interactions, one can derive a field theory that gives an effective description of the massless degrees of freedom in a space-time that has the critical dimension of the string theory. Since the mass gap is of the order of the string tension, corrections to the effective model are of the order of the Planck scale. The resulting effective theory contains a gravitational sector, corresponding to the string state with spin 2. Moreover, the GSO-condition turns out to be enough to prove space-time supersymmetry, meaning that the resulting effective field theory is supersymmetric. So if we stay well below the Planck scale, a superstring model is effectively described by a ten-dimensional supergravity theory. If a superstring model includes open strings, the spectrum is expected to include so-called D-branes, extended objects on which the open string can have its endpoints. In the effective supergravity theories these D-branes are related to differential forms that couple to the gravitational fields. Moreover, there are indications that eleven-dimensional supergravity plays an important role in this context [3].

The question now arises what the physics between the Planck-scale and the standard model looks like. Two parts of the answer have already been given: superstrings lead to ten-dimensional supergravity and the standard model can be formulated at higher energy scales by extension to a supersymmetric model, possibly a superGUT.

To understand what lies in between, one has to realize that superstrings can be embedded in a ten-dimensional space that has special properties. In fact, the only way in which we can explain the evidently four-dimensional nature of 'our' space-time, is by assuming that the remaining six dimensions reside in a very small, compact submanifold of the ten-dimensional space-time. Such a compactification of space-time [2] enables one to reformulate ten-dimensional supergravity in terms of four-dimensional fields and integrate out the degrees of freedom that have a mass of the order of the inverse compactification scale. Depending on the particulars of the compact submanifold, part of the ten-dimensional supersymmetry is preserved in the four-dimensional theory, which can therefore again be a theory of (extended or simple) supergravity. Moreover, such a compactification gives rise to gauge and matter fields and correspondingly a coupling of super-Yang-Mills models to supergravity. Again the idea is that the length scale at which the four-dimensional description is valid is much larger then the scale of the compactification. In the limit of flat space-time, which entails a decoupling of the gravitational degrees of freedom, the model is further reduced to a rigidly supersymmetric Yang-Mills model (among which are the superGUT's and the supersymmetric extensions of the standard model). Breaking of supersymmetry and possibly also part of the gauge symmetry, leads to non-supersymmetric Yang-Mills models, such as the standard model.

In this thesis, we investigate four-dimensional field theories that exhibit rigid or local $N = 2$ extended supersymmetry, in order to find the most general supersymmetric way in which $N = 2$ multiplets can be coupled. So in the hierarchy sketched above, we are considering models that can arise after $N = 2$ compactifications of ten-dimensional superstring models. Physically relevant compactifications display a lower degree of supersymmetry. Consequently, many of the questions that this thesis deals with are motivated from a string theoretical point of view, as opposed to a more phenomenological, standard-model oriented standpoint.

In recent years, $N = 2$ supersymmetric models have received a lot of attention, because they provide a rich theoretical setting for the study of non-perturbative phenomena in field theory. This surge of interest was first of all inspired by the work of Seiberg and Witten [83, 84], who derived exact solutions for the $N = 2$ supersymmetric Wilson effective actions of broken SU(2) vector-multiplet models, with and without coupling to hypermultiplets. Their approach can be extended to higher gauge groups and has implications, for instance, for three-dimensional $N = 4$ supersymmetric field theories. The Seiberg-Witten approach relies on a number of basic properties of $N = 2$ supersymmetric field theories. First of all, they use the fact that $N = 2$ supersymmetry relates vector gauge fields to complex scalar fields. More specifically, they use the geometry of the non-linear sigma-model of these scalars, which is called special geometry, and its relation to electric-magnetic duality and integer shifts of the theta-angle. Secondly, they invoke an $N = 2$ non-renormalization theorem to determine the perturbative corrections to the tree-level vector-multiplet couplings. The non-perturbative aspect of the approach lies in the break-down of the effective description when BPS-states become massless. The mass of a BPS-state is determined by the value of the $N = 2$ central charge. Based on the singularity structure of the sigma-model manifold, the problem of solving the effective couplings can be mapped to a problem in the algebraic geometry of Riemann surfaces, which can then be solved by mathematical means.

A second source of inspiration for the study of $N = 2$ models, is the interest in string dualities. There are strong arguments suggesting that certain compactifications of the five superstring theories are related by dualities, *i.e.* a certain weakly-coupled string model with an appropriate choice of vacuum, can be viewed as a strong-coupling limit of another string model with another vacuum. For example, the $E_8 \times E_8$ heterotic string theory compactified on $\mathrm{K3} \times \mathrm{T}^2$ is thought to be dual to type-IIA string theory compactified on a Calabi-Yau manifold that can be described as a K3-fibration. A central role in the discussion of string dualities is played by the spectrum of non-perturbative excitations. To test the various string-duality relations, one can check the correspondence in a four-dimensional setting for $N = 2$ compactifications, like the above example, again with a strong emphasis on the non-perturbative aspects.

Of course, much of these four-dimensional $N = 2$ models was already known for quite some time. Soon after the first steps had been made in $N = 1$ supersymmetry, the first $N = 2$ models were constructed. However, the off-shell formulation and coupling of multiplets of extended supersymmetry, especially the extended supergravity multiplets, remained difficult. A very productive approach to tackle this problem in the context of $N = 2$ supersymmetry turned out to be a formulation in terms of conformal supergravity. Like Poincaré supergravity is based on an algebra of supersymmetries that intertwine with the Poincaré group, conformal supergravity is based on a supersymmetric extension of the conformal group, which includes the Poincaré group as a subgroup. Therefore the off-shell representations and couplings of conformal supergravity are much more restricted and structured than those of Poincaré supergravity. At the same time, however, every Poincaré supergravity model can be turned into a gauge-equivalent superconformal model when so-called compensating fields are introduced. So for every Poincaré supergravity model there are gauge-equivalent superconformal models, which enable one to employ the added symmetry of the superconformal framework to analyze supergravity couplings in a much more structured fashion. Moreover, the different sets of auxiliary fields that are used to formulate off-shell Poincaré supergravity can be concisely represented by different compensating multiplets in the conformal framework. It is therefore beneficial to formulate supersymmetric actions first as invariants of the full conformal supersymmetry, and then later to break to Poincaré supersymmetry by means of a gauge choice.

For this reason we use superconformal methods in this thesis. They are explained in chapter 2 in some detail. Besides an introduction to $N = 2$ supersymmetry and the Weyl multiplet, we consider the off-shell chiral- and vector multiplets. We also construct an action for vector multiplets coupled to conformal supergravity and discuss the gauge equivalence to Poincaré supersymmetry. In chapter 3 we continue along those lines, analyzing the couplings of rigidly supersymmetric vector multiplets, their duality transformations and the geometry of the scalar target-space in both the globally and locally supersymmetric case. Chapter 4 deals with hypermultiplet models, whose scalar target-spaces can be hyperkähler or quaternionic spaces, depending on whether one considers rigid (or superconformal) couplings or a coupling to Poincaré supergravity. One of the questions that arose in the context of string compactifications is that of the mirror map in four dimensions. When considering compactifications of a type-II superstring theory on a mirror pair of Calabi-Yau manifolds, one finds that the moduli spaces of the four-dimensional vector- and hypermultiplets are interchanged. Similarly, the interchange is effected by a compactification of type-IIA and type-IIB models on the *same* Calabi-Yau manifold. One can study this interchange in a four-dimensional context by the so-called classical mirror map, which we construct in chapter 5 by means of a dimensional reduction after which the vector- and hypermultiplet models are related by a simple duality transformation. Finally in chapter 6, we discuss a supermultiplet that contains a scalar field, a vector field, an antisymmetric tensor field and a pair of Majorana fermions. We expand on the possible couplings of this multiplet, the so-called vector-tensor multiplet, to gauge fields and to supergravity. This work was initially motivated by the fact that the dilaton of heterotic string theory is expected to reside in a vector-tensor multiplet after compactification to four dimensions. However, it has become clear that its relevance is mainly found in the reduction of six-dimensional string compactifications.

The rest of this chapter gives an introduction to some of the concepts that play an important role in the chapters that follow. The aim is to clarify these concepts at a level that is as basic as possible, in order to be able to refer back to these explanations at a later stage, where they are used in a different context and possibly in a more complicated way.

1.2 Supersymmetry: the basics

As was stated in the previous section, supersymmetry transformations take bosons into fermions and fermions into bosons. In a quantum theory this means that if we split up the Hilbert space into bosonic states $|\text{boson}\rangle$, and fermionic states $|\text{fermion}\rangle$, there are *supersymmetry generators* (or supercharges) Q, acting as:

$$Q\,|\text{boson}\rangle = |\text{fermion}\rangle, \qquad Q\,|\text{fermion}\rangle = |\text{boson}\rangle. \tag{1.1}$$

The consequences of such a boson-fermion symmetry are, of course, far-reaching. It effects the statistics of the transformed state and correspondingly, it means that the spin of the transformed state is changed by a half unit. Hence, supersymmetry generators themselves have spin one-half and form spinor representations of the Lorentz group, as opposed to the usual generators of symmetry transformations which are Lorentz scalars.

At first sight this seems to contradict a theorem by Coleman and Mandula [5], which states that for every non-trivial relativistic field theory, under some very mild assumptions, *all the symmetries of the S-matrix commute with the generators of the Poincaré group.* Essentially, this means that all internal symmetry generators are Lorentz scalars and there is no hope of unifying the internal symmetries of a theory with the Poincaré group. However, one of the assumptions that they make, is that the symmetry generators form a closed algebra under *commutation* relations, thus restricting themselves to Lie groups of symmetry transformations. Hence, it is clear from (1.1) that the Coleman-Mandula theorem has no bearing on supersymmetries: in a Fock-space representation of the Hilbert space, a generator Q would be a sum of products of either a fermionic creation and a bosonic annihilation operator or a bosonic creation and a fermionic annihilation operator. The statistics of the fermionic operators imply that supersymmetry generators will obey *anticommutation* relations.

Therefore an extension of the Coleman-Mandula theorem was called for and beautifully presented by Haag, Łopuszański and Sohnius in [6]. Their findings are the following: bosonic symmetry generators still have to obey the Coleman-Mandula theorem, *i.e.* there are no symmetry generators that can change the spin of the states by non-zero integers. Hence the bosonic part of the symmetry algebra splits up into Poincaré and internal symmetries, the latter again being Lorentz scalars. Fermionic symmetry generators commute with translations and form spinor representations of the Lorentz group. In four dimensions, the smallest such spinor has four real components as a result of a Majorana (or a chiral) constraint (for definitions and conventions on spinor notation, Dirac gamma matrices etcetera, we refer to appendix A). The anticommutation relation between supersymmetry generators is fixed and takes the form:

$$\{Q_\alpha\,,\bar{Q}_\beta\,\} = 2(\gamma_\mu)_{\alpha\beta}P^\mu, \tag{1.2}$$

where α, β denote spinor indices and γ_μ is a gamma matrix. Equation (1.2) reflects the central relation in supersymmetry: it expresses the fact that the anticommutator of two supersymmetry transformations closes into the four-momentum, the generator of translations in the Poincaré group. The set of commutation and anticommutation relations between Poincaré generators and supercharges is called the *Poincaré superalgebra*.

With reference to chapter 2, we note that in two respects the analysis of Haag *et al.* went somewhat further. First of all, they left room for a non-trivial action of the internal symmetry group on the supersymmetry generators. If the supercharges form a reducible Lorentz representation that decomposes into N, $(N > 1)$ Majorana spinors instead of forming just one, we speak of (N-)extended supersymmetry. Since the internal symmetries commute with the Lorentz group, these different Majorana supercharges may transform among each other and thus form a representation of the internal symmetry group. Secondly, Haag *et al.* noted that in the case of a theory that does not have an intrinsic scale (such as a mass or a dimensionful coupling constant) the bosonic part of the symmetry algebra is given by all transformations that leave the Minkowski metric invariant *modulo* a scale transformation. The generators of these transformations form the Lie algebra of the so-called conformal group and the corresponding supersymmetric extension is called the *superconformal algebra*. We come back to both these subjects at length in the second chapter.

Let us now consider in some more detail the general properties of a supersymmetric model. First of all, note that the four-momentum commutes with the supercharges. This means that the mass-operator P^2 is an invariant of supersymmetry. In other words, given a state of mass m, all states it is related to by supersymmetry also have mass m [7].

Secondly, note that we can contract (1.2) with γ_0 and take the trace over the spinor index to find:

$$\sum_\alpha \Big(Q_\alpha Q_\alpha^\dagger + Q_\alpha^\dagger Q_\alpha \Big) = 8P_0 = 8H\,. \tag{1.3}$$

where H is the Hamiltonian of the model. Note that the *l.h.s.* of this equation is always positive. So given an arbitrary eigenstate $|\phi\rangle$ of H in the theory, we find:

$$H|\phi\rangle = E_\phi|\phi\rangle\,, \qquad E_\phi \geq 0\,, \tag{1.4}$$

proving that the energy spectrum of a supersymmetric model is positive definite.

Also note that one can determine from the lowest energy in the spectrum of H whether supersymmetry is spontaneously broken or not. The state with lowest energy, the vacuum $|0\rangle$, satisfies:

$$H|0\rangle = 0, \quad \text{if and only if} \quad Q|0\rangle = 0\,. \tag{1.5}$$

So depending on whether the energy of the vacuum is zero, the vacuum is invariant under supersymmetry or not.

A third property of a supersymmetric model is the fact that, within a finite-dimensional, linear supersymmetry representation of non-zero energy, the number of fermionic states equals the number of bosonic states. The proof of this statement is very simple: suppose that the dimension of the fermionic subspace is greater than that of the bosonic subspace. Since Q acts linearly, this means that in that case the kernel of Q is non-trivial. But this contradicts the assumption that the representation has non-zero energy. Analogously, one proves that the dimension of the bosonic subspace can not be greater than the dimension of the fermionic subspace and consequently they are equal.

Obviously, the same argument does not apply to representations with zero energy, the vacuum sector. The vacuum states are singlets, unless supersymmetry is spontaneously broken. To determine whether or not spontaneous symmetry breaking is at all possible, one can consider the so-called Witten index [8], which is defined as the trace over the Hilbert space of the operator $(-1)^F$. This operator by definition has eigenvalues $+1$ for a bosonic and -1 for a fermionic state. Decomposing the Hilbert space into subspaces of definite energy, we note that only the vacuum sector ($E = 0$) contributes to the Witten index:

$$\mathrm{Tr}\,(-1)^F = \mathrm{Tr}_{\,E=0}\,(-1)^F, \tag{1.6}$$

because the number of bosonic states equals the number of fermionic states if $E \neq 0$. This implies that when $\mathrm{Tr}\,(-1)^F \neq 0$, supersymmetry can not be spontaneously broken. It is argued in [8] that if we change the parameters defining the model, such as the volume, the bare masses and the bare coupling constants, without breaking supersymmetry explicitly, the value of the Witten index remains the same. Namely, a supersymmetric change of the parameters causes states to move in and out of the vacuum sector in boson-fermion pairs only[1]. According to Witten, this means that we can calculate $\mathrm{Tr}\,(-1)^F$ reliably in some convenient limit (such as small volume, large bare masses and weak coupling) and the result can be applied to the situation of interest, for instance large volume, small bare masses and strong coupling. Witten claims that even if we calculate $\mathrm{Tr}\,(-1)^F$ using some approximation, for example a loop-expansion, the result is valid non-perturbatively, because the corrections to the approximation again only lead to boson-fermion pair shifts in the spectrum.

However, for this thesis, the most important characteristic of supersymmetric models has to do with the gauging of supersymmetries. Suppose that we start with a field theory that exhibits *rigid* supersymmetry. From this theory we could try to construct a model with *local* supersymmetry by minimal coupling to a gauge field ψ_μ. Besides being a spinor this so-called Rarita-Schwinger field carries a four-index and its spin adds up to 3/2. The fact that the supersymmetry transformations have been made local, implies that their anticommutator, which of course is also a symmetry, generates local translations, *c.f.* (1.2). In other words, a

[1]There are two important caveats to be made here: firstly, if the change of parameters is not of an adiabatic nature, *i.e.* it does not induce a continuous deformation of the spectrum, then the above argument does not apply. For a more detailed discussion, we refer to [8]. Besides that, it is important that the model under consideration has a mass gap: if the spectrum is continuous then the argument fails.

gauge theory of supersymmetry is automatically invariant under general coordinate transformations. Hence, such locally supersymmetric theories are referred to as supergravity theories. The field ψ_μ turns out to be related to the graviton by supersymmetry and is called the gravitino.

In the above we have restricted ourselves to the consequences of supersymmetry in a quantum mechanical system. In a field theory, it is possible to represent supersymmetry on the fields. To demonstrate this, we turn to some examples. We look at three representations of the $N = 1$ superalgebra on fields in a four-dimensional dimensional field theory, namely the scalar, vector and tensor supermultiplets.

The $N = 1$ scalar multiplet consists of a scalar field A, a pseudo-scalar B, and a single Majorana fermion ζ. The free action for these fields is given by:

$$S = \int d^4x \left(-\partial_\mu A\, \partial^\mu A - \partial_\mu B\, \partial^\mu B - \tfrac{1}{2}\bar{\zeta}\partial\!\!\!/\zeta \right), \tag{1.7}$$

where $\partial\!\!\!/ = \gamma^\mu \partial_\mu$. Note that this action is invariant under the global supersymmetry variations:

$$\begin{aligned} \delta_Q(\epsilon)\, A &= \bar{\epsilon}\zeta\,, \\ \delta_Q(\epsilon)\, B &= -i\bar{\epsilon}\gamma_5\zeta\,, \\ \delta_Q(\epsilon)\, \zeta &= (\partial\!\!\!/ A + i\gamma_5 \partial\!\!\!/ B)\epsilon\,. \end{aligned} \tag{1.8}$$

Here ϵ is the parameter of the supersymmetry transformation, which, for obvious reasons, is a Majorana fermion as well. The above free model can be generalized in a supersymmetric way to include a mass and an interaction term, thus arriving at the so-called Wess-Zumino model [9].

With a simple field redefinition $X = A + iB$, it is possible to write the multiplet in a chiral form as follows:

$$\begin{aligned} \delta_Q(\epsilon)\, X &= 2\,\bar{\epsilon}_L \zeta_L, \qquad & \delta_Q(\epsilon)\, \bar{X} &= 2\,\bar{\epsilon}_R \zeta_R, \\ \delta_Q(\epsilon)\, \zeta_L &= \partial\!\!\!/ X \epsilon_R, \qquad & \delta_Q(\epsilon)\, \zeta_R &= \partial\!\!\!/ \bar{X} \epsilon_L, \end{aligned} \tag{1.9}$$

where $\zeta_{L,R} = \frac{1}{2}(\mathbb{I} \pm \gamma_5)\zeta$, the left- and right-handed chiral projections of the scalar multiplet fermion. For this reason, the $N = 1$ scalar multiplet is often called the chiral multiplet. Note that the chiral decomposition of the Majorana fermion has as its bosonic counterpart a decomposition into holomorphic and antiholomorphic scalars. This feature, although seemingly insignificant at this point, plays a prominent role in the supersymmetric non-linear sigma-models that we discuss in the next section and in chapters 3, 4 and 5.

To check whether the transformations (1.8) form a representation of the anticommutation relation (1.2), we have to perform two subsequent supersymmetry variations:

$$[\,\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)\,]\, X = 2\,\bar{\epsilon}_{[2}\gamma^\mu \epsilon_{1]}\, \partial_\mu X\,, \tag{1.10}$$

(note the change from anticommutator to commutator, due to the exchange of fermions) and analogously for $\bar{X}$. For ζ_R and ζ_L, however, we also find terms proportional to the fermionic field equation. So the multiplet $(X, \bar{X}, \zeta_L, \zeta_R)$ forms a representation of the superalgebra, *only* when we impose the fermionic field equation. In general, we speak of an *on-shell* multiplet when it forms a representation of the supersymmetry algebra, provided the equations of motion hold. If we add a new complex scalar Y and change the transformation rules for the multiplet $(X, \bar{X}, \zeta_L, \zeta_R, Y, \bar{Y})$ as follows:

$$\begin{aligned} \delta_Q(\epsilon)\, X &= 2\bar{\epsilon}_L \zeta_L\,, \\ \delta_Q(\epsilon)\, \zeta_L &= \partial\!\!\!/ X \epsilon_R + Y \epsilon_L\,, \\ \delta_Q(\epsilon)\, Y &= \bar{\epsilon}_R \partial\!\!\!/ \zeta_L\,, \end{aligned} \tag{1.11}$$

then the supersymmetry algebra closes without the need for field equations and we speak of an *off-shell* representation. Its form is independent of the form of the action. A supersymmetric action is found by adding to (1.7) the term $\frac{1}{2}Y\bar{Y}$. Note that the field equations of Y and $\bar{Y}$ are algebraic. When resubstituted into the action, we retrieve the action (1.7) for components of the on-shell multiplet. Such non-propagating fields that are needed to close the supersymmetry algebra without reference the action, are called *auxiliary* fields.

In fact, the addition of auxiliary degrees of freedom is related to the counting of bosonic and fermionic dimensions in a supersymmetry representation. Recall that the Dirac equation projects out half of the polarizations of the spinor it acts on. Namely, suppose that we are looking for positive-energy plain-wave solutions of the Dirac equation of the form:

$$\zeta(x) = \mathrm{e}^{-ik\cdot x} u(k). \tag{1.12}$$

If we now impose the Dirac equation, $(\partial\!\!\!/ + m)\zeta = 0$, with the mass m possibly zero, we find:

$$(-ik_\mu \gamma^\mu + m)u(k) = 0, \tag{1.13}$$

Note that $(\partial\!\!\!/ - m)(\partial\!\!\!/ + m)\zeta = 0$ ensures that $k^2 = -m^2$, so k_μ is indeed the four-momentum. Note that if $m \neq 0$, we can choose to go to the rest frame, in which (1.13) becomes:

$$(-i\gamma_0 + \mathbb{I})u(k) = 0. \tag{1.14}$$

Since $(-i\gamma_0)^2 = \mathbb{I}$, the above represents a projection. Given a representation of the gamma-matrices, one can now explicitly write down two solutions of (1.14) representing the physical positive-energy polarizations of a massive fermion with four-momentum k_μ. One easily generalizes the above reasoning to negative energies and to massless representations. So an off-shell Dirac fermion has eight degrees of freedom, an off-shell Majorana fermion has four, because the Majorana condition eliminates half. An on-shell Dirac fermion has four degrees of freedom, two positive-energy solutions and two with negative energy. The Majorana constraint again eliminates half, meaning that an on-shell Majorana fermion has two degrees of freedom.

This is related to the counting as follows: if we consider the scalar multiplet as an on-shell representation, we count two real bosonic degrees of freedom, namely A and B, and two fermionic degrees of freedom. So *on-shell counting* leads to a multiplet with $2 + 2$ degrees of freedom. However, if we do *off-shell counting* we see that nothing changes for the bosonic fields, but the number of degrees of freedom of the fermionic field goes from two to four. This would imply a discrepancy of two degrees of freedom. This is where the degrees of freedom of the auxiliary field Y come in: adding two bosonic degrees of freedom restores the balance and we say that the chiral multiplet has $4+4$ off-shell degrees of freedom.

The second example is the so-called $N=1$ on-shell vector multiplet. It contains an abelian gauge field A_μ and a Majorana fermion χ. The action is given by:

$$S = \int d^4x \left(-\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - \tfrac{1}{2} \bar{\chi} \partial\!\!\!/ \chi \right), \tag{1.15}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, the field strength of the vector gauge field. The supersymmetry transformations under which the action is invariant are given by:

$$\begin{aligned} \delta_Q(\epsilon)\, A_\mu &= \bar{\epsilon}\gamma_\mu \chi\,, \\ \delta_Q(\epsilon)\, \chi &= -\sigma_{\mu\nu} F^{\mu\nu} \epsilon\,, \end{aligned} \tag{1.16}$$

where $\sigma_{\mu\nu} = \frac{1}{4}[\gamma_\mu, \gamma_\nu]$, the generator of Lorentz transformations for spinors. Again we note that the commutator of two supersymmetry variations on the fermion closes into a translation only modulo a fermionic field equation. In this case, we also find an extra term in the supersymmetry commutator of the bosonic field:

$$[\,\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)\,]\, A_\mu = 2\, \bar{\epsilon}_{[2} \gamma^\nu \epsilon_{1]}\, \partial_\nu A_\mu + \partial_\mu(-2\, \bar{\epsilon}_{[2} \gamma^\nu \epsilon_{1]}\, A_\nu). \tag{1.17}$$

But since A_μ is a gauge field, the last term in (1.17) does not represent a physical change of the field A_μ. In other words, the representation of supersymmetry on a gauge field has to satisfy the algebra (2.30) only as far as its physical modes are concerned. We say that the superalgebra closes modulo a gauge transformation. Note that if the gauge field had been coupled to some field Z that transforms under the gauge group, then the gauge transformation resulting from the commutator of two supersymmetry variations *should be represented on Z with the same ϵ-dependent parameter*. Only then does the last term in (1.17) give a proper representation of the gauge group on both the gauge field and the field Z. This means that the fermion χ that is in one multiplet with the gauge field, does not transform under the gauge transformation. In fact, this is due to the abelian nature of our example: if we use a non-abelian group, all fields related to the gauge field by supersymmetry transform in the adjoint representation of the group.

Considering the numbers of degrees of freedom, we recall that the vector gauge field has three off-shell degrees of freedom (the longitudinal polarization can be gauged away) and the field equation eliminates another polarization, leading to two on-shell degrees of freedom. So the on-shell vector multiplet is balanced, having $2+2$ degrees of freedom. In an off-shell representation χ has four degrees of freedom, meaning that the off-shell vector multiplet involves one real auxiliary scalar field, D, and like in the previous example its field equation imposes $D=0$.

The last example is the so-called $N = 1$ tensor multiplet. Its field content is given by a real scalar ϕ, a Majorana fermion λ and an antisymmetric tensor (or two-form) gauge field $B_{\mu\nu}$. The free action for these fields is given by:

$$S = \int d^4x \left(-\tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \tfrac{1}{2}\bar{\lambda}\not{\partial}\lambda + H_\mu H^\mu \right), \tag{1.18}$$

where $H^\mu = \frac{i}{2}\varepsilon^{\mu\nu\rho\sigma}\partial_\nu B_{\rho\sigma}$, the field strength of the tensor gauge field. Note that H_μ is invariant under the gauge transformation:

$$\delta_{\text{tensor}}(\Lambda)\,B_{\mu\nu} = \partial_\mu\Lambda_\nu - \partial_\nu\Lambda_\mu. \tag{1.19}$$

The supersymmetry transformation rules of the tensor multiplet are given by:

$$\begin{aligned} \delta_Q(\epsilon)\,\phi &= \bar{\epsilon}_L\lambda_L + \bar{\epsilon}_R\lambda_R\,, \\ \delta_Q(\epsilon)\,B_{\mu\nu} &= 2\bar{\epsilon}_L\sigma_{\mu\nu}\lambda_L + 2\bar{\epsilon}_R\sigma_{\mu\nu}\lambda_R\,, \\ \delta_Q(\epsilon)\,\lambda_L &= (\not{\partial}\phi - i\not{H})\epsilon_R\,, \end{aligned} \tag{1.20}$$

under which the action (1.18) is invariant. Again, the commutator of two supersymmetry transformations closes into a gauge transformation for the tensor field. However, if we apply the commutator of two supersymmetry transformations to the fermion λ, we do not find a term proportional to the fermionic field equation. This indicates that the multiplet does not change its form if we take it off-shell. Indeed, both on- and off-shell counting are balanced, leading to $2 + 2$ and $4 + 4$ degrees of freedom respectively. So the off-shell $N = 1$ tensor multiplet does not involve auxiliary degrees of freedom. The off-shell counting for the tensor gauge field goes as follows: an antisymmetric two-tensor has 6 independent components. The gauge transformations have a parameter Λ_μ, which has four components. However, since a change of Λ_μ by a derivative $\partial_\mu\theta$ does not affect (1.19), the actual number of gauge degrees of freedom in $B_{\mu\nu}$ is three. Consequently, the off-shell number of degrees of freedom of the tensor field is $6 - 3 = 3$. The on-shell counting again involves a discussion of the equation of motion and the physical polarizations. As it turns out, the tensor gauge field has only one physical polarization, *i.e.* only one on-shell degree of freedom.

1.3 Supersymmetric non-linear sigma-models

Having seen the basic concepts and the most relevant multiplets of simple supersymmetry, we now concentrate on the more general couplings described by $D = 4$, $N = 1$ supersymmetric non-linear sigma-models [14, 15, 60]. In these models, we consider a number of $N = 1$ chiral multiplets, the scalars of which serve as a coordinatization of a Riemannian manifold. As we shall see, the manifolds that can be described by such a supersymmetric sigma-model are so-called Kähler manifolds. In the following chapters, we shall often refer to the concepts introduced in this section, because every multiplet of extended supersymmetry that has a complex scalar field with a chiral supersymmetry variation, *contains* an $N = 1$ scalar multiplet, which is combined with other $N = 1$ multiplets to form a representation of the larger extended supersymmetry algebra[2].

Unfortunately, it is neither possible to discuss supersymmetric non-linear sigma-models without the use of some basic concepts in complex differential geometry, nor to give a comprehensive introduction to this subject within the limits of this section. Therefore we assume some familiarity with the basics of (complex) manifolds. An introduction to differential geometry in general, written for theoretical physicists, can be found, for example, in [13].

Let us begin by considering four-dimensional *non*-supersymmetric sigma-models, which describe couplings of n real scalar fields ϕ^a. The action is proportional to $\partial_\mu\phi^a\partial^\mu\phi^b$, contracted with an $n \times n$ matrix g_{ab}, which can depend on the values of the scalar fields:

$$S = -\int d^4x\, g_{ab}(\phi)\, \partial_\mu\phi^a\partial^\mu\phi^b. \tag{1.21}$$

The scalar fields in the model are interpreted as local coordinates of an n-dimensional, Riemannian manifold $\mathscr{M}$ and $g_{ab}(\phi)$ plays the role of the metric. Note that the action (1.21) is invariant under diffeomorphisms on $\mathscr{M}$.

Before we continue with the discussion of supersymmetric sigma-models, we very briefly review some concepts in Riemannian geometry that play a role in the remainder of this section and in the following chapters. First of all, the existence of a metric implies the possibility to define a unique, torsion-free, metric-compatible connection on the tangent bundle $\mathrm{T}\mathscr{M}$, the *Levi-Civita connection*. Covariant derivatives with respect to the Levi-Civita connection are denoted with D_a. The corresponding curvature is called the *Riemann tensor*. Parallel transport along closed loops starting and ending at some point p in $\mathscr{M}$ with respect to the Levi-Civita connection, gives rise to a linear transformation of the tangent space $\mathrm{T}_p\mathscr{M}$ onto itself, generated by the Riemann tensor. If $\mathscr{M}$ is arc-wise connected, the Lie group of all such transformations is the same at every point in $\mathscr{M}$ and is called the *holonomy group*. Since the metric is covariantly constant, the holonomy group of a manifold with a metric of positive signature is contained in $\mathrm{O}(n)$. Note that we say 'contained', because the curvature tensor determines which part of $\mathrm{O}(n)$ is actually realized; *e.g.* on a flat manifold, the curvature vanishes and the holonomy group consists only of the identity. Furthermore, we note that on some manifolds, there exist globally defined diffeomorphisms that leave the metric invariant, called *isometries*. In infinitesimal form, an isometry implies the existence of a so-called *Killing vector field* k, such that the Lie derivative of the metric along k vanishes. In coordinates, this requirement leads to the *Killing equation*, given by $D_ak_b + D_bk_a = 0$. Of course, two isometries do not necessarily commute. The Lie group of all isometries is the *isometry group* $\mathrm{Isom}(\mathscr{M})$ of $\mathscr{M}$.

When we try to extend the above model to a supersymmetric sigma-model involving $D = 4$, $N = 1$ chiral multiplets [14], we immediately find that the manifold $\mathscr{M}$ is of *even*

[2]For a more detailed explanation of the combination of $N = 1$ multiplets into multiplets of extended supersymmetry, we refer to section 2.1.

dimension. Namely, if n is the number of Majorana fermions, then on-shell counting leads us to introduce $2n$ scalar fields as coordinates for $\mathscr{M}$. Furthermore, the positive- and negative-chiral components are related by complex conjugation, so the coordinates can, at least locally, be assembled in complex coordinates X^i and $\bar{X}^{\bar{\imath}}$, transforming chirally under supersymmetry. In geometrical terms, this implies the existence of a globally defined rank $(1,1)$-tensor J, called an *almost complex structure*, which can be used to split the tangent space into a holomorphic and an antiholomorphic part. The positive- and negative-chiral components of the fermion field, ζ_L^i and $\zeta_R^{\bar{\imath}}$, transform under diffeomorphisms as holomorphic and antiholomorphic vector fields respectively. Note that the kinetic terms for the fermion fields in the action, which are proportional to $\bar{\zeta}_L^i \not{D} \zeta_R^{\bar{\jmath}}$ and its complex conjugate, imply that the metric can be chosen *Hermitian*, *i.e.* the components g_{ij} and $g_{\bar{\imath}\bar{\jmath}}$ are equal to zero. In coordinate-independent form, the Hermiticity condition is written as:

$$g(x,y) = g(Jx, Jy), \tag{1.22}$$

for any two vector fields x and y. Correspondingly, an almost complex manifold $\mathscr{M}$ with a metric that can be brought in Hermitian form, $g_{i\bar{\jmath}} = \overline{g_{\bar{\imath}j}}$, is called a *Hermitian manifold.*

With this knowledge, we are in a position to write down the action:

$$S = \int d^4x \left(-g_{i\bar{\jmath}}\, \partial_\mu X^i \partial^\mu \bar{X}^{\bar{\jmath}} - g_{i\bar{\jmath}}\, \bar{\zeta}_L^i \not{D} \zeta_R^{\bar{\jmath}} - g_{\bar{\imath}j}\, \bar{\zeta}_R^{\bar{\imath}} \not{D} \zeta_L^j + \tfrac{1}{8} R_{i\bar{\jmath}k\bar{l}}\, \bar{\zeta}_L^i \gamma_\mu \zeta_R^{\bar{\jmath}}\, \bar{\zeta}_L^k \gamma_\mu \zeta_R^{\bar{l}} \right), \tag{1.23}$$

where R is the Riemann tensor of the Hermitian sigma-manifold. The action (1.23) is invariant under the supersymmetry variations:

$$\begin{aligned} \delta_Q(\epsilon)\, X^i &= 2\bar{\epsilon}_L \zeta_L^i, \\ \delta_Q(\epsilon)\, \zeta_L^i &= \not{\partial} X^i \epsilon_R - \delta_Q(\epsilon) X^j\, \Gamma^i{}_{jk}\, \zeta_L^k, \end{aligned} \tag{1.24}$$

where Γ is a connection that is to be specified momentarily.

On a *complex* Hermitian manifold, a unique, metric-compatible connection Γ for the tangent bundle exists, such that the complex structure is covariantly constant: $\nabla J = 0$. However, this so-called *Hermitian connection* is not necessarily torsion-free and can therefore differ from the Levi-Civita connection. The compatibility of the Hermitian connection with the complex structure implies that covariant translation respects the holomorphic/antiholomorphic decomposition. So in the coordinate-system $(X^i, \bar{X}^{\bar{\imath}})$, only:

$$\Gamma^i{}_{jk} = g^{\bar{l}i}\, \partial_j g_{k\bar{l}}, \tag{1.25}$$

and its complex conjugate are non-zero and the mixed components of the Hermitian connection vanish.

Note that the covariant derivatives on the fermions and the Riemann tensor are based on the Levi-Civita connection. As it turns out[3], supersymmetry of the action and closure of the

[3] In view of the fact that a detailed explanation in the $N = 2$ supersymmetric case is given in section 4.3, we refrain from elaboration on the $N = 1$ derivation and concentrate on its implications instead.

supersymmetry algebra require that the (almost) complex structure is covariantly constant *with respect to the Levi-Civita connection*:

$$D_a J^b{}_c = 0. \tag{1.26}$$

Hence, we find that the Hermitian connection is *equal* to the Levi-Civita connection and the almost complex structure is promoted to a complex structure.

On a Hermitian manifold a $(1,1)$-form ω called the Kähler form can be defined, the components of which are given by lowering an index of the complex structure: $\omega(x,y) = g(Jx,y)$ for any two vector fields x and y. By definition, a *Kähler manifold* is a Hermitian manifold whose Kähler form ω is closed:

$$\mathrm{d}\omega = 0. \tag{1.27}$$

One can easily show that the conditions (1.26) and (1.27) are equivalent, implying the following alternative definition of a Kähler manifold: *a Hermitian manifold is Kähler if and only if the almost complex structure J is covariantly constant with respect to the Levi-Civita connection.* From this we conclude that $N = 1$ rigidly supersymmetric sigma-models are formulated on Kähler manifolds. In section 4.3, we discuss the $N = 2$ supersymmetric analog of the above model, and find that the relevant class of sigma-manifolds is formed by hyperkähler manifolds, which have three covariantly constant complex structures.

Let us therefore explore the geometry of Kähler manifolds somewhat further. The Kähler property (1.27) implies a local integrability condition for the Hermitian metric, *i.e.* in a coordinate patch with coordinates $(X^i, \bar{X}^{\bar{\imath}})$, the metric can be written as:

$$g_{i\bar{\jmath}}(X,\bar{X}) = \partial_i \bar{\partial}_{\bar{\jmath}} K(X,\bar{X}), \tag{1.28}$$

where K is the so-called *Kähler potential*. Note that (1.28) does not determine the Kähler potential uniquely, because the *l.h.s.* of (1.28) does not change if we add a purely holomorphic or antiholomorphic expression to K. Such a change of the Kähler potential is called a *Kähler transformation*. Since g transforms as a tensor and keeps its Hermitian form under any holomorphic coordinate transformation $X \to X'(X)$, the Kähler potential can be changed by a Kähler transformation:

$$K'(X',\bar{X}') = K(X,\bar{X}) + f(X) + \bar{f}(\bar{X}), \tag{1.29}$$

i.e. the Kähler potential is *not* a function on $\mathscr{M}$. It has to be noted that a far more elegant description of $N = 1$ sigma-models [14] can be given using superspace methods. In that case the Kähler potential arises as the integrand of the superspace integration that gives the action.

Next we consider the holonomy group of a Kähler manifold. First of all, notice that parallel transport does not mix holomorphic and antiholomorphic indices, because the complex structure is covariantly constant. Hence the holonomy group is contained in $\mathrm{U}(n) \subset \mathrm{O}(2n)$.

On a Kähler manifold, the only non-vanishing, independent components of the Riemann tensor are:

$$R^i{}_{j\,\bar{k}l} = \bar{\partial}_{\bar{k}}\Gamma^i{}_{jl}, \tag{1.30}$$

and its complex conjugate. Other components can be found by means of the symmetry relations:

$$R_{\bar{\imath}j\,\bar{k}l} = -R_{j\bar{\imath}\,\bar{k}l} = -R_{\bar{\imath}j\,l\bar{k}}\,, \qquad R_{\bar{\imath}j\,\bar{k}l} = R_{\bar{k}l\,\bar{\imath}j}\,, \qquad R_{\bar{\imath}j\,\bar{k}l} = R_{\bar{\imath}l\,\bar{k}j}\,. \tag{1.31}$$

The first two symmetries are common to all Hermitian complex manifolds, but the last symmetry is a consequence of the Kähler geometry. Note that, as such, the Riemann tensor of a Kähler manifold has exactly the right symmetries for the four-fermion coupling in (1.23).

If a compact Kähler manifold $\mathscr{M}$ is *Ricci-flat*, *i.e.* the Ricci-tensor $R_{i\bar{\jmath}} = R^k{}_{i\,k\bar{\jmath}}$ vanishes, then $\mathscr{M}$ is called a *Calabi-Yau manifold*. The Ricci-flatness condition corresponds to the fact that the holonomy group of a Calabi-Yau manifold is contained in $\mathrm{SU}(n) \subset \mathrm{U}(n)$. In fact, the requirement that the holonomy group is contained in $\mathrm{SU}(n)$ is sometimes used as defining condition for a Calabi-Yau manifold[4]. Calabi-Yau manifolds play a central role in $N = 2$ compactifications of type-II string theories and we come back to them in chapter 5.

The coupling (1.23) does not represent the most general coupling of $N = 1$ scalar multiplets. Namely, from the scalar multiplets a separate invariant of supersymmetry may be constructed, given by:

$$\mathscr{L}_{\mathrm{SP}} = -g^{i\bar{\jmath}}(\partial_i W)(\bar{\partial}_{\bar{\jmath}}\bar{W}) - \tfrac{1}{2}\Big(\bar{\zeta}^i_L(D_i\partial_j W)\zeta^j_L + \text{h.c.}\Big), \tag{1.32}$$

where $W(X)$ is a holomorphic expression in terms of the scalar fields X, called the *superpotential*. Note that if W is quadratic, (1.32) leads to mass-terms in the action for the scalar multiplet. Furthermore, if for some $i \in \{1, \ldots, n\}$, the vacuum expectation value of $\partial_i W$ is non-zero, then supersymmetry is spontaneously broken, because in that case the vacuum energy is non-zero.

Couplings of $N = 1$ scalar multiplets to $N = 1$ vector multiplets can be formulated using the isometry group of the manifold $\mathscr{M}$. Any isometry of $\mathscr{M}$ represents a *global* symmetry of the sigma-model. Minimal coupling of vector fields to Killing vector fields realizes (part of) the isometry group as a local symmetry group of the model [17]. So, given a certain Kähler sigma-model, the gauge groups that can be coupled are the subgroups of the isometry group.

With respect to the isometry groups of Kähler manifolds, the following point has to be noted: when we use a set of coordinates, such that the metric takes its Hermitian form, the Killing equation can be decomposed into two independent conditions:

$$\begin{aligned} D_i k^{\bar{m}} g_{j\bar{m}} + D_j k^{\bar{m}} g_{i\bar{m}} &= 0, \\ D_i k^m g_{\bar{\jmath}m} + D_{\bar{\jmath}} k^{\bar{m}} g_{i\bar{m}} &= 0, \end{aligned} \tag{1.33}$$

[4]Similarly, the quaternionic and hyperkähler manifolds that are used in chapter 4 can be defined as manifolds whose holonomy group is contained in $\mathrm{Sp}(1) \times \mathrm{Sp}(n)$ and $\mathrm{Sp}(n)$ respectively.

where k is the Killing vector field. Since we would like the isometry to preserve the complex structure, we require that it is a holomorphic diffeomorphism. This means that the Killing vector is a holomorphic vector field, *i.e.*:

$$\partial_i k^{\bar{\jmath}} = 0, \qquad \partial_{\bar{\imath}} k^i = 0. \tag{1.34}$$

Note that on a Kähler manifold, the Levi-Civita connection does not mix holomorphic and antiholomorphic indices, making (1.34) a covariant statement. This observation will be generalized to the case of a hyperkähler manifold in section 4.4, where we consider the minimal coupling of so-called tri-holomorphic isometries.

As far as the coupling to supergravity is concerned [19, 16, 17], we suffice to say that a consistent coupling requires that the Kähler manifold is a so-called *Hodge manifold* or *Kähler manifold of restricted type* [16]. A Hodge manifold $\mathscr{M}$ is a Kähler manifold with the extra condition that there exists a line bundle on $\mathscr{M}$ with the cohomology of the Kähler form equal to twice the (integer) first Chern class of the line bundle. The superpotential is interpreted as a section of this line bundle.

1.4 Duality transformations

Duality transformations play a prominent role throughout the following chapters. For that reason, we discuss three duality transformations in some detail, namely the vector-vector and tensor-scalar dualities in four dimensions and the vector-scalar duality in three dimensions, with particular emphasis on their relation to supersymmetry. At the end of this section, we briefly discuss the relation with the central charge in $N = 2$ supersymmetric models.

First we concentrate on the vector-vector duality in four space-time dimensions. We consider an abelian gauge field A_μ with Lagrangian density:

$$4\pi\mathscr{L} = -\frac{1}{4}\Big(\frac{4\pi}{g^2}F_{\mu\nu}F^{\mu\nu} + i\frac{\theta}{2\pi}F_{\mu\nu}\tilde{F}^{\mu\nu}\Big). \tag{1.35}$$

Here g is a coupling constant (which can be absorbed in the normalization of the vector field) and θ is the theta-angle. Note that the $F\tilde{F}$-term amounts to a total derivative, which means that θ can be shifted arbitrarily[5]. We retain this term here not only to illustrate its role in duality transformations, but also because in the context of general $N = 2$ supersymmetric abelian vector multiplet models, the $F\tilde{F}$-term does not have a constant coefficient and therefore does not give rise to a total derivative. We can combine the coupling constant and

[5]The $F\tilde{F}$-term leads to a non-zero contribution in the action, only if the gauge field attains a non-trivial configuration on the S^3 at (Euclidean) infinity. For instance, when the above model describes the effective action of a pure SU(2) Yang-Mills model that has been broken to a U(1) subgroup, then the $F\tilde{F}$-term in the action equals the Pontryagin-index $(32\pi^2)^{-1}\int F\wedge F$, which gives the instanton-number. In that case, the shift symmetry of the theta-angle is broken to a discrete shift. This also explains the overall normalization factor of 4π for the Lagrangian density. (See also the remarks made at the end of section 3.2.)

theta-angle in one complex constant τ, defined by:

$$\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2}\,, \tag{1.36}$$

with which the Lagrangian density (1.35) can be written in the form:

$$4\pi\mathscr{L} = -\tfrac{1}{4}i\Big(\tau F^-_{\mu\nu}F^{-\,\mu\nu} - \bar\tau F^+_{\mu\nu}F^{+\,\mu\nu}\Big), \tag{1.37}$$

where $F_{\mu\nu}$ has been decomposed in its selfdual and anti-selfdual components (see appendix A for a definition).

A duality transformation is now effected by integrating out the field strength $F_{\mu\nu}$. However, $F_{\mu\nu}$ is subject to a constraint, the Bianchi identity, expressing the fact that F is a closed two-form:

$$\varepsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0. \tag{1.38}$$

After we impose (1.38) by means of a Lagrange multiplier term in the action, the field strength $F_{\mu\nu}$ can be treated as an unconstrained field, thus enabling us to integrate it out. Note that (1.38) has a free Lorentz index μ, implying that the Lagrange multiplier is a vector field, which we denote by A^D_μ. The Lagrange multiplier term that is to be added to (1.37) takes the form:

$$\begin{aligned} 4\pi\mathscr{L}_B &= \tfrac{1}{2}iA^D_\mu\varepsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} \\ &= -\tfrac{1}{2}i\Big((F^D)^-_{\mu\nu}F^{-\,\mu\nu} - (F^D)^+_{\mu\nu}F^{+\,\mu\nu}\Big) + i\partial_\mu((F^{-\,\mu\nu} - F^{+\,\mu\nu})A^D_\nu). \end{aligned} \tag{1.39}$$

where $(F^D)_{\mu\nu} = 2\partial_{[\mu}A^D_{\nu]}$. Looking at the first line in (1.39) we see that if we change A^D_μ by the addition of a term $\partial_\mu\Lambda$, the resulting change in the Lagrange density vanishes identically. Hence A^D_μ is subject to a gauge transformation. A trivial, yet important observation at this stage is the fact that the *field equation* of A^D imposes the *Bianchi identity* of the field strength.

The sum of the Lagrange density (1.37) and the Lagrange multiplier (1.39) can be written in the form:

$$\begin{aligned} 4\pi(\mathscr{L} + \mathscr{L}_B) &= -\tfrac{1}{4}i\tau(F^-_{\mu\nu} + \frac{1}{\tau}(F^D)^-_{\mu\nu})(F^{-\,\mu\nu} + \frac{1}{\tau}(F^D)^{-\,\mu\nu}) \\ &\quad -\tfrac{1}{4}i\Big(-\frac{1}{\tau}\Big)(F^D)^-_{\mu\nu}(F^D)^{-\,\mu\nu} + \text{h.c.}\,, \end{aligned} \tag{1.40}$$

where we have dropped the total derivative in the second line of (1.39). The unconstrained two-form field $F_{\mu\nu}$ arises in the action only quadratically and can be integrated out of the path integral as a (translated) gaussian. The resulting action depends only on $(F^D)_{\mu\nu}$ and takes the form:

$$4\pi\mathscr{L}_{\text{Dual}} = -\tfrac{1}{4}i\Big(\tau^D\,(F^D)^-_{\mu\nu}(F^D)^{-\,\mu\nu} - \bar\tau^D\,(F^D)^+_{\mu\nu}(F^D)^{+\,\mu\nu}\Big), \tag{1.41}$$

where the new coupling constant and theta-angle, encoded in the complex constant τ^D, are related to the original g and θ through:

$$\tau^D = -\frac{1}{\tau}. \tag{1.42}$$

Note that the abelian gauge field A_μ entered the Lagrange density only through its field strength $F_{\mu\nu}$. Had there been a coupling of A_μ to some field transforming under the local abelian gauge transformations associated with A_μ, then we would not have been able to integrate out $F_{\mu\nu}$. More particularly, if there had been a charged field in the model or if the gauge group had been non-abelian, A_μ would have appeared in the action coupled to other fields, thus inhibiting the gaussian integration over $F_{\mu\nu}$.

Also important to stress is the fact that a duality transformation is *not* a symmetry of the model, because the coupling constant and theta-angle are *changed*. A duality transformation maps one description of the model to another, equivalent description. Note that the field equation and the Bianchi identity *change roles* when we perform the above duality transformation. The field equation for the Lagrange multiplier field A^D_μ is equivalent to the Bianchi identity for the field strength $F_{\mu\nu}$. Performing this duality transformation once more, thus mapping the description in terms of $(F^D)_{\mu\nu}$ back to the original description, we see that the opposite is also true: the field equation for the field A_μ is equivalent to the Bianchi identity for $(F^D)_{\mu\nu}$. As such, the above duality transformation can be viewed as the interchange of roles between field equation and Bianchi identity. This will become important in section (3.2), where we generalize the duality transformations to include *all transformations that transform the set of field equations and Bianchi identities into an equivalent set.* Note that this group of transformations also includes shifts of the theta-angle.

Next, we consider the extension of the above duality transformation to a supersymmetric model. First we consider the case of $N = 1$ supersymmetry and at the end of this section we briefly look at the implications for an $N = 2$ supersymmetric field theory. The $N = 1$ vector-multiplet action (1.15) can be extended to include a coupling constant g and theta-parameter by means of a selfdual/antiselfdual decomposition of the field strength $F_{\mu\nu}$. Substituting this decomposition of $F_{\mu\nu}$ into the fermionic transformation rule in (1.16) and using (A.5), we find that the positive- and negative-chiral projections of χ transform only into the antiselfdual and selfdual parts of $F_{\mu\nu}$ respectively. Correspondingly, $F^-_{\mu\nu}$ transforms only into χ_L and $F^+_{\mu\nu}$ only into χ_R. Introduction of τ in the action (1.15) leads to:

$$\begin{aligned} 4\pi\mathscr{L}_{\rm Susy} &= -\tfrac{1}{4}i\Big(\tau F^-_{\mu\nu}F^{-\,\mu\nu} - \bar\tau F^+_{\mu\nu}F^{+\,\mu\nu}\Big) \\ &\quad -\tfrac{1}{4}i(\tau-\bar\tau)\Big(\bar\chi_R \partial\!\!\!/ \chi_L + \bar\chi_L \partial\!\!\!/ \chi_R\Big). \end{aligned} \tag{1.43}$$

When we perform a duality transformation as before, we are immediately confronted with the fact that we do not know the supersymmetry variation of the Lagrange multiplier A^D_μ and what the supersymmetry variation of the fermion is in terms of A^D_μ or $F^D_{\mu\nu}$. In other words, after the duality transformation on the vector field strength, the other fields in the model are no longer in the standard parameterization of an $N = 1$ vector multiplet. Furthermore, the action is no longer manifestly supersymmetric, because the coupling τ in front of the vector kinetic term is inverted, but the fermionic term keeps its form.

As far as the supersymmetry variation of A^D_μ is concerned, we note that it can be derived from the supersymmetry of the action. Namely, after we introduce the Langrange multiplier term, the field $F_{\mu\nu}$ no longer satisfies the Bianchi identity. Upon supersymmetry variation of the action (1.43) *plus* the Lagrange multiplier, we find a term in the variation of (1.43), proportional to the Bianchi identity, which has to be cancelled by the variation of the Lagrange multiplier term (possibly modulo a total derivative). Since the latter is given by the variation of A_μ^D contracted with the Bianchi identity, this fixes $\delta_Q A_\mu^D$. Of course, there are some subtleties in the above reasoning, due to the fact that we do not know the variation of the part of the field strength that does not satisfy the Bianchi identity, but in the relatively simple case at hand they do not change the argument. Furthermore, the supersymmetry variation of the fermion χ depends on the field strength $F_{\mu\nu}$. However, when integrating out $F_{\mu\nu}$, we obtain an expression for $F_{\mu\nu}$ in terms of the other fields, *i.e.* $F^-_{\mu\nu} = -(\tau)^{-1}(F^D)^-_{\mu\nu}$. Substitution in the fermionic transformation rule shows that the field $-\tau\chi_L$ transforms into $(F^D)^-_{\mu\nu}$ in exactly the same way as χ_L transforms into $F^-_{\mu\nu}$. Therefore we extend the action of the duality transformation to the fermion field and define $\chi^D_L = -\tau\chi_L$ and $\chi^D_R = -\bar{\tau}\chi_R$. In terms of $(F^D)_{\mu\nu}$, χ^D and τ^D, the action and supersymmetry transformation rules again take the form (1.43) and (1.16).

The above can be summerized as follows: when making the duality transformation, we change the field representation of the spin-1 sector of the supersymmetric model under consideration. But at the same time, manifest supersymmetry of the model requires that we change the other fields accordingly. We shall encounter a similar situation when we consider the $N = 2$ supersymmetric extension of the above $N = 1$ vector multiplet model. However, $N = 2$ vector multiplets contain complex scalar fields[6], which parameterize a sigma-manifold[7]. Therefore duality transformations in the context of $N = 2$ vector multiplet models, which are called symplectic reparameterizations[8], not only induce fermion redefinitions, but also recoordinatizations of the sigma-manifold.

Duality transformations such as the one described in the above example are easily generalized to other dimensions and different ranks of the field strength tensor. Consider the $N = 1$ supersymmetric tensor multiplet model described in section 1.2. The Bianchi identity for the three-form field strength $H_{\mu\nu\rho} = \partial_{[\mu}B_{\nu\rho]}$ takes the form $\varepsilon^{\mu\nu\rho\sigma}\partial_\mu H_{\nu\rho\sigma} = 0$. In terms of the Poincaré dual one-form of $H_{\mu\nu\rho}$, given by $H_\mu = \frac{i}{2}\varepsilon_{\mu\nu\rho\sigma}\partial_\nu B_{\rho\sigma}$, the Bianchi-identity takes the form $\partial_\mu H^\mu = 0$. Note that the higher rank of the tensor gauge field implies that the Bianchi identity is a Lorentz scalar and correspondingly, we introduce a *scalar* Lagrange multiplier a, which we call an *axion* for reasons to be explained shortly. The Lagrange multiplier term then takes the form:

$$\mathscr{L}_B = a\,\partial_\mu H^\mu. \tag{1.44}$$

Note that any change in the Lagrange multiplier of the form $a(x) \to a(x) + \text{constant}$, changes

[6]see section 2.3
[7]See sections 3.1 and 3.3
[8]See section 3.2

$\mathscr{L}_B$ only by a total derivative.

When we add $\mathscr{L}_B$ to the tensor multiplet Lagrangian density, given in (1.18), and perform the duality transformation, the kinetic term $+H_\mu H^\mu$ for the tensor gauge field is replaced by a scalar kinetic term $-\partial_\mu a\, \partial^\mu a$. So the duality transformation exchanges the single on-shell degree of freedom of the tensor gauge field for a single scalar degree of freedom and the dual action is of the form (1.7). One can derive the supersymmetry variations of ϕ, λ and a, employing the method explained in the context of the $N = 1$ vector multiplet. Not surprisingly, we find that the resulting model describes a scalar multiplet, *c.f.* (1.9). Note that constant shifts in $a(x)$ represent a symmetry of the dual model.

In the case where the two-form is coupled to a vector field V_μ through a Chern-Simons coupling, *i.e.* an additional term $V_{[\mu}\partial_\nu V_{\rho]}$ in the three-form field strength, the Bianchi-identity takes the form: $\partial_\mu H^\mu = \frac{1}{4} i F_{\mu\nu} \tilde{F}^{\mu\nu}$. Then the Lagrange multiplier term leads to an $F\tilde{F}$-term for the vector field strength, in which the field a assumes the role of the theta-angle, whence also the name axion. Note that the shift of the axion by a constant, called a *Peccei-Quinn symmetry*, is still a symmetry of the dual action. However, as we have seen previously, such continuous shift symmetries can be broken to a discrete subgroup by instanton effects.

In the case at hand, the Peccei-Quinn symmetry of a does not look very remarkable, the more because the (free) scalar ϕ also possesses a shift-symmetry. However, the Peccei-Quinn symmetry of the axion a arises as a result of its duality to a tensor gauge field and generalizes to much more complicated models, such as the $N = 2$ vector-tensor multiplet and its dual vector-multiplet models, discussed in section 6.6.

A duality transformation that is employed in chapter 5, is the duality between a vector gauge field and a scalar in *three* dimensions. To appreciate this and to give an introduction to the methods employed in section 5.2, we start by considering the four-dimensional vector-gauge model (1.37) and perform a so-called *dimensional reduction*. This entails the following steps: we consider the theory not in four-dimensional flat Minkowski space, but on $\mathbb{R}^3 \times S^1$, a circle-compactification, where the circle is space-like with radius R. We choose space-time coordinates such that x^3, say, parameterizes the circle S^1. Then we decompose all fields in non-trivial representations of the *four*-dimensional Lorentz group $\mathrm{SO}(3,1)$ (spin > 0), into representations of the *three*-dimensional Lorentz group $\mathrm{SO}(2,1)$ that is associated with the coordinates x^0, x^1, x^2. Furthermore we assume that the dependence of all fields on x^3 vanishes. Then we take the limit $R \to 0$, normalizing all fields such that the resulting theory can be viewed as an effective three-dimensional description. The procedure sketched above is called *dimensional reduction*. The generalization to gravity-coupled models requires that the model is formulated on a space-time that has a space-like $\mathrm{U}(1)$ isometry. When we assume that all fields do not depend on the coordinate along the isometric direction and we perform similar steps as in the above flat case, the space-time dimension is reduced by one, resulting in an effectively three-dimensional model [101].

In the case of the vector-gauge model (1.37), dimensional reduction leads to the following three-dimensional action:

$$4\pi\mathscr{L}_3 = -\tfrac{1}{4}i(\tau\bar{W}_\mu\bar{W}^\mu - \bar{\tau}W_\mu W^\mu) \tag{1.45}$$

where $W_\mu = \partial_\mu A - \frac{1}{2}\varepsilon_{\mu\nu\rho}F^{\nu\rho}$. Here A is the component of the vector field along the x^3-direction, $F^{\nu\rho}$ is the three-dimensional field strength and $\varepsilon^{\mu\nu\rho}$ is the three-dimensional Levi-Civita symbol. The Bianchi identity is given by[9] $\varepsilon^{\mu\nu\rho}\partial_\mu F_{\nu\rho} = 0$. The Bianchi identity for the vector field strength in three dimensions is a scalar identity, so we introduce a scalar Lagrange multiplier B. The Lagrange multiplier term then takes the form:

$$\mathscr{L}_B = B\,\varepsilon^{\mu\nu\rho}\partial_\mu F_{\nu\rho}. \tag{1.46}$$

Integrating out $F_{\mu\nu}$, we obtain a Lagrangian density that depends on the scalars A and B only:

$$\mathscr{L}_{\text{dual}} = -\frac{i}{\tau-\bar{\tau}}(\partial_\mu B - \tau\partial_\mu A)(\partial^\mu B - \bar{\tau}\partial^\mu A). \tag{1.47}$$

Note that like the axion in the tensor-scalar duality in four dimensions, the field B is subject to a shift-symmetry $B(x) \to B(x) + b$, where b is a constant. Moreover, in this case, the scalar A can also be shifted by a constant, $A(x) \to A(x) + a$, corresponding to four-dimensional gauge transformations with gauge parameter $\Lambda(x) = a\,x^3$.

Effectively, we have replaced the two on-shell degrees of freedom of the four-dimensional vector gauge field by two scalar degrees of freedom in a three-dimensional model. Note that we could have performed a dimensional reduction on a four-dimensional model containing two scalar fields, to obtain an action of the form (1.47). Hence, this construction defines a map between (the coupling constants of) a *four*-dimensional vector gauge model and a *four*-dimensional model describing two scalar fields. Due to the relatively simple nature of the (non-supersymmetric) models in the above, this mapping is a trivial exercise and does not lead to any interesting conclusions. However, in chapter 5, the above steps are performed for general abelian $N = 2$ vector-multiplet models, which are mapped to four-dimensional hypermultiplet couplings. The duality transformations that play a role in four-dimensional $N = 2$ vector-multiplet models are inherited by the constructed hypermultiplet couplings. Not only does this raise some interesting geometrical questions concerning the non-linear sigma-models describing the scalar sectors of the respective models, it also corresponds to a (classical version of) a duality of $N = 2$ compactifications of type-II string theories, called mirror-symmetry.

Finally, some remarks concerning the relation of duality transformations to $N = 2$ supersymmetry and in particular the central charge[10] are in place. One can show [21] that the value of the central charge for a given on-shell field or state is lower-bounded by its mass. This is called the *BPS bound.* States for which the central charge equals the mass are called

[9] Note that this is the only component of the four-dimensional Bianchi identity (1.38) that remains after dimensional reduction, because $\partial_3 F_{\mu\nu} = 0$

[10] For a definition, see section 2.1.

BPS-saturated. Furthermore, in many theories the central charge is related to other conserved quantities, such as the electric (and magnetic) charges in an abelian gauge model [54]. As was said earlier, besides inverting the coupling constant, the duality transformation in such a model exchanges the roles of the field equation and the Bianchi identity. Note that an electrically charged field configuration $A_\mu(\vec{x}\,)$ gives rise to a contribution to the *r.h.s.* of the field equation and a magnetic monopole gives a contribution to the *r.h.s.* of the Bianchi identity. Therefore a duality transformation induces a transformation among electric and magnetic charges. However, the central charge, which is defined without reference to the particular field representation of the model, is invariant under duality. Hence the central charge (or more specifically, the spectrum of BPS-saturated states) provides a very powerful tool in the study of duality transformations in models with extended supersymmetry (see *e.g.* [83, 84] and for a review, see [22] and references therein).

In section 5.5, we use the central charges that arise in the context of dimensionally reduced $N = 2$ abelian vector-multiplet models and relate them to the central charge of four-dimensional vector-multiplet models [100] and, through the duality transformation, with the central charges of the dimensionally reduced hypermultiplet model.

1.5 Gauge equivalence

In theories of extended supergravity, one is confronted with models that have a large number of fields. Since every supercharge has four components, the number of degrees of freedom in field representations typically grows exponentially with N (this statement is quantified in section 2.3). Moreover, the off-shell formulations of extended Poincaré supergravity are characterized by complicated non-linear terms in the supersymmetry transformation rules. Secondly, the requirement of local invariance under supersymmetry is a very general one: it is comparable with the requirement of general coordinate invariance in non-supersymmetric field theories. To construct *general* actions for models with local extended supersymmetry, it is therefore desirable to have a framework that is more restrictive, so that one can construct supergravity theories in a more structured and systematic fashion.

One of the ways in which the construction can be facilitated is through the use of a so-called *gauge-equivalent* formulation, *i.e.* a model with a larger symmetry group, in which a suitable gauge choice leads to the model that one would like to describe. The advantage lies in the fact that the larger symmetry group poses more stringent requirements on the transformation rules and the coupling terms, leading to relatively simple expressions. In the case of a theory of extended Poincaré supergravity, it is advantageous to recast the theory in a form that is symmetric under the larger *superconformal* algebra. This is accomplished by making a field redefinition that introduces additional local conformal symmetries and new degrees of freedom at the same time, in such a way that the total number of degrees of freedom is not changed. The field redefinition is made such that it allows gauge choices that break the extra, conformal symmetries and lead back to Poincaré theories. This means that the

Poincaré model is equivalent to the conformal model, because they differ only by a gauge transformation. The extra fields that one introduces are called *compensating fields*.

To clarify the concepts underlying gauge equivalence, we treat in this section two examples of gauge-equivalent descriptions, namely one in which we reformulate the Proca field theory as an abelian gauge theory and a somewhat more complicated example in which we consider a gauge-equivalent scale-invariant version of the Einstein-Hilbert action.

The Proca field is a massive vector field V_μ, the dynamics of which are prescribed by the Lagrangian density:

$$\mathscr{L}_{\text{Proca}} = -\tfrac{1}{4}(\partial_\mu V_\nu - \partial_\nu V_\mu)^2 - \tfrac{1}{2}m^2 V_\mu V^\mu. \tag{1.48}$$

Note that the mass term breaks the gauge invariance that we know from the massless vector field: $V_\mu \to V_\mu + \partial_\mu \Lambda$. As is well-known, the Proca field can be split up into transversal and longitudinal degrees of freedom, through the non-local projections:

$$(V_\perp)_\mu = \Big(\eta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\Box}\Big) V^\nu, \quad (V_\parallel)_\mu = \frac{\partial_\mu \partial_\nu}{\Box} V_\nu\,, \tag{1.49}$$

satisfying $\partial_\mu (V_\perp)^\mu = 0$, $\partial_\mu (V_\parallel)^\mu = \partial_\mu V^\mu$. Note that the above projections commute with Lorentz transformations, meaning that the Proca field forms a reducible representation of the Lorentz group, namely the $(\frac{1}{2}, \frac{1}{2}) \oplus (0,0)$ representation. The question now arises if we can decompose the Proca field according to its Lorentz-irreducible components, without non-local expressions like (1.49). To answer that question we consider the following field redefinition, which was introduced by Stueckelberg [10]:

$$V_\mu = A_\mu - \frac{1}{m}\partial_\mu \phi\,. \tag{1.50}$$

Note that the scalar field ϕ is used to describe the longitudinal degree of freedom. The transversal degrees of freedom are described by the gauge field A_μ. The corresponding gauge transformation acts also on ϕ, in such a way that V_μ remains invariant:

$$\delta\, A_\mu(x) = \partial_\mu \Lambda(x)\,, \qquad \delta\, \phi(x) = m\Lambda(x)\,. \tag{1.51}$$

Substitution of the redefinition (1.50) into the Lagrangian (1.48), leads to the new Lagrangian:

$$\mathscr{L}_{\text{Equiv}} = -\tfrac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \tfrac{1}{2}(D_\mu \phi)^2, \tag{1.52}$$

where $D_\mu \phi = \partial_\mu \phi - m A_\mu$. The new action is invariant under the abelian gauge transformation (1.51). Note that we have not changed the number of degrees of freedom: even though there is a new scalar field $\phi(x)$, we have at the same time introduced a local gauge symmetry, parameterized by $\Lambda(x)$. Effectively, we have decomposed the Proca field in its spin-1 and its spin-0 part. We can get back to the description in terms of the Proca field, by means of the condition:

$$\phi(x) = 0\,, \tag{1.53}$$

which breaks the gauge freedom and casts ϕ from the description simultaneously. Alternatively, ϕ can be absorbed into A_μ through a ϕ-dependent gauge transformation, after which ϕ decouples from the theory because of gauge invariance. In fact, such a field-dependent gauge transformation that decouples a degree of freedom can be regarded as a definition of gauge equivalence. Because the new model differs from the original model only by a gauge transformation, it is called a *gauge-equivalent* representation.

In the second example we consider the Einstein-Hilbert action for pure gravity in d dimensions:

$$S_{\text{EH}} = -\int d^dx\,\sqrt{-g}\,R. \tag{1.54}$$

As an introduction to the conformal methods we employ in the second chapter, we derive in this example a gauge-equivalent action that displays invariance under local scale transformations. We start by introducing so-called Weyl rescalings (or dilatations) with parameter $\Lambda_D(x)$, under which the space-time metric $g_{\mu\nu}$ transforms as:

$$\delta_{\text{D}}(\Lambda_D)\,g_{\mu\nu} = -2\Lambda_D\,g_{\mu\nu}. \tag{1.55}$$

Using the standard definitions of the space-time volume measure, the Levi-Civita connection, the curvature tensor and the Ricci tensor, one can show that under (1.55), we have:

$$\begin{aligned}\delta_D(\sqrt{-g}) &= -\tfrac{1}{2}\sqrt{-g}\,g_{\mu\nu}\,\delta_D g^{\mu\nu}\\ &= -d\,\Lambda_D\,\sqrt{-g},\\ g^{\mu\nu}\,\delta_D R_{\mu\nu} &= D_\mu(D_\nu(\delta_D g^{\mu\nu}) - g^{\rho\sigma}D^\mu(\delta_D\,g_{\rho\sigma}))\\ &= -2(d-1)\Box\Lambda_D\,,\end{aligned} \tag{1.56}$$

where $\Box = D^\mu D_\mu$ is the covariant d'Alembertian. With these variations, one easily shows that the Einstein-Hilbert action transforms under Weyl rescalings as:

$$\begin{aligned}\delta_D\,S_{\text{EH}} &= -\int d^dx\,\Big(\delta_D(\sqrt{-g})\,R + \sqrt{-g}\,R_{\mu\nu}\,\delta_D\,g^{\mu\nu} + \sqrt{-g}\,g^{\mu\nu}\delta_D\,R_{\mu\nu}\Big)\\ &= -\int d^dx\,\sqrt{-g}\,\Big((2-d)\Lambda_D R - 2(d-1)\Box\Lambda_D\Big).\end{aligned} \tag{1.57}$$

Note that in this case, the variation of the Ricci tensor only leads to a total derivative. Obviously, the Einstein-Hilbert action is not invariant under local scale transformations.

To find a scale-invariant action we introduce a (compensating) real scalar field a, which transforms under dilatations as:

$$\delta_D(\Lambda_D)\,a = \tfrac{1}{2}(d-2)\Lambda_D a, \tag{1.58}$$

One can check using (1.56) that the action:

$$S'_{\text{EH}} = -\int d^dx\,\sqrt{-g}\Big(a^2 R - \frac{4(d-1)}{d-2}\partial_\mu a\,\partial^\mu a\Big), \tag{1.59}$$

is invariant under local scale transformations. We have to choose the ‘wrong’ sign for the kinetic term of the scalar field in order to keep the right sign for the scalar curvature. Note that we have not changed the number of degrees of freedom of the model: the metric $g_{\mu\nu}$ describes $\frac{1}{2}d(d-1)$ degrees of freedom, which decompose into a $\frac{1}{2}(d-2)(d+1)$ dimensional massive spin-2 representation of the d-dimensional Lorentz group, and a scalar that represents the scale of the metric. In the new action $S'_{\rm EH}$ the scale has been split off in the form of the scalar a, just like we split off the longitudinal part of the Proca field in the previous example. One finds back the original action (1.54), by imposing:

$$a(x) = 1, \tag{1.60}$$

which breaks scale invariance. The above gauge choice is a simple way to demonstrate the gauge equivalence, but it is not essential: alternatively, one could scale away a by making a finite Weyl rescaling with $\exp(\Lambda_D) = a^{-1}$. This corresponds to the fact that we could have obtained the action (1.59) by defining a new scale-invariant metric:

$$g_{\mu\nu}^{\rm new} = a^{\frac{4}{d-2}}\, g_{\mu\nu}. \tag{1.61}$$

making the decomposition in scaling part and spin-2 part explicit, analogous to the definition (1.50).

Note that the above construction can be generalized to other models that involve gravitational degrees of freedom, through multiplication by suitable powers of a for every term that is not scale invariant. For example, the Einstein-Hilbert action with a cosmological constant is gauge-equivalent to a model described by the action (1.59) with a scale-invariant scalar potential of the form:

$$V(a) = r\, a^{\frac{2d}{d-2}}, \tag{1.62}$$

where r is the cosmological constant. However, the compensating field a must be present in the model: note that it is possible to write down a scale-invariant action in four space-time dimensions without the compensating field:

$$S = \int d^4x\, \sqrt{-g}\Big(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\Big), \tag{1.63}$$

but this action is second order in the curvature and depends only on the spin-2 part of the metric.

Having seen the two examples above, we are now in a position to outline the strategy we are going to follow in the second chapter. First we exploit the relatively large symmetry algebra that is described by the highest-spin part of the fields in the gauge theory of the Poincaré superalgebra. Based on the arguments put forth by Haag *et al.*, it is not very surprising that this larger algebra is the superconformal algebra. So in section 2.2, we first construct the so-called Weyl multiplet, which contains the gauge fields for the superconformal algebra. These fields form the analogs of the massless gauge field A_μ in the Proca model and the scale-invariant metric $g_{\mu\nu}^{\rm new}$ used in the above examples. Coupling the superconformal gauge fields to an appropriate set of compensating fields then leads to a model that is gauge-equivalent to Poincaré supergravity. Again, we stress that due to the larger symmetry algebra the expressions found in superconformal models are *simpler* than those found in a Poincaré model. It is therefore relatively easy to construct general couplings in conformal supergravity and use them to construct general couplings in Poincaré supergravity, via a procedure similar to the above.

With this knowledge we turn to the construction of conformal supergravity and its couplings to other multiplets in the next chapter.

Chapter 2

Supersymmetry and Supergravity

2.1 $N = 2$ supersymmetry

In section 1.2, it was noted that in a supersymmetric model, it is the possible that there are several supersymmetries instead of just one. The number of supersymmetries is usually denoted with N, so a four-dimensional theory with N-extended supersymmetry has N distinct Majorana supercharges Q^i, with i ranging from 1 to N. Individually, all supercharges Q^i have to satisfy the restrictions posed earlier for the $N = 1$ superalgebra. However, it has not yet been prescribed how supercharges Q^i, Q^j with $i \neq j$ anticommute. As alluded to in section 1.2, these anticommutation relations are also fixed [6], and given by:

$$\{Q^i, \bar{Q}^j\} = 2\gamma^\mu P_\mu \delta^{ij} + Z^{ij}, \tag{2.1}$$

where $Z^{ij} = z^{ij} + i\gamma_5 (z')^{ij}$ consists of a Lorentz scalar and a pseudoscalar, both antisymmetric in i and j. Note that separately, the supercharges indeed satisfy the $N = 1$ algebra.

The generator Z is new to us, in the sense that it has no $N = 1$ equivalent. One easily shows that the Jacobi identities of the algebra impose:

$$[\, Q^i, Z^{jk}\,] = 0, \qquad [\, P_\mu, Z^{jk}\,] = 0, \qquad [\, M_{\mu\nu}, Z^{jk}\,] = 0, \tag{2.2}$$

where $M_{\mu\nu}$ are the generators of Lorentz transformations. So Z commutes with all other generators of the Poincaré superalgebra and for that reason it is called the *central charge*. Due to its antisymmetry, the central charge has $\frac{1}{2}N(N-1)$ independent components. So in the case of $N = 2$ supersymmetry, there is only one central charge, corresponding to the fact that every antisymmetric 2×2 matrix can be written as a multiple of ε^{ij}. Note that the action of the central charge is very simple: given an irreducible representation of the superalgebra, the action of the central charge is nothing other than a transformation into an isomorphic representation. According to Schur's lemma, in a linear, finite-dimensional, irreducible representation of supersymmetry, a central-charge transformation acts as a multiple of the identity. However, in chapters 4 and 6 we shall see two examples of (infinite-dimensional) supersymmetry representations that do not have a multiplicative action of the central charge.

Another noteworthy aspect of the algebra of extended supersymmetry is the fact that it gives a natural representation of non-abelian (global) symmetries on the representations of the superalgebra. To show how this comes about, we ignore the central-charge term in (2.1) for the moment. The remaining term on the *r.h.s.* is invariant under the unitary group $\mathrm{U}(N)$, which commutes with the Lorentz group[1]. So a rearrangement of the supercharges Q^i by a $\mathrm{U}(N)$ transformation does not change the form of their anticommutation relation. The group of transformations of the supercharges that leave the algebra invariant is called the *automorphism group*. The above $\mathrm{U}(N)$-factor is contained in the automorphism as the group of automorphisms that commute with the Lorentz group and is usually denoted $\mathrm{U}(N)_R$. Note that compatibility with the Majorana condition on the supercharges requires that the $\mathrm{U}(N)_R$ transformations act in a chiral way: the chiral projections $Q^i_L = \frac{1}{2}(\mathbb{I}+\gamma_5)Q^i$ and $Q^i_R = \frac{1}{2}(\mathbb{I}-\gamma_5)Q^i$ transform in conjugate representations $\mathbf{N}$ and $\bar{\mathbf{N}}$ of $\mathrm{U}(N)_R$. For this reason, we introduce a notation for the chiral projections of the supercharges that also infers the representation of $\mathrm{U}(N)_R$: we denote $Q_i = Q^i_L$ and $Q^i = Q^i_R$ with the upper index transforming in the $\mathbf{N}$ and the lower index in the $\bar{\mathbf{N}}$ representation. Similarly, the chiral projections of other Majorana fermions are assigned upper or lower indices, although not always the positive chirality is associated with the $\bar{\mathbf{N}}$ representation. For instance, the chirally projected supersymmetry parameters ϵ^i and ϵ_i are left and right chiral respectively (for other examples, see the tables in appendix C). In the presence of a central charge the invariance of the antisymmetric Z^{ij} poses another restriction on top of unitarity, in which case the part of the automorphism group that commutes with the Lorentz group is given[2] by $\mathrm{Sp}(\frac{1}{2}N)$, the group of *symplectic* unitary transformations. So in the case of $N=2$, we have $\mathrm{U}(2)_R \cong \mathrm{SU}(2)_R\times\mathrm{U}(1)_R$ in the absence of a central charge and $\mathrm{Sp}(1)_R \cong \mathrm{SU}(2)_R$ if a central charge is present.

Of course, the representation theory of N-extended supersymmetry also reflects this automorphism property. Since the supercharges can undergo a $\mathrm{U}(N)_R$ transformation, the components of a multiplet automatically arrange themselves in representations of $\mathrm{U}(N)_R$. For instance, if a ($\mathrm{U}(N)_R$ singlet) scalar X transforms into a fermion ζ under Q^1 and into χ under Q^2, then ζ and χ are related by the same $\mathrm{U}(N)_R$ transformations that relate Q^1 and Q^2.

In fact, that last example provides a useful way of constructing $N=2$ supersymmetry representations from $N=1$ building blocks. Let us consider the scalar and the vector multiplet introduced in section 1.2 simultaneously. We can make the chiral projections in the transformation rule (1.16) of the vector-multiplet fermion χ. Now we assume that the two Majorana fermions ζ_L and χ_L form an $\mathrm{SU}(2)_R$ doublet. This means that we can write them

[1]For $N=4$ there is a subtlety: there the part of the automorphism group that commutes with the Lorentz group is $\mathrm{SU}(4)$, because the $\mathrm{U}(4)$ has an overlap with the helicity group by a $\mathrm{U}(1)$-factor [6, 22].

[2]A $\mathbb{Z}_2$ subgroup of $\mathrm{U}(N)_R$ also remains unbroken, but we restrict ourselves here to the continuous group of automorphisms.

collectively as Ω_i. Similarly, we write ζ_R and χ_R as Ω^i. As said, the position of the index i corresponds to chirality and choice of $\bar{\mathbf{2}}$ or $\mathbf{2}$ representation. Note that the 'first' supersymmetry Q_1 (with supersymmetry parameter ϵ^1) relates Ω_1 to X. Similarly, the second supersymmetry Q_2 (with ϵ^2) relates Ω_2 and Ω^2 with A_μ. Now we combine the $N = 1$ multiplets into an $N = 2$ multiplet by requiring $\mathrm{SU}(2)_R$ invariance of the transformation rules. This means that the action of $\mathrm{SU}(2)_R$, which in the transformation rules is reflected by a unitary transformation of the supersymmetry parameters, is compensated by an $\mathrm{SU}(2)_R$ transformation of the fermions in the multiplet. So the on-shell $N = 2$ transformation rules are:

$$\begin{aligned} \delta_Q(\epsilon)\, X &= 2\,\bar{\epsilon}^i\Omega_i, \\ \delta_Q(\epsilon)\, \Omega^i &= \partial\!\!\!/ X\epsilon^i + \varepsilon^{ij}\sigma\cdot F^-\epsilon_j, \\ \delta_Q(\epsilon)\, A_\mu &= \bar{\epsilon}^i\gamma_\mu\Omega_i + \bar{\epsilon}_i\gamma_\mu\Omega^i, \end{aligned} \tag{2.3}$$

and a supersymmetric action is simply the sum of (1.7) and (1.15). Closure of the $N = 2$ superalgebra is guaranteed by the closure of $N = 1$ supersymmetry on the building blocks and $\mathrm{SU}(2)_R$ invariance. In principle, this leaves the possibility of a central charge, but as it turns out, the supersymmetry commutator closes into a translation (and a gauge transformation, *c.f.* (1.17)) only. The above multiplet is known as the $N = 2$ on-shell vector-multiplet and we come back to it at length in section 2.3 and in chapter 3. With reference to formula (2.38), we note that off-shell counting indicates that three real scalars are necessary to extend this on-shell representation to an off-shell representation.

Using a similar construction, it is possible to combine two $N = 1$ scalar multiplets into a single $N = 2$ multiplet, called the hypermultiplet. In that case, the scalars are related through a representation of $\mathrm{SU}(2)_R$. Moreover, the massive version of this multiplet has an on-shell central charge, which complicates the above construction somewhat. We come back to the hypermultiplet in chapter 4.

One can even construct an on-shell multiplet for the gauge fields in $N = 2$ supergravity in this way, by combining the $N = 1$ supergravity multiplet, containing a spin-3/2 gravitino and the spin-2 graviton field, with an $N = 1$ representation that contains an abelian gauge field and a spin-3/2 Rarita-Schwinger field that assumes the role of the second gravitino. The resulting on-shell $N = 2$ supergravity multiplet thus describes the graviton, an $\mathrm{SU}(2)_R$ doublet of gravitini and an abelian vector gauge field that is called the *graviphoton* [23]. However, an on-shell representation is inconvenient if one is interested in constructing couplings to other multiplets. Because the representation of supersymmetry itself depends on the form of the action, one is forced to monitor the invariance of the action *and* the representations of the supersymmetry algebra simultaneously, throughout the procedure for construction of an invariant action. An off-shell representation of the $N = 2$ Poincaré supergravity multiplet is therefore preferable.

An off-shell representation of $N = 1$ supergravity was found in [24] and a way to combine and transform $N = 1$ multiplets, called $N = 1$ tensor calculus, was found in [25]. However, the off-shell representation of the $N = 2$ supergravity multiplet turned out to be difficult to find [26]. In [27, 28] an off-shell representation with $40 + 40$ degrees of freedom was found, closing the superalgebra at the linearized level.

2.2 Conformal supergravity and the Weyl multiplet

As was argued in section 1.5, $N = 2$ supergravity models, which are locally symmetric under the Poincaré superalgebra, are most conveniently considered within the framework of a gauge theory for the superconformal algebra, called $N = 2$ conformal supergravity (for reviews, see [34, 35, 36, 37]). The superconformal algebra forms the supersymmetric extension of the conformal group and contains a generator for scale transformations. Since quantum effects introduce a scale, its direct applicability to a quantum-mechanical description of supergravity is limited. However, we emphasize that superconformal methods are relevant for the construction of off-shell models of Poincaré supergravity and are *not* to be taken as an appropriate setting for the exploration of their quantum-mechanical aspects.

The multiplet that describes the superconformal gauge fields was already noticed with the introduction of the off-shell $N = 2$ Poincaré supergravity multiplet. The off-shell representation of the latter contains an irreducible submultiplet, consisting of $24+24$ degrees of freedom that describe the highest-spin components of the graviton, the gravitini and other fields [28]. The aim of this section is the introduction of this superconformal gauge multiplet, which is called the Weyl multiplet [29]. In this section we do not follow the historical steps that led to its discovery, but instead we build up the multiplet from the superconformal algebra.

Relativistic models that have no intrinsic scale such as a mass or a dimensionful coupling constant, are (classically) invariant under the conformal group. This group is an extension of the Poincaré group, characterized by the requirement that its transformations leave the light-cone invariant. Besides the translations and Lorentz transformations in the Poincaré group, which are generated by P_a and M_{ab} and leave a Minkowski inner-product invariant, the conformal group contains scale transformations, generated by D, and so-called special conformal transformations, generated by K_a. Both D and K transformations multiply every Minkowski inner-product by a scale factor. The commutation relations of the Poincaré Lie algebra, given by:

$$\begin{aligned} [\,P_a, M_{bc}\,] &= 2P_{[b}\,\eta_{c]a}, \\ [\,M_{ab}, M_{cd}\,] &= 2\eta_{[a}{}^{[c} M_{b]}{}^{d]}, \end{aligned} \tag{2.4}$$

are extended with the relations:

$$\begin{aligned} [\,K_a, M_{bc}\,] &= 2K_{[b}\,\eta_{c]a}, \\ [\,D, P_a\,] &= P_a, \\ [\,D, K_a\,] &= -K_a, \end{aligned}$$

$$[\,P_a, K_b\,] \;=\; \eta_{ab}D - 2M_{ab}, \tag{2.5}$$

Note that K_a, like P_a, transforms as a four-vector under the Lorentz group, but has opposite dilatational weight.

As was already noted by Haag *et al.* [6], incorporating supersymmetries in a conformal model leads to a much larger algebra then the Poincaré superalgebra. Where the $N = 2$ Poincaré algebra closes after adding the supercharges Q^i, the superconformal algebra requires the addition of two more fermionic generators S^i, called S-supersymmetries. The anticommutator of Q^i and S^j requires the inclusion of $\mathrm{U}(2)_R$ in the algebra. We decompose $\mathrm{U}(2)_R$ into $\mathrm{SU}(2)_R \times \mathrm{U}(1)_R$ and denote the corresponding generators as V_Λ, $\Lambda = 1, 2, 3$ and A. In addition to (2.4) and (2.5), the $N = 2$ superconformal algebra has the following non-zero commutators:

$$[\,M_{ab}, Q^i\,] = \tfrac{1}{2}\sigma_{ab}Q^i, \qquad [\,M_{ab}, S^i\,] = \tfrac{1}{2}\sigma_{ab}S^i, \tag{2.6}$$

expressing the fact that the Q^i and S^i transform as spinors under the Lorentz group. Furthermore, we establish the representations for the $\mathrm{SU}(2)_R$ factor in the automorphism group:

$$\begin{aligned} [\,V_\Lambda, V_\Sigma\,] \;&=\; -2\,\varepsilon_{\Lambda\Sigma}{}^{\Xi}\,V_\Xi\,, \\ [\,V_\Lambda, Q\,]^i = i(\sigma_\Lambda)^i{}_j Q^j, \qquad & [\,V_\Lambda, Q\,]_i = i(\sigma_\Lambda)_i{}^j Q_j\,, \\ [\,V_\Lambda, S\,]^i = i(\sigma_\Lambda)^i{}_j S^j, \qquad & [\,V_\Lambda, S\,]_i = i(\sigma_\Lambda)_i{}^j S_j\,, \end{aligned} \tag{2.7}$$

and we give the weights of the fermionic generators under dilatations and chiral $\mathrm{U}(1)_R$ transformations:

$$\begin{aligned} [\,D, Q^i\,] = \frac{1}{2}Q^i, \qquad & [\,D, S^i\,] = -\frac{1}{2}S^i \\ [\,A, Q^i\,] = -\frac{i}{2}Q^i, \qquad & [\,A, S^i\,] = \frac{i}{2}S^i. \end{aligned} \tag{2.8}$$

The bosonic generators P_a and K_a relate the fermionic generators as follows:

$$[\,K_a, Q^i\,] = \gamma_a S^i, \qquad [\,P_a, S^i\,] = \tfrac{1}{2}\gamma_a Q^i. \tag{2.9}$$

Regarding the anticommutators, there is the chiral form of of the supersymmetry anticommutator:

$$\{Q^i, \bar{Q}_j\} = -(\mathbb{I} - \gamma_5)\gamma^a P_a \delta^i{}_j\,, \tag{2.10}$$

and a similar anticommutator for the S-generators that closes into the generator of special conformal transformations:

$$\{S^i, \bar{S}_j\} = -\tfrac{1}{2}(\mathbb{I} + \gamma_5)\gamma^a K_a \delta^i{}_j\,. \tag{2.11}$$

Finally, there is the anticommutator of a Q and an S generator:

$$\{Q^i, \bar{S}_j\} = \tfrac{1}{2}(\mathbb{I} - \gamma_5)\Big(2\sigma^{ab}M_{ab} + D - iA - 2V^i{}_j\Big). \tag{2.12}$$

The central charge is the result of the anticommutator:

$$\{Q^i, \bar{Q}^j\} = \tfrac{1}{2}(\mathbb{I} - \gamma_5)\varepsilon^{ij} Z. \tag{2.13}$$

where Z is to be understood as a complex generator. In the above relations, we have used a notation that combines the chirality and the representation of the automorphism group in the upper or lower position of the $\mathrm{SU}(2)_R$ index, as defined in the previous section.

In preparation of a superconformal gauge theory, we now proceed to define vector bundles of the superconformal algebra over space-time, through the definition of connections for each of the generators in the superconformal algebra. We omit the generator for central charge transformations and come back to it at the end of this section. Note that we can not yet call this a gauge theory, because we do not give an action, nor do we make choices for the specifics of the bundle. Corresponding to the generators:

$$P^a, \quad M^{ab}, \quad D, \quad K^a, \quad Q^i, \quad S^i, \quad (V_\Lambda)^i{}_j, \quad A, \tag{2.14}$$

we choose the connections:

$$e_\mu{}^a, \quad \omega_\mu{}^{ab}, \quad b_\mu, \quad f_\mu{}^a, \quad \tfrac{1}{2}\psi_\mu{}^i, \quad \tfrac{1}{2}\phi_\mu{}^i, \quad -\tfrac{1}{2}\mathcal{V}_\mu{}^i{}_j, \quad A_\mu. \tag{2.15}$$

Using the structure constants of the superconformal algebra, as given in the above commutator and anticommutator relations, we can now write down transformation rules, covariant derivatives, curvatures and Bianchi identities for the superconformal connections. Although the inclusion of the explicit expressions would be more precise, we have chosen not to burden the discussion with endless sets of formulas for the sake of clarity and conciseness and refer to [29]. For future reference, however, we include here the expression for the P-curvature:

$$R_{\mu\nu}(P)^a = 2\partial_{[\mu} e_{\nu]}^a + 2b_{[\mu} e_{\nu]}^a - 2\omega_{[\mu}{}^{ab} e_{\nu]\,b} - (\bar{\psi}_{[\mu}^{\,i} \gamma^a \psi_{\nu]\,i} + \text{h.c.}). \tag{2.16}$$

At this point it is important to note that the superconformal transformations describe only a symmetry of the superconformal bundle over space-time and *not* a symmetry of space-time itself. In particular, the generators P^a generate an internal symmetry and are, as yet, in no way related to general coordinate transformations or their action on the Minkowski frame for the tangent bundle. In this way, the anticommutator of two supersymmetry transformations closes into an the internal symmetry P and not into a general coordinate transformation. However, in a theory that describes supergravity the latter must be the case.

Hence we pursue the following strategy: first we identify the subbundle on which the P and M generators act with the Minkowski frame for the tangent bundle of space-time. Then we impose a set of constraints, that relate the P transformations to general coordinate transformations. The solutions of these constraints make certain superconformal connections dependent on the remaining ones. The constraints are *not* invariant under supersymmetry, so they change the supersymmetry transformations of the various connections. These changes can be fine-tuned in such a way that the anticommutator of two supersymmetry transformations closes into a (covariant) general coordinate transformation, replacing the P transformation. As we shall see, this fine-tuning requires the presence of additional covariant fields in the resulting multiplet.

Let's go over this in some more detail. We first identify the Minkowski space on which the P's act with the Minkowski frame for the tangent space of space-time. This means that we make a choice for the translational connection $e_\mu{}^a$: we require that e acts as an invertible map from the tangent bundle to the Minkowski frame, *i.e.* we interpret $e_\mu{}^a$ as the vierbein. Lorentz transformations relate the different (equivalent) choices of frame we can make. Furthermore, we relate the action of a general coordinate transformation to the action of the P transformations. Note that when we make an infinitesimal coordinate transformation $x^\mu \to x^\mu + \xi^\mu(x)$, the translational connection/vierbein $e_\mu{}^a$ transforms as:

$$\begin{aligned}\delta_{\text{g.c.t.}}(\xi)\, e_\mu{}^a &= \xi^\nu\, \partial_\nu e_\mu{}^a - \partial_\mu \xi^\nu\, e_\nu{}^a \\ &= \delta_P(\Lambda_P)\, e_\mu{}^a + (\delta_D(\Lambda_D) + \delta_Q(\epsilon) + \delta_M(\varepsilon^{ab}))\, e_\mu{}^a - \xi^\nu R_{\mu\nu}(P)^a, \end{aligned} \tag{2.17}$$

where $\Lambda_P^a(\xi, e) = \xi^\mu e_\mu{}^a$ and analogous contractions between ξ and the appropriate connection for the other transformation parameters. Given that the vierbein is invertible, we can therefore interpret every P transformation on e as a general coordinate transformation with the necessary covariantization terms, *if we set the $R(P)$ curvature to zero.* Equation (2.17) then reduces to:

$$\delta_P(\Lambda)\, e_\mu{}^a = \delta_{\text{g.c.t.}}(e^\mu{}_a \Lambda^a) + \Big(\delta_D(\Lambda_D(e, \Lambda, b)) + \delta_Q(\epsilon(e, \Lambda, \psi)) + \delta_M(\varepsilon^{ab}(e, \Lambda, \omega))\Big)\, e_\mu{}^a. \tag{2.18}$$

The anticommutator of two supersymmetry transformations now closes into a *covariant* general coordinate transformation, which we denote by δ^{cov} henceforth.

Of course, this identification has drastic consequences for the various geometric quantities involved in the construction of the superconformal bundle. Considering the explicit form of the P-curvature, *c.f.* (2.16), we see that due to the invertibility of the vierbein, the constraint $R(P) = 0$ can be solved algebraically for the connection $\omega_\mu{}^{ab}$, leading to an expression $\omega = \omega(e, b, \psi)$, which can be found in appendix B. For this reason, the above constraint is a so-called *conventional constraint* [30], meaning that it has an algebraic solution, as opposed to constraints that lead to differential equations. However, the dependent connection $\omega(e, \psi, b)$ does not transform under supersymmetry in the same way as the original, independent connection. This corresponds to the fact that the constraint $R(P) = 0$ is not supersymmetry invariant. Consequently, the transformation rule of the dependent connection differs from that of the independent connection by the addition of covariant terms. This also means that we have additional terms (proportional to the supersymmetry connection ψ_μ) in the curvature $R(M)$ to make it covariant with respect to the new supersymmetry transformation of ω. We denote this covariant M-curvature by $\hat{R}(M)$. We find that, after substitution of the solution $\omega(e, \psi, b)$, $\hat{R}_{\mu\nu}(M)^{ab}$ is related to (the superconformally covariant version of) the Riemann curvature of space-time, as follows:

$$\hat{R}_{\mu\nu}(M)^{ab} = R_{\mu\nu}{}^{\rho\sigma} e_\rho{}^a e_\sigma{}^b, \tag{2.19}$$

which shows that now the Minkowski frame bundle is indeed related to the tangent bundle in the required fashion. Furthermore, the superconformal Bianchi-identities take a new form because of the vanishing $R(P)$ terms. In particular, the Bianchi identity for $R(P)$ itself turns into an algebraic equation which enables us to express $R(D)$ in terms of $R(M)$.

However, the above constraint is not the only one we must impose. In fact, conformal and superconformal gravity can not be formulated in a translationally invariant way if the fields $f_\mu{}^a$ and ϕ_μ^i are independent [31]. Besides, we want the multiplet that results at the end of the construction to be minimal in the sense that it contains as few independent fields as possible. Moreover, we note that equation (2.18) relates covariant general coordinate transformations and P transformations *for* $e_\mu{}^a$ *only.* If we want similar results for the remaining connections, we have to impose additional constraints. These constraints have to be such that they force a change of the Q transformations of the remaining connections, leading to closure into covariant general coordinate transformations. In order to leave the action of the other superconformal transformations on the connections unchanged, we require that the constraints are invariant under the rest of the superconformal transformations.

The above requirements do not fix the constraints uniquely. However, a set of conventional constraints that has the required properties is given by:

$$\begin{aligned}
&R_{\mu\nu}(P)^a = 0,\\
&\gamma^\mu(\hat{R}_{\mu\nu}(Q)^i + \sigma_{\mu\nu}\chi^i) = 0,\\
&e_{\nu\,b}\hat{R}_{\mu\nu}(M)_a{}^b - i\tilde{\hat{R}}_{\mu a}(A) + \tfrac{1}{8}T^+_{ab\,ij}T^{-\,ij}_{\mu b} - \tfrac{3}{2}De_{\mu a} = 0.
\end{aligned} \tag{2.20}$$

They lead to algebraic dependency of f and ϕ. Again the hats in (2.20) denote extra covariantizations due to the changed supersymmetry transformations of the various connections. The expressions for f and ϕ in terms of the unconstrained fields and the precise form of the covariant curvatures is given in appendix B.

As alluded to earlier, the constraints involve new fields, namely a $\mathrm{SU}(2)_R$ doublet of Majorana spinors χ^i, an (anti)selfdual, (ij)-antisymmetric tensor T^{ij}_{ab} and a real scalar D. They are necessary to fine-tune the changes to the supersymmetry transformations in the desired fashion and to be able to render the set of constraints invariant under the other superconformal transformations. There is another way in which we can see that the multiplet of superconformal gauge fields requires the addition of the fields χ, T and D: if we count the bosonic degrees of freedom, we find $16 + 4 + 12 + 4$ for $e_\mu{}^a$, b_μ, $\mathcal{V}_\mu{}^i{}_j$ and A_μ, *minus* the $4 + 6 + 1 + 4 + 3 + 1$ gauge degrees of freedom for the conformal and internal gauge transformations, leading to a total of 17 independent (off-shell) bosonic degrees of freedom. However the fermionic degrees of freedom add up to 32 for $\psi_\mu{}^i$, minus 16 for Q- and S-supersymmetries, a total of 16 independent (off-shell) fermionic degrees of freedom. As we have seen in section 1.2, when off-shell counting leads to a discrepancy, the addition of auxiliary fields is called for. Inclusion of the set $T^{ij}_{\mu\nu}$, D and χ^i ($(6+1)+8$ degrees of freedom) leads to a multiplet of $24+24$ degrees of freedom, which is the number of components found in the conformal submultiplet of off-shell $N=2$ Poincaré supergravity.

Without going into the details of the lengthy calculations involved in imposing the conventional constraints (2.20), we suffice by giving the transformation rules of the resulting multiplet under Q-supersymmetry, S-supersymmetry and special conformal transformations. The variations under the remaining transformations in the superconformal algebra can be deduced from the weights of the fields and their Lorentz representation. The various independent fields in the *Weyl multiplet* transform as follows:

$$\begin{aligned}
\delta e_\mu{}^a &= \bar\epsilon^i\gamma^a\psi_{\mu\,i} + \text{h.c.}\,,\\
\delta\psi_\mu{}^i &= 2\mathscr{D}_\mu\epsilon^i - \tfrac14\sigma\cdot T^{ij}\gamma_\mu\epsilon_j - \gamma_\mu\eta^i\,,\\
\delta b_\mu &= \tfrac12\bar\epsilon^i\phi_{\mu\,i} - \tfrac34\bar\epsilon^i\gamma_\mu\chi_i - \tfrac12\bar\eta^i\psi_{\mu\,i} + \text{h.c.} + \Lambda^a_K\, e_{\mu\,a}\,,\\
\delta A_\mu &= \tfrac12 i\bar\epsilon^i\phi_{\mu\,i} + \tfrac34 i\bar\epsilon^i\gamma_\mu\chi_i + \tfrac12 i\bar\eta^i\psi_{\mu\,i} + \text{h.c.}\,,\\
\delta\mathcal{V}_\mu{}^i{}_j &= 2\,\bar\epsilon_j\phi_\mu{}^i - 3\,\bar\epsilon_j\gamma_\mu\chi^i + 2\,\bar\eta_j\psi_\mu{}^i - (\text{h.c.}\,;\,\text{traceless})\,,\\
\delta T^{ij}_{ab} &= 8\,\bar\epsilon^{[i}\hat R_{ab}(Q)^{j]}\,,\\
\delta\chi^i &= -\tfrac16\sigma^{ab}\not{D}T^{ij}_{ab}\epsilon_j + \tfrac13\hat R(\text{SU}(2))^i{}_j\cdot\sigma\epsilon^j - \tfrac23 i\hat R(\text{U}(1))\cdot\sigma\epsilon^i\\
&\quad + D\,\epsilon^i + \tfrac16\sigma\cdot T^{ij}\eta_j\,,\\
\delta D &= \bar\epsilon^i\not{D}\chi_i + \text{h.c.}\,,
\end{aligned}\tag{2.21}$$

and the dependent fields transform as:

$$\begin{aligned}
\delta\omega_\mu{}^{ab} &= -\bar\epsilon^i\sigma^{ab}\phi_{\mu\,i} - \tfrac12\bar\epsilon^i T^{ab}_{ij}\psi_\mu{}^j + \tfrac32\bar\epsilon^i\gamma_\mu\sigma^{ab}\chi_i\\
&\quad + \bar\epsilon^i\gamma_\mu\hat R^{ab}(Q)_i - \bar\eta^i\sigma^{ab}\psi_{\mu\,i} + \text{h.c.} + 2\Lambda^{[a}_K\, e_\mu{}^{b]}\,,\\
\delta\phi_\mu{}^i &= -2f_\mu{}^a\gamma_a\epsilon^i - \tfrac14\not{D}T^{ij}\cdot\sigma\gamma_\mu\epsilon_j + \tfrac32[(\bar\chi_j\gamma^a\epsilon^j)\gamma_a\psi_\mu{}^i - (\bar\chi_j\gamma^a\psi_\mu{}^j)\gamma_a\epsilon^i]\\
&\quad + \tfrac12\hat R(\text{SU}(2))^i{}_j\cdot\sigma\gamma_\mu\epsilon^j + i\hat R(\text{U}(1))\cdot\sigma\gamma_\mu\epsilon^i + 2\mathscr{D}_\mu\eta^i + \Lambda^a_K\gamma_a\psi_\mu{}^i\,,\\
\delta f_\mu{}^a &= -\tfrac12\bar\epsilon^i\psi_\mu{}^j\, D_b T^{ba}_{ij} - \tfrac34 e_\mu{}^a\bar\epsilon^i\not{D}\chi_i - \tfrac34\bar\epsilon^i\gamma^a\psi_{\mu\,i}\,D\\
&\quad + \bar\epsilon^i\gamma_\mu D_b\hat R^{ba}(Q)_i + \tfrac12\bar\eta^i\gamma^a\phi_{\mu\,i} + \text{h.c.} + \mathscr{D}_\mu\Lambda^a_K\,.
\end{aligned}\tag{2.22}$$

The derivative $\mathscr{D}_\mu$ denotes a derivative that is covariant with respect to local Lorentz transformations, chiral $\text{U}(2)_R$ and dilatations. The derivative D_μ is covariant with respect to the full superconformal algebra. For further details concerning the definitions of covariant derivatives, we refer to appendix B.

Recapitulating, we have constructed a multiplet containing $24+24$ independent off-shell degrees of freedom, that has the appropriate gauge fields and algebra for a description of conformal $N=2$ supergravity. This multiplet is minimal in the sense that no additional constraints can be imposed to further lower the number of independent degrees of freedom. The Weyl multiplet is the smallest massive multiplet that describes spin-2 degrees of freedom.

Furthermore, the constraints introduce general coordinate transformations in the algebra, at the expense of P transformations.

To see how such a constrained gauge theory of the superconformal algebra can be related to Poincaré supergravity, we consider the following simplified construction. We start with the (non-supersymmetric) conformal group, generated by P_a, M_{ab}, K_a and D with commutation relations given by (2.4) and (2.5). Like in the above, we consider a bundle with connections $e_\mu{}^a$, $\omega_\mu{}^{ab}$, $f_\mu{}^a$ and b_μ. Again we identify this bundle with the Minkowski frame for the tangent bundle of space-time and impose the conventional constraint $R(P) = 0$ to relate P transformations to covariant general coordinate transformations. This makes $\omega_\mu{}^{ab}$ dependent on the other connections. Similarly, we render $f_\mu{}^a$ dependent by imposing the constraint:

$$e^\nu{}_b\, R_{\mu\nu}(M)^{ab} = 0. \tag{2.23}$$

Using the invertibility of $e_\mu{}^a$, (2.23) can be solved for $f_\mu{}^a$, leading to:

$$f_\mu{}^a = \tfrac{1}{2}(R_{\mu\nu} - \tfrac{1}{6} g_{\mu\nu} R) e^{\nu\, a}, \tag{2.24}$$

where $R_{\mu\nu}$ is the Ricci tensor and R is the scalar curvature. The next step is the introduction of a scalar field a that is K-invariant and has Weyl weight $w = 1$, which couples to the conformal gauge fields through the definition of a conformally covariant derivative $D_\mu a = \partial_\mu a - b_\mu a$ and the invariant action:

$$\mathscr{L}_{\text{conf}} = -e\, a \Box a, \tag{2.25}$$

where $\Box = D^\mu D_\mu$, the conformally covariant d'Alembertian. Because the dilatational connection transforms under K through $\delta_K b_\mu = \Lambda_{K\,\mu}$, the second covariant derivative in the d'Alembertian requires a connection for special conformal covariance:

$$D_\mu(D^a a) = (\partial_\mu - 2b_\mu) D^a a - \omega_\mu{}^{ab} D_b a + f_\mu{}^a a. \tag{2.26}$$

Hence, we find that the d'Alembertian transforms as follows under special conformal transformations:

$$\delta_K(D^a D_a a) = e^{\mu\, a}\Big(-2\Lambda_{K\,\mu} D_a a - 2\Lambda_{K\,a} D_\mu a + e_{\mu\, a} \Lambda_K^b D_b a\Big) = 0. \tag{2.27}$$

So the Lagrangian density (2.25) is indeed K-invariant. However, the only independent field that has a non-zero action of the special conformal transformations is b_μ. This means that if we substitute the dependent expressions for $f_\mu{}^a$ and $\omega_\mu{}^{ab}$, the b_μ dependence of the Lagrangian density vanishes. We conclude that we may rewrite the Lagrangian density in the form:

$$\mathscr{L}_{\text{conf}} = -e\, a \Box^{\text{grav}} a - e f_\mu{}^\mu a^2, \tag{2.28}$$

where $\Box^{\text{grav}}$ denotes the standard general-coordinate covariant d'Alembertian. Substituting (2.24) and making a partial integration, the Lagrangian density takes the form:

$$\mathscr{L}_{\text{conf}} = e\, \partial_\mu a\, \partial^\mu a - \tfrac{1}{6} e a^2\, R. \tag{2.29}$$

Comparison with (1.59) now demonstrates the crucial point in the approach: by formulation of a suitable coupling of the constrained conformal gauge model to a compensating scalar field a, we arrive at a conformally invariant action that is gauge-equivalent to the Einstein-Hilbert action. It should be noted that any attempt to write down conformally invariant actions using only the independent fields $e_\mu{}^a$ and b_μ would result in actions like (1.63) that pertain to the higher-spin degrees of freedom only.

Similar arguments apply to superconformal models: if we couple the Weyl multiplet to an appropriate set of compensating fields, we can formulate superconformally invariant actions that are gauge-equivalent to models for Poincaré supergravity. The Weyl multiplet describes the highest-spin components of the fields that are contained in the Poincaré supergravity multiplet. In order to give a full description of Poincaré supergravity, we have to couple it to supermultiplets that contain a suitable set of compensating fields to break the superconformal symmetry to super-Poincaré. Hence, we postpone the supersymmetric extension of the above gauge equivalence until after the construction of the $N = 2$ vector multiplet and its coupling to conformal supergravity.

Coming back to the central charge, we see from the anticommutator (2.13) that a local theory of supersymmetry also implies local central charge transformations. In order to write down a supergravity theory that includes a non-zero central charge, we therefore need a gauge field for Z. One easily sees that the inclusion of such a gauge field in the set (2.15) would not have changed anything in the ensuing discussion, because the central charge commutes with all other transformations. So the central-charge gauge field is in a separate supermultiplet, which turns out to be a vector multiplet. The model describing the Weyl multiplet and a vector multiplet that gauges the central charge is called the $N = 2$ *minimal field representation*. In section 2.4 we shall use the minimal field representation to demonstrate a supersymmetric generalization of the above conformal construction.

2.3 The chiral and vector multiplets

As was argued at the end of the previous section, a model that describes supergravity requires more than just the Weyl multiplet. A local central charge and the inclusion of a suitable set of compensating fields both require a multiplet that contains a vector gauge field. Therefore, we need a coupling of this multiplet to conformal supergravity. In this section we derive the superconformal transformation rules for such a multiplet. The way in which we do the derivation may seem elaborate: we could have just stated the superconformal transformation rules and move on to the construction of superconformal actions in the next section. However, many aspects of this derivation return in chapter 6 in a more complicated setting. For that reason, we first give a general discussion of the construction of field representations of the superconformal algebra and their coupling to the Weyl multiplet. Then we deal with the vector multiplet as an example of the approach.

The first step in the construction of a superconformal multiplet is the determination of the field content. This usually entails the formulation of (an educated guess at) the supersymmetry transformation laws of the components. At this stage we simplify matters as much as possible. For instance, initially we look for supersymmetry transformations that are *rigid* without worrying too much about the local case. Similarly, we do not make any attempts to couple to other symmetries at this point, nor do we worry about possible central charges. The objective is to get started in the right direction, not to arrive at final results.

In many cases, the next step is the counting of off-shell degrees of freedom. This often entails the introduction of a central charge hierarchy (to be discussed in chapters 4) or the introduction of a suitable set of auxiliary fields. In the example of the vector multiplet, our first guess turns out to describe a multiplet with too many degrees of freedom, so we formulate constraints to limit that number. After this stage, we usually have a set of rigid supersymmetry transformation laws that close off-shell, possibly with a non-trivial action of the central charge.

Then we proceed to couple the multiplet to conformal supergravity, meaning that we extend the rigid supersymmetry transformations to local transformations under the full conformal group. This is done in three steps: first we assign scaling and chiral $\mathrm{U}(1)_R$ weights to the fields in such a way that we immediately satisfy the commutators (2.8). It is possible that such assignments can not be made without the need for a compensating field. We come back to this situation in chapter 6. Note that at this point, we know most of the transformations in the superconformal algebra: the only real unknowns are S-supersymmetry and special conformal transformations, although there could be unexpected additions to the Q-supersymmetry transformation rules as well. This means that we can 'covariantize' the derivatives that are always present in the rigid supersymmetry transformations, with respect to all conformal transformations, through the addition of terms proportional to the gauge fields in the Weyl multiplet. However, because we can not be sure about transformations under Q, S and K, we do not commit ourselves on these covariantizations yet.

The next step is the most important one: we check to see if the modified transformation rules form a representation of the superconformal algebra. Most of the commutators, for instance those involving Lorentz transformations, chiral transformations and dilatations, are represented in the correct manner by construction. However, relations such as (1.10) and (1.17), that represent the fermionic part of the algebra still have to be checked. The fact that the construction has been started with a simpler, rigid supersymmetry representation ensures that at least the above 'covariantized' supersymmetry transformations rules are a reasonable guess as far as Q-supersymmetry is concerned.

We know what the desired representation looks like from the algebra of the Weyl multiplet: the commutator of two supersymmetry transformations leads to:

$$[\,\delta_Q(\epsilon_1),\delta_Q(\epsilon_2)\,]=\delta^{\mathrm{cov}}(\xi)+\delta_M(\varepsilon)+\delta_K(\Lambda_K)+\delta_S(\eta)+\delta_{\mathrm{gauge}}\,, \tag{2.30}$$

where the parameters of the various transformations on the *r.h.s.* are given by:

$$\begin{aligned}
\xi^\mu &= 2\,\bar\epsilon_2^i\gamma^\mu\epsilon_{1\,i} + \text{h.c.}\,,\\
\varepsilon^{ab} &= \bar\epsilon_1^i\epsilon_2^j\,T_{ij}^{ab} + \text{h.c.}\,,\\
\Lambda_K^a &= \bar\epsilon_1^i\epsilon_2^j\,D_bT_{ij}^{ba} - \tfrac{3}{2}\,\bar\epsilon_2^i\gamma^a\epsilon_{1\,i}\,D + \text{h.c.}\,,\\
\eta^i &= 6\,\bar\epsilon_{[1}^i\epsilon_{2]}^j\,\chi_j\,.
\end{aligned} \tag{2.31}$$

Note that the changes in the supersymmetry transformations due to the constraints lead to a slightly changed form of the algebra and parameters that depend on the added fields χ, T and D. We come back to the δ_{gauge} term at the end of this section. The commutator of a Q- and an S-supersymmetry transformation is represented through:

$$\begin{aligned}
[\,\delta_S(\eta),\delta_Q(\epsilon)\,] &= \delta_M\Big(2\bar\eta^i\sigma^{ab}\epsilon_i + \text{h.c.}\Big) + \delta_D\Big(\bar\eta_i\epsilon^i + \text{h.c.}\Big)\\
&\quad + \delta_{\text{U}(1)}\Big(i\bar\eta_i\epsilon^i + \text{h.c.}\Big) + \delta_{\text{SU}(2)}\Big(-2\bar\eta^i\epsilon_j - (\text{h.c.}\,;\,\text{traceless})\Big)\,.
\end{aligned} \tag{2.32}$$

Finally, we generate K transformations by the commutator of two S-supersymmetry transformations:

$$[\,\delta_S(\eta_1),\delta_S(\eta_2)\,] = \delta_K(\Lambda_K^a)\,,\qquad \text{with } \Lambda_K^a(\eta_1,\eta_2) = \bar\eta_{2\,i}\gamma^a\eta_1^i + \text{h.c.}\,. \tag{2.33}$$

Usually the QQ-commutator is checked first, because we have relative certainty about the Q-supersymmetry transformations. If the resulting terms cannot be cast in the appropriate form, the supersymmetry transformations have to be changed and we reiterate until they lead to (2.30). After closure of the Q-supersymmetry commutator, one can read off the S transformation on the *r.h.s.*. At this point, matters are usually settled: the rest of the algebra does not require additional changes to the transformations laws and we have indeed found a representation of local conformal supersymmetry. Note that in most cases the S-supersymmetry takes a very simple form and special conformal transformations usually evaluate to zero.

For the construction of the vector multiplet, we start with a brief (and far from complete) introduction to *superspace*. Superspace is an extension of space-time by the addition of anticommuting (or Grassman) coordinates θ_α^i to the usual coordinates x^μ, where α is a spinor index and $i = 1,\ldots,N$, in N-extended superspace. A translation of the Grassman coordinates, $\theta^i \to \theta^i + \epsilon^i$, together with a transformation of x^μ into the Grassman coordinates, is interpreted as a supersymmetry transformation. The anticommutator of two of these shifts closes into a shift of the 'bosonic' coordinates x^μ, thus forming a representation of the supersymmetry algebra. Supersymmetry multiplets are now denoted as so-called superfields, functions depending not only on x^μ, but also on the Grassman coordinates. Since the Grassman coordinates anticommute, a Taylor expansion in the coordinates θ_α^i is finite. For every term in the θ-expansion we have an x-dependent coefficient, corresponding to the usual components in a field representation of supersymmetry. Because there are terms in the expansion with an even and terms with a odd number of Grassman variables, the corresponding component fields are alternatingly bosonic and fermionic. If we make a Grassmanian shift, like above, the expansion in terms of the new θ's induces a supersymmetry transformation on the component fields.

Since a Majorana spinor consists of four real components, the θ-expansion of a general $N = 2$ complex superfield $\Phi(x, \theta)$ contains $2 \cdot 2^{4N} = 256 + 256$ degrees of freedom. However, from earlier experience, we know that the number of components in irreducible representations of $N = 2$ supersymmetry is considerably smaller. For this reason, we immediately impose the Lorentz-invariant constraint that Φ depends only on θ's with positive chirality. Consequently, the number of Grassman variables on which Φ depends is reduced by half, leaving room for $16 + 16$ degrees of freedom in the θ-expansion of Φ. Such a Φ is called a chiral superfield. Note that a similar analysis in the context $N = 1$ supersymmetry leads to the chiral multiplet, which was discussed in section 1.2.

Due to the properties of chiral spinors under Lorentz transformations, the degrees of freedom of a chiral superfield automatically arrange into representations of the Lorentz group. The components are given by: two complex scalars A and C, two $\mathrm{SU}(2)_R$ doublets of Majorana spinors Ψ^i and Λ^i, an $\mathrm{SU}(2)_R$ triplet of complex scalars B^{ij} (symmetric in i, j) and an antisymmetric two-tensor F_{ab}. A shift of fermionic coordinates in superspace generates the following rigid supersymmetry transformation laws for the various components:

$$\begin{aligned}
\delta_Q(\epsilon)\, A &= \bar{\epsilon}^i \Psi_i \,, \\
\delta_Q(\epsilon)\, \Psi_i &= 2\partial\!\!\!/ A\, \epsilon_i + B_{ij}\epsilon^j + \sigma \cdot F^- \varepsilon_{ij}\epsilon^j \,, \\
\delta_Q(\epsilon)\, B_{ij} &= 2\bar{\epsilon}_{(i}\partial\!\!\!/ \Psi_{j)} + 2\varepsilon_{k(i}\bar{\epsilon}^k \Lambda_{j)} \,, \\
\delta_Q(\epsilon)\, F^-_{ab} &= \varepsilon^{ij}\bar{\epsilon}_i \partial\!\!\!/ \sigma_{ab}\Psi_j + \bar{\epsilon}^i \sigma_{ab}\Lambda_i \,, \\
\delta_Q(\epsilon)\, \Lambda_i &= -\sigma^{ab}\partial\!\!\!/ F^-_{ab}\epsilon_i + \varepsilon^{kj}\partial\!\!\!/ B_{ij}\epsilon_k + \varepsilon_{ij} C \epsilon^j \,, \\
\delta_Q(\epsilon)\, C &= -2\varepsilon^{ij}\bar{\epsilon}_i \partial\!\!\!/ \Lambda_j .
\end{aligned} \tag{2.34}$$

Note that the first two components of this so-called $N = 2$ *chiral multiplet* transform like the scalar and spinor components of the on-shell vector multiplet (2.3) that we have constructed in section 2.1. Moreover, the chiral multiplet contains an (anti)selfdual tensor $F^{\pm}_{ab}$, which in principle can serve as a field strength for the gauge field in a vector multiplet. This completes the first step in the program for multiplet construction: we have found a representation of rigid supersymmetry that at least shows some of the features that we expect of a multiplet that contains a gauge field.

However, the tensor F does not satisfy a Bianchi identity, making an interpretation as field strength of a gauge field pointless. So we impose the Bianchi identity:

$$\partial^a F^+_{ab} = \partial^a F^-_{ab}, \tag{2.35}$$

expressing the fact that F is a closed two-form. Hence we can find (at least locally) a one-form W, such that $F = \mathrm{d}W$. However, the constraint (2.35) is not invariant under supersymmetry.

In fact, supersymmetry generates three new constraints that have to be satisfied in order for (2.35) to make sense. Namely, we can not interpret the supersymmetry transformation law for F, the fourth line in (2.34), as $\mathrm{d}(\delta_Q(\epsilon)W)$, unless:

$$\Lambda_i = -\varepsilon_{ij} \partial\!\!\!/ \Psi^j , \tag{2.36}$$

Comparing $\delta_Q(\epsilon)\Lambda$ and $\delta_Q(\epsilon)\Psi$, we see that this in turn leads to the two constraints:

$$C = -2\,\Box A^* , \qquad B_{ij} = \varepsilon_{ik}\varepsilon_{jl} B^{kl} , \tag{2.37}$$

where $B^{ij} = (B_{ij})^*$, the complex conjugate of B_{ij}. The rigid supersymmetry transformation rules for the unconstrained fields, which we rename $(X, \Omega^i, W_\mu, Y_{ij})$, now take the form:

$$\begin{aligned}
\delta_Q(\epsilon)\, X &= \bar{\epsilon}^i \Omega_i \,, \\
\delta_Q(\epsilon)\, \Omega_i &= 2\, \partial\!\!\!/ X\, \epsilon_i + \varepsilon_{ij} \sigma \cdot F^- \epsilon^j + Y_{ij} \epsilon^j \,, \\
\delta_Q(\epsilon)\, W_\mu &= \bar{\epsilon}^i \gamma_\mu \Omega_i + \bar{\epsilon}_i \gamma_\mu \Omega^i \,, \\
\delta_Q(\epsilon)\, Y_{ij} &= 2\, \bar{\epsilon}_{(i} \partial\!\!\!/ \Omega_{j)} + 2\, \varepsilon_{ik}\varepsilon_{jl}\, \bar{\epsilon}^{(k} \partial\!\!\!/ \Omega^{l)} \,.
\end{aligned} \tag{2.38}$$

The third line implies that the field strength $F = 2\partial_{[\mu} W_{\nu]}$ transforms as:

$$\delta_Q(\epsilon)\, F_{\mu\nu} = -2\, \varepsilon^{ij} \bar{\epsilon}_i \gamma_{[\mu} \partial_{\nu]} \Omega_j + \text{h.c.} \,, \tag{2.39}$$

The fact that we can impose a set of constraints means that the $N = 2$ chiral multiplet is a reducible representation of the superalgebra: the chiral constraint on the superfield does not restrict the multiplet to an irreducible representation. This is indicative of the limited value of the superspace approach in the context of extended supersymmetry: the need for numerous supersymmetric constraints and the indirect description in terms of superfields hamper its applicability to a large extent. Moreover, the description of multiplets with a non-trivial central charge is somewhat cumbersome in the context of the superspace formalism [53].

The primary difference between (2.38) and (2.3) is the appearance of the field Y_{ij}. Counting degrees of freedom, we note that (2.3) gives the transformation rules for $4+4$ on-shell degrees of freedom, whereas (2.38) describes $8 + 8$ off-shell degrees of freedom. In other words, the vector multiplet we found earlier is an on-shell version of the multiplet we have constructed here, the so-called *rigid off-shell vector multiplet*. At this point, we have completed the second step in the program for superconformal multiplet construction: we have restricted our 'first guess' to an irreducible off-shell multiplet of rigid supersymmetry.

The third step in our program is the coupling of the multiplet to conformal supergravity. As was said earlier, we have to assign appropriate weights to the different components, thus satisfying the commutators of chiral and dilatational transformations with the supersymmetry generators, *c.f.* (2.8). First we consider the chiral superfield: if we assign chiral and dilatational weights w and c to A, the weights of the other fields are fixed: Ψ_i has weights $(w{+}1/2, c{+}1/2)$, B_{ij} and F^- both have weights $(w{+}1, c{+}1)$, Λ_i has weights $(w{+}3/2, c{+}3/2)$ and C has weights $(w+2, c+2)$. We could now proceed to couple the chiral multiplet to conformal supergravity, as has been done in [33][3]. In that case, we would find that in order to represent the whole superconformal algebra, we have to impose $c = -w$. Anticipating a similar result for the vector multiplet, we take $c = -w$ from this point onward. Note that the constraints (2.35), (2.36) and (2.37) can only be imposed consistently on a chiral multiplet with chiral weight $c = -1$ (and consequently $w = 1$). So the form of the constraints fixes the weights of the components of the vector multiplet uniquely. At this point we have a multiplet that transforms not only under rigid supersymmetry, but also under global chiral $\mathrm{U}(2)_R$ transformations and dilatations. The chiral and dilatational weights of the component fields of the vector multiplet are listed in table C.II.

Having sorted out the dilatational and chiral weights, we come to the coupling of the Weyl multiplet. We turn the rigid symmetries we have found into local ones and require that they satisfy the superconformal algebra. The first step is, of course, the covariantization of the derivatives. When we assume that for instance the supersymmetry parameter ϵ is space-time dependent, we have to compensate the $\partial_\mu\epsilon$ term that arises after a second supersymmetry variation on the fermionic transformation rule in (2.38), by suitable addition of terms proportional to the gravitino $\psi_\mu{}^i$. Similarly we have to add connection terms for the other superconformal transformations. *De facto* what we are doing is covariantizing all derivatives with respect to the superconformal algebra. If we now check the closure of the anticommutator of two supersymmetries, *c.f.* (2.30), on the scalar X, we find that the covariant translation $\delta^{\rm cov}$ is represented by a superconformally covariant derivative:

$$[\,\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)\,]\, X = 2\,\bar\epsilon_2^i\gamma^a\epsilon_{1\,i}\, D_a X, \tag{2.40}$$

Next we turn to the closure of the QQ-commutator on the fermion Ω_i. There we immediately run into a complication, because we have not specified the form of the field strength $\mathscr{F}$ yet. However, we require that this field strength transforms covariantly under supersymmetry, *i.e.*:

$$\delta_Q(\epsilon)\,\mathscr{F}_{ab} = -2\,\varepsilon^{ij}\bar\epsilon_i\gamma_{[a}D_{b]}\Omega_j + \text{h.c.}\,, \tag{2.41}$$

analogous to (2.39). With this assumption the algebra closes on Ω_i as well, albeit with an S-supersymmetry transformation for the fermion, proportional to the scalar X. Then we check the QQ-commutator of vector field W_μ. From the discussion of the $N = 1$ vector multiplet, we expect to find a gauge term, *c.f.* (1.17). However, in the derivation we made there, we freely took derivatives across ϵ's, because we assumed them to be globally constant. In this case, closure of the QQ-commutator requires the explicit addition of a $\psi_\mu{}^i$ term in the supersymmetry variation of W_μ, in order to complete the total derivative that forms the gauge transformation. Furthermore, closure requires an explicit definition of the covariant field strength, to be given shortly. After checking the supersymmetry commutator on Y_{ij} (which already in the rigid case required the use of the Bianchi identity for the field strength),

[3]The resulting superconformal generalization of the transformation rules (2.34) can be found in appendix C, *c.f.* (C.3).

we arrive at the following superconformal transformation rules:

$$\begin{aligned}
\delta X &= \bar{\epsilon}^i \Omega_i \,, \\
\delta \Omega_i &= 2 \not{D} X \, \epsilon_i + \varepsilon_{ij} \sigma \cdot \mathscr{F}^- \epsilon^j + Y_{ij} \epsilon^j + 2X \eta_i \,, \\
\delta W_\mu &= \bar{\epsilon}^i \gamma_\mu \Omega_i + + 2\bar{X} \, \varepsilon_{ij} \bar{\epsilon}^i \psi_\mu{}^j + \text{h.c.} \,, \\
\delta Y_{ij} &= 2 \, \bar{\epsilon}_{(i} \not{D} \Omega_{j)} + 2 \, \varepsilon_{ik} \varepsilon_{jl} \, \bar{\epsilon}^{(k} \not{D} \Omega^{l)} \,,
\end{aligned} \tag{2.42}$$

where the covariant field strength $\mathscr{F}$ is given by:

$$\mathscr{F}_{\mu\nu} = 2\partial_{[\mu} W_{\nu]} - \left(\varepsilon_{ij} \bar{\psi}_{[\mu}^{\;i} \gamma_{\nu]} \Omega^j + \varepsilon_{ij} \bar{X} \bar{\psi}_\mu{}^i \psi_\nu{}^j + \tfrac{1}{4} \varepsilon_{ij} \bar{X} T^{ij}_{\mu\nu} + \text{h.c.} \right) , \tag{2.43}$$

satisfying the Bianchi identity:

$$D^b \Big(\mathscr{F}^+_{ab} - \mathscr{F}^-_{ab} + \tfrac{1}{4} X T_{ab\,ij} \varepsilon^{ij} - \tfrac{1}{4} \bar{X} T^{ij}_{ab} \varepsilon_{ij} \Big) = \tfrac{3}{4} \Big(\bar{\chi}^i \gamma_a \Omega^j \varepsilon_{ij} - \bar{\chi}_i \gamma_a \Omega_j \varepsilon^{ij} \Big) . \tag{2.44}$$

Note that the QQ-commutator on W_μ closes into a gauge transformation δ_{gauge} in addition to the other terms in (2.30):

$$[\, \delta_Q(\epsilon_1), \delta_Q(\epsilon_2) \,] \, W_\mu = \ldots + \partial_\mu \theta, \quad \text{with} \quad \theta = 4\bar{X} \, \varepsilon_{ij} \bar{\epsilon}_2^i \epsilon_1^j + \text{h.c.} \,, \tag{2.45}$$

If we have several vector multiplets, there is the possibility of associating them with a non-abelian gauge group. Enumerating the generators of the group with I, the transformation rules are extended with terms proportional to the structure constants $f_{IJ}{}^K$ of the gauge group (see formula (C.4)). The QQ-commutator again closes into a gauge transformation, acting on the vector multiplet components through the adjoint representation generated by the structure constants.

The above represents the off-shell coupling of a single (abelian) vector multiplet to the Weyl multiplet. It is with a supermultiplet like the above that one gauges central charge transformations and it is this multiplet together with the Weyl multiplet that forms the $N = 2$ minimal field representation, mentioned earlier.

2.4 Superconformal actions

The construction of off-shell representations of the superconformal algebra, such as the vector multiplet constructed in the previous section, forms only the first step in the formulation of a model of conformal supergravity. An equally important ingredient, of course, is the action of the model, which has to be invariant under local superconformal transformations. So it is desirable to have at our disposal the necessary methods for the construction of invariant actions, or more generally, superconformal invariants. It is the aim of this section to clarify some of these methods and give an example in the form of an action for the $N = 2$ minimal field representation.

Given an off-shell representation of the algebra, the most straightforward method for the construction of supersymmetric (or superconformal) invariants consists of a number of simple steps, namely: first we write down one or more terms that have to be in the Lagrangian density and call this expression $\mathscr{L}_0$. Then we vary $\mathscr{L}_0$ with respect to supersymmetry, calling the result $\delta_Q\mathscr{L}_0$. Next, we add to $\delta_Q\mathscr{L}_0$ a term $\delta_Q\mathscr{L}_1$, such that the sum is a total derivative and we try to find the smallest expression $\mathscr{L}_1$ whose variation *contains* $\delta_Q\mathscr{L}_1$. Defining a new Lagrangian $\mathscr{L}_0' = \mathscr{L}_0 + \mathscr{L}_1$, we can go back to the first step until no further additions are necessary. Usually, this so-called *Noether procedure* ends after a finite number of iterations. In the case that we are dealing with on-shell supersymmetry, the situation becomes more complicated: in that case every addition to the Lagrange densities can contribute to a field equation that is needed to close the supersymmetry algebra. Note that the Noether procedure becomes more and more laborious when the number of fields increases, when the variations become more complicated or when more symmetries are to be checked. Especially increasing the number of symmetries complicates matters, because the additions made to compensate one symmetry in turn have to be compensated with respect to all others.

For this reason, alternative methods for the construction of supersymmetric invariants are called for. Note that some components in supermultiplets transform into total derivatives themselves. For instance, the C component of the (rigid) chiral multiplet transforms as:

$$\delta_Q(\epsilon)\, C = \partial_\mu(-2\varepsilon^{ij}\bar{\epsilon}_i\gamma^\mu\Lambda_j). \tag{2.46}$$

Consequently, a very simple invariant of rigid supersymmetry for a chiral multiplet can be given:

$$S_{\text{inv}} = \int d^4x\, C. \tag{2.47}$$

In some cases, combinations of components of one or several multiplets can be found that display the same property that they transform into a total derivative. Such a combination is called a *density formula*. Needless to say, an invariant of the superconformal algebra has to transform into a total derivative under all the superconformal transformations. In particular, the measure used in the definition of an action, is invariant under diffeomorphisms, but has dilatational weight -4. Correspondingly, in the superconformal density formula analogous to (2.47), which takes the form [33]:

$$\begin{aligned} e^{-1}\mathscr{L} \;=\;& C - \varepsilon^{ij}\bar{\psi}_i\cdot\gamma\Lambda_j - \tfrac{1}{4}\bar{\psi}_{\mu\, i}\sigma\cdot T_{jk}\gamma^\mu\Psi_l\varepsilon^{ij}\varepsilon^{kl} \\ & -\tfrac{1}{16}A(T_{ab\, ij}\varepsilon^{ij})^2 - \bar{\psi}_{\mu\, i}\sigma^{\mu\nu}\psi_{\nu\, i}B_{kl}\varepsilon^{ik}\varepsilon^{jl} \\ & \bar{\psi}_{\mu\, i}\psi_{\nu\, j}\varepsilon^{ij}(F^{-\,\mu\nu} - \tfrac{1}{2}AT^{\mu\nu}_{kl}\varepsilon^{kl}) \\ & -\tfrac{1}{2}\varepsilon^{ij}\varepsilon^{kl}e^{-1}\varepsilon^{\mu\nu\rho\sigma}\bar{\psi}_{\mu\, i}\psi_{\nu\, j}(\bar{\psi}_{\rho\, k}\gamma_\sigma\Psi_l + \bar{\psi}_{\rho\, k}\psi_{\sigma\, l}A), \end{aligned} \tag{2.48}$$

the weight of the C component has to be 4, meaning that the chiral superconformal density formula can only be used for chiral multiplets of weight $w = 2$.

However, the notion of a density formula alone does not help us: the construction of a density formula in principle requires the Noether procedure. Only in the case that we can use a *known* superconformal density formula in the construction of an action for a *new* multiplet have we really gained something. For instance, suppose that we have a new superconformal multiplet M and we succeed in the construction of an S-invariant, K-invariant, complex, weight $w = 2$ scalar A' from the components of M and the supersymmetry variation of A' depends only on the positive-chiral ϵ^i. This means that A' transforms as follows:

$$\delta_Q(\epsilon)\, A' = \bar{\epsilon}^i \Psi'_i, \tag{2.49}$$

where Ψ'_i is a newly defined $\mathrm{SU}(2)_R$ doublet of spinors that depend on the components of M. Further supersymmetry variations of Ψ' give rise to new combinations F'_{ab}, Λ'_i and C' and the set of all primed fields forms a chiral multiplet. Consequently, we can use the superconformal density formula (2.48) to construct an invariant. However, since C' is also an expression in terms of the components of M, we have in fact constructed an invariant of M. The question of whether this invariant is suitable as an action is secondary at this point.

The construction of one representation of the algebra from another is therefore of the utmost importance. If we manage to construct a suitable chiral multiplet (or any other multiplet for which a density formula is known) from a new multiplet, we have immediately found a superconformally invariant or even an invariant action for that new multiplet. The collection of tricks that has been developed to obtain one multiplet from another is known as *multiplet calculus*. Superconformal multiplet calculus is discussed extensively in [34, 35, 36, 37] and references therein.

Of course, there are other density formulae than just the chiral density formula. In practice, it is desirable to have a density formula that is based on an irreducible multiplet, because then the number of components that plays a role in the multiplet calculus is minimal. In the previous section we have seen that the chiral multiplet is reducible, containing the vector multiplet as an irreducible submultiplet. There is, of course, another such irreducible submultiplet, called the *linear multiplet*, which we shall consider shortly. From the linear multiplet we can derive a density formula which we shall use frequently in the the following chapters.

Consider the constraints (2.35), (2.36) and (2.37) and define:

$$\begin{aligned}
L_{ij} &= B_{ij} - \varepsilon_{ik}\varepsilon_{jl} B^{kl}\,, \\
\varphi_i &= \Lambda_i + \varepsilon_{ij} \partial\!\!\!/ \Psi^j\,, \\
G &= C + 2\Box A^*\,, \\
E_b &= \partial^a (F^+_{ab} - F^-_{ab})\,.
\end{aligned} \tag{2.50}$$

Note that L satisfies $L_{ij} = \varepsilon_{ik}\varepsilon_{jl} L^{kl}$. The constraints transform amongst each other under the variations (2.34), indicating a way to find a representation of rigid supersymmetry for the above defined components, which can then be extended to a representation of the superconformal algebra with the procedure described in the previous section. The transformation rules under Q- and S-supersymmetry are given by:

$$\delta\, L_{ij} = 2\,\bar{\epsilon}_{(i}\varphi_{j)} + 2\,\varepsilon_{ik}\varepsilon_{jl}\,\bar{\epsilon}^{(k}\varphi^{l)}\,,$$

$$\begin{aligned}
\delta\,\varphi^i &= \not{D}L^{ij}\epsilon_j + \not{E}\varepsilon^{ij}\epsilon_j - G\epsilon^i + 2g\,\bar{X}L^{ij}\varepsilon_{jk}\epsilon^k + 2\,L^{ij}\eta_j\,,\\
\delta\,G &= -2\,\bar{\epsilon}_i\not{D}\varphi^i - \bar{\epsilon}_i\Big(6\chi_j L^{ij} + \tfrac{1}{2}\varepsilon^{ij}\varepsilon^{kl}\sigma\cdot T_{jk}\varphi_l\Big)\\
&\quad +2g\,\bar{X}\Big(\varepsilon^{ij}\bar{\epsilon}_i\varphi_j - \varepsilon_{ij}\bar{\epsilon}^i\varphi^j\Big) - 2g\,\bar{\epsilon}_i\Omega^j L^{ik}\varepsilon_{jk} + 2\bar{\eta}_i\varphi^i\,,\\
\delta\,E_a &= 2\,\varepsilon_{ij}\bar{\epsilon}^i\sigma_{ab}D^b\varphi^j + \tfrac{1}{4}\bar{\epsilon}^i\gamma_a\Big(6\varepsilon_{ij}\chi_k L^{jk} - \tfrac{1}{2}\sigma\cdot T_{ij}\varepsilon^{jk}\varphi_k\Big)\\
&\quad +2g\,\bar{X}\bar{\epsilon}^i\gamma_a\varphi_i + g\,\bar{\epsilon}^i\gamma_a\Omega^j L_{ij} + \tfrac{3}{2}\bar{\eta}^i\gamma_a\varphi^j\varepsilon_{ij} + \text{h.c.}\,. \qquad (2.51)
\end{aligned}$$

The weights of the components are listed in table C.II. Note that we have included a coupling to a vector multiplet $(X, \Omega^i, W_\mu, Y^{ij})$, weighed by a coupling constant g. This coupling constant serves only to identify the terms that represent the coupling and can be absorbed in the normalization of the vector multiplet components. From the discussion of the vector multiplet we know that the QQ-commutator leads to a gauge transformation, *c.f.* (2.45). Therefore, the QQ-commutator for the components of a linear multiplet that is in a representation of the gauge group, should result in a gauge transformation as well, as was discussed in the $N = 1$ context after formula (1.17). The g-proportional terms in (2.51) lead to:

$$[\,\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)\,] = \ldots + \delta_{\text{gauge}}(4\bar{X}\,\varepsilon_{ij}\bar{\epsilon}_2^i\epsilon_1^j + \text{h.c.})\,, \qquad (2.52)$$

i.e. a gauge transformation on the linear multiplet components with the same composite parameter as the one found in (2.45).

The observant reader may have noticed that the off-shell counting of degrees of freedom in the linear multiplet seems to result in $9+8$ bosonic and fermionic degrees of freedom. Indeed, we have to impose one (real, scalar) constraint in order to close the superconformal algebra, namely:

$$2\,D_a E^a = g\Big(\tfrac{1}{2}Y^{ij}L_{ij} - 2XG - 2\bar{\Omega}^i\varphi_i\Big) - 3\bar{\varphi}^i\chi^j\varepsilon_{ij} + \text{h.c.}\,. \qquad (2.53)$$

In the case that $g = 0$, this constraint can be solved and E^a can be written as the (supercovariant) field strength of a two-rank tensor gauge field $E_{\mu\nu}$. The solution takes the form:

$$E^a = \tfrac{1}{2}ie^{-1}e_\mu{}^a\varepsilon^{\mu\nu\rho\sigma}D_\nu E_{\rho\sigma}\,, \qquad (2.54)$$

the superconformal variation of which is given by:

$$\delta_Q(\epsilon)\,E_{\mu\nu} = -2i\bar{\epsilon}^i\sigma_{\mu\nu}\varphi^j\varepsilon_{ij} - iL_{ij}\varepsilon^{jk}\bar{\epsilon}^i\gamma_{[\mu}\psi_{\nu]\,k} + \text{h.c.}\,. \qquad (2.55)$$

The resulting multiplet is known as the $N = 2$ tensor multiplet.

The density formula for the linear multiplet mentioned earlier, can be constructed from a linear multiplet and an abelian vector multiplet. The linear multiplet may be in a representation of an abelian factor of the gauge group, in which case the vector multiplet must be associated with that abelian factor. Otherwise the linear multiplet must be gauge invariant. The density formula is given by:

$$\begin{aligned}
e^{-1}\mathscr{L} &= XG - \Big(\tfrac{1}{4}Y^{ij} + \tfrac{1}{2}\bar{\psi}_\mu{}^i\gamma^\mu\Omega^j + \bar{X}\bar{\psi}_\mu{}^i\sigma^{\mu\nu}\psi_\nu{}^j\Big)L_{ij} + \bar{\varphi}^i\Big(\Omega_i + X\gamma^\mu\psi_{\mu\,i}\Big)\\
&\quad -\tfrac{1}{2}W_a\Big(E^a + 2\bar{\varphi}^i\sigma^{ab}\psi_b{}^j\varepsilon_{ij} - \tfrac{1}{2}\varepsilon^{abcd}\,\bar{\psi}_{b\,k}\gamma_c\psi_d{}^i L_{ij}\varepsilon^{jk}\Big) + \text{h.c.}\,. \qquad (2.56)
\end{aligned}$$

Note that the abelian vector multiplet $(X, W_\mu, \Omega_i, Y_{ij})$ can be the one associated with the central-charge transformations.

As an example, we now construct the superconformally invariant action of $N = 2$ minimal conformal supergravity. To that end, we first have to define a linear multiplet in terms of the fields in the vector multiplet gauging the central charge. So we are looking for a $w = 2$, $c = 0$ $\mathrm{SU}(2)_R$ triplet of scalars that can serve as the L_{ij} component. A possible choice is given by Y_{ij}, which fixes the other linear multiplet components in terms of vector multiplet components as follows:

$$\begin{aligned} L_{ij} &= -\tfrac{1}{2} Y_{ij} \,, \\ \varphi_i &= -\tfrac{1}{2} \not{D} \Omega_i \,, \\ G &= \Box_C \bar{X} - \tfrac{1}{4} \mathscr{F}^+_{ab} T^{+\,ab}_{ij} \varepsilon^{ij} + 2 \bar{\chi}_i \Omega^i \,, \\ E_a &= -\tfrac{1}{2} D^b \Big(\mathscr{F}^+_{ba} - \tfrac{1}{4} X T^+_{ba\,ij} \varepsilon^{ij} \Big) + \tfrac{3}{4} \bar{\chi}^i \gamma_a \Omega^j \varepsilon_{ij} + \text{h.c.} \,, \end{aligned} \tag{2.57}$$

where $\Box_C = D^a D_a$, the superconformally covariant d'Alembertian. To obtain an action, we substitute the above linear multiplet components in the density formula (2.56). Furthermore, we do a partial integration taking the derivative in the expression for E_a to the gauge field and we complete the resulting expression to a covariant field strength. This leads to the following superconformal vector multiplet Lagrangian:

$$\begin{aligned} e^{-1} \mathscr{L} = \; & X \Box_C \bar{X} - \tfrac{1}{4} \mathscr{F}^+_{ab} \mathscr{F}^{+\,ab} - \tfrac{1}{2} \bar{\Omega}_i \not{D} \Omega^i + \tfrac{1}{8} Y_{ij} Y^{ij} \\ & + \tfrac{1}{8} X \mathscr{F}^+_{ab} T^{+\,ab}_{ij} \varepsilon^{ij} + \tfrac{3}{2} X \, \bar{\chi}_i \Omega^i \\ & + \tfrac{1}{2} X \bar{\psi}_i \cdot \gamma \not{D} \Omega^i + \tfrac{1}{4} \bar{\psi}_i \cdot \gamma \, \Omega_j Y^{ij} + \tfrac{1}{4} \varepsilon^{ij} \bar{\psi}_i \cdot \gamma \, \sigma \cdot \mathscr{F}^- \Omega_j \\ & - \tfrac{1}{64} X^2 (T^+_{ab\,ij} \varepsilon^{ij})^2 + \tfrac{1}{8} X \bar{\psi}_{\mu\,i} \sigma \cdot T^+_{jk} \gamma^\mu \Omega_l \varepsilon^{ij} \varepsilon^{kl} \\ & + \tfrac{1}{4} \bar{\psi}_{\mu\,i} \sigma^{\mu\nu} \psi_{\nu\,j} (2X \, Y^{ij} - \varepsilon^{ik} \varepsilon^{jl} \, \bar{\Omega}_k \Omega_l) \\ & - \tfrac{1}{4} \bar{\psi}_{a\,i} \psi_{b\,j} \varepsilon^{ij} (2X \mathscr{F}^+_{ab} - \tfrac{1}{2} X^2 T^+_{ab\,kl} \varepsilon^{kl} - \tfrac{1}{2} \varepsilon^{kl} \, \bar{\Omega}_k \sigma_{ab} \Omega_l) \\ & + \tfrac{1}{8} \varepsilon^{ij} \varepsilon^{kl} \, e^{-1} \varepsilon^{\mu\nu\rho\sigma} \bar{\psi}_{\mu\,i} \psi_{\nu\,j} (2X \bar{\psi}_{\rho\,k} \gamma_\sigma \Omega_l + \bar{\psi}_{\rho\,k} \psi_{\sigma\,l} X^2) \\ & + \text{h.c.} \,, \end{aligned} \tag{2.58}$$

which can be interpreted as an action for the $N = 2$ minimal field representation. Another way to arrive at this result is by the definition of a chiral multiplet starting with $A' = X^2$ and applying the superconformal chiral density formula (2.48).

Next we demonstrate how the Lagrangian density (2.58) of the minimal field representation is related to a model for Poincaré supergravity. Note that, like in the Lagrangian density (2.25), the scalar kinetic term is still written in a manifestly covariant way. However, the action is K-invariant and hence b_μ-independent after substitution of the dependent expressions for $\omega_\mu{}^{ab}$, $f_\mu{}^a$ and $\phi_\mu{}^i$. Consequently, we can write the first term in the action as:

$$e^{-1} \mathscr{L} = X \Box \bar{X} + \text{h.c.} + 2 X \bar{X} (-\tfrac{1}{6} R + D) + \ldots \,, \tag{2.59}$$

where $\Box$ is covariantized with respect to all symmetries except dilatations, R is the scalar curvature and D is the auxiliary D field in the Weyl multiplet. In fact, we can break K-invariance by simply setting the dilatational connection to zero:

$$K\text{-gauge:} \qquad b_\mu = 0. \tag{2.60}$$

In order for this constraint to be invariant under all remaining transformations in the algebra, we have to modify the transformation laws under dilatations, Q-supersymmetry and S-supersymmetry by a compensating, field-dependent K transformation. However, since all other independent fields are K-invariant, this does not modify the transformation rules (2.21) and (2.42). To obtain the supersymmetric Einstein-Hilbert action, we have to normalize the coefficient of the scalar curvature term appropriately. The choice:

$$D\text{-gauge:} \qquad X\bar{X} = \text{constant}, \tag{2.61}$$

breaks scale-invariance, analogous to the choice (1.60). Note that the constant on the *r.h.s.* of (2.61) is dimensionful and related to Newton's constant. Chiral $\mathrm{U}(1)_R$ can be broken by further specifying X, for instance by imposing:

$$\mathrm{U}(1)\text{-gauge:} \qquad X = \bar{X}. \tag{2.62}$$

At this point, we could have chosen to redefine supersymmetry transformations by adding compensating, field-dependent scale- and $\mathrm{U}(1)_R$ transformations, to make (2.61) and (2.62) supersymmetric. However, we choose to fix $\delta_Q X = 0$, by imposing:

$$S\text{-gauge:} \qquad \Omega_i = 0. \tag{2.63}$$

Note that this condition seems to break both Q- and S-supersymmetry, *c.f.* $\delta_Q \Omega_i$ in (2.42). However, for a certain field-dependent choice of S-parameter, a combination of Q- and S-supersymmetry remains unbroken. This combination, which we interpret as the supersymmetry in the Poincaré superalgebra, takes the form:

$$\delta_Q^{\text{Poincaré}} = \delta_Q(\epsilon) + \delta_S(\eta) + \delta_K(\Lambda_K), \tag{2.64}$$

where we have also included the K transformation needed to maintain $b_\mu = 0$. A formula like (2.64), expressing a combination of symmetry transformation rules that remains unbroken after a certain gauge-choice, is called a decomposition rule. The parameter of S-supersymmetry is given by the following field-dependent expression:

$$\eta_i = -i\not{A}\epsilon_i - \frac{1}{2X}(\varepsilon_{ij}\sigma \cdot \mathscr{F}^- + Y_{ij})\epsilon^j, \tag{2.65}$$

where X takes the constant value specified in (2.61). So in the resulting Poincaré supergravity model, the fields $\mathscr{F}_{ab}$ and Y_{ij} enter in the supersymmetry variations of, for instance, the gravitini. In fact, the field $\mathscr{F}_{ab}$ assumes the role of the graviphoton field strength and the field Y_{ij} becomes an auxiliary field in the supergravity multiplet, as do A_μ and $\mathcal{V}_\mu{}^i{}_j$. Formula (2.65) clearly demonstrates why Poincaré supergravity is more complicated than its conformal counterpart: not only is it based on a larger field representation ($40+40$ components), but the Poincaré supersymmetry variations have a more complicated, non-linear structure, induced by the presence of extra, field-dependent transformations in the decomposition rule (2.64).

In the above example, we can not fully demonstrate the gauge equivalence to a model of Poincaré supergravity, because the minimal field representation does not contain the appropriate fields to act as compensators for the chiral $\mathrm{SU}(2)_R$ invariance. Still, although not generic, a local SU(2) symmetry can occur in a model of Poincaré supergravity and does not pose an essential problem.

The insufficiency is demonstrated by the fact that the D field that is part of the Weyl multiplet appears linearly in the action, giving rise to an inconsistency in the equations of motion. A solution can be found by the introduction of an additional compensating multiplet. In fact, we could have anticipated this by counting the degrees of freedom: we know that the smallest off-shell representation of Poincaré supergravity contains $40 + 40$ degrees of freedom [29], whereas the Weyl multiplet and the compensating vector multiplet in the minimal field representation add up to a total of $32 + 32$ off-shell degrees of freedom. Hence, the extra compensating multiplet has to contain $8 + 8$ off-shell degrees of freedom. As it turns out [34], three viable choices for this extra multiplet are a non-linear multiplet, a tensor multiplet or a hypermultiplet [38]. Note that all three contain scalar fields in a non-trivial representation of $\mathrm{SU}(2)_R$, enabling us to impose a gauge-fixing condition for this symmetry. The resulting Poincaré supergravity models are equivalent as far as their on-shell behavior is concerned, but they differ substantially in their off-shell field content. Classically they will lead to the same results, but differences may occur upon consideration of their quantum-mechanical aspects.

Chapter 3

Vector Multiplets and Special Geometry

In the previous chapter, we have discussed $N = 2$ conformal supergravity and the gauge equivalence to Poincaré supergravity. We introduced the Weyl multiplet and the vector multiplet and we discussed the coupling of the Weyl multiplet to a single vector multiplet. The aim of the current chapter is the derivation of the couplings between *several* vector multiplets, first in a background of the Weyl multiplet and later in flat space-time. These couplings provide a rich environment for the study of duality transformations in supergravity models and lead to a specific class of supersymmetric non-linear sigma-models, the geometry of which is known as 'special geometry'.

Although in principle it is possible to formulate superconformal models that do not involve a vector-multiplet coupling (see *e.g.* the action (1.63)), in practice, every $N = 2$ supergravity model has a vector-multiplet sector. In particular, $D = 4$, $N = 2$ superstring compactifications contain vector multiplets: in the $\mathrm{K3} \times \mathrm{T}^2$ compactification of heterotic string theory, non-abelian gauge fields play a role. In the type-II compactifications on Calabi-Yau manifolds, which will be discussed in more detail in section 5.1, the moduli space of the compactification manifold decomposes into two pieces, the $(1,1)$- and $(2,1)$-moduli, each of which is coordinatized by scalar fields in the four-dimensional effective supergravity model. One of these moduli spaces is described by the scalars of vector multiplets and displays special geometry, whereas the other (containing also the dilaton) falls into the hypermultiplet sector and is a quaternionic space. The couplings of hypermultiplets form the subject of chapter 4.

Also in rigidly $N = 2$ supersymmetric field theory vector multiplets play an important role. Rigidly $N = 2$ supersymmetric non-abelian vector-multiplet models have a corresponding Wilson effective abelian model in which the gauge group is broken to its maximal abelian subgroup. The sigma-manifold of the effective model, which displays so-called rigid special geometry, is given by the possible choices for the vacuum expectation values of the scalar fields. Of course, the effective description breaks down at certain points of the space of vacua, where degrees of freedom that were not included in the effective description become massless.

One of the important developments in recent years, has been the work of Seiberg and Witten [83], who have determined the singularities of the space of vacua and the behavior of the vector-multiplet coupling in their vicinity, for rigidly $N = 2$ supersymmetric SU(2) Yang-Mills theory. Matching the respective singularity structures, they showed that this space of vacua can be viewed as the moduli spaces of certain families of Riemann surfaces, similar to the Calabi-Yau moduli spaces in the case of type-II compactifications. This correspondence can be used to solve the effective model exactly, *i.e.* including non-perturbative contributions.

We shall find in section 3.1 that for every holomorphic, second-degree homogeneous expression F in terms of the vector-multiplet scalars (called a prepotential), we can define a superconformal action for vector-multiplets. In section 3.2, we show that the prepotential is not unique: every choice for F is related to others through so-called symplectic reparameterizations, which form the $N = 2$ supersymmetric generalization of the electric-magnetic duality discussed in chapter 1. A more geometrical standpoint is taken in section 3.3, where we discuss the gauge equivalence with a model of Poincaré supergravity and special geometry. In section 3.4 we simplify matters by reducing the supergravity-coupled model studied in the preceding sections to an on-shell rigidly supersymmetric vector-multiplet model. Of course, the consequences of this reduction for the symplectic behavior of the model and the rigid version of special geometry receive due attention.

3.1 Superconformal vector multiplet couplings

In this section we construct the coupling of $n + 1$ vector multiplets $(X^I, \Omega_i^I, W_\mu^I, Y_{ij}^I)$, $I = 0, 1, \ldots, n$. The resulting action is invariant under the superconformal algebra and describes vector multiplet couplings in the background of the Weyl multiplet. We define n to be one less than the number of vector multiplets in the superconformal model, because after imposing suitable gauge choices, similar to the ones made at the end of section 2.4, this model describes n vector multiplets coupled to Poincaré supergravity, to be discussed in section 3.3. If the Weyl background is chosen to be in the flat space-time limit, the model reduces to a description of rigidly supersymmetric vector-multiplet couplings, which we shall consider in section 3.4. So, by making specific background choices, both the Poincaré and rigid vector-multiplet couplings can be obtained from the model presented in this section, which therefore plays the role of a 'master' formulation throughout the rest of the chapter.

In section 2.4, we have given two density formulae: one involved a linear multiplet, *c.f.* (2.56), and the other was constructed from the highest component of a chiral multiplet, *c.f.* (2.48). The former was used for the construction of the action (2.58) for a single vector multiplet and it will be used again in chapter 6 for the construction of the action for the vector-tensor multiplet. In this section we use the latter to construct the action of several vector multiplets. Note that in order to define the required chiral multiplet, we first of all need a scalar A' that transforms according to the first line of (C.3). The most general choice we can make is given by:

$$A' \propto F(X), \tag{3.1}$$

where F is a holomorphic expression in terms of all the vector multiplet scalars $X^0, \ldots, X^n$. Note that we do not allow any $\bar{X}^I$-dependence, because in that case the supersymmetry variation of the scalar A' would depend on the negative-chiral supersymmetry parameter ϵ_i and the multiplet that arises upon subsequent supersymmetry variations of A' is not a chiral multiplet. In order that the resulting chiral multiplet has the appropriate dilatational and $\mathrm{U}(1)_R$ weights for the coupling to conformal supergravity, this so-called *prepotential* $F(X)$ must be homogeneous of second degree: a simultaneous rescaling of all the vector multiplet scalars X^I by a complex factor λ results in a quadratic rescaling of the prepotential, as follows:

$$F(\lambda X) = \lambda^2 F(X). \tag{3.2}$$

The above implies the following useful relations between F and its derivatives:

$$\begin{aligned} F(X) &= \tfrac{1}{2} F_I X^I , \\ F_I &= F_{IJ} X^J , \\ F_{IJK} X^K &= 0, \\ F_{IJK} &= -F_{IJKL} X^L , \end{aligned} \tag{3.3}$$

where, by definition, F with indices $I, J, \ldots$ is given by:

$$F_{I_1 \cdots I_k} = \frac{\partial}{\partial X^{I_1}} \cdots \frac{\partial}{\partial X^{I_k}} [F(X)]. \tag{3.4}$$

For the moment, the gauge group associated with the vector fields W_μ^I is taken to be abelian. We come back to the non-abelian case at a later stage. The proportionality factor in front of $F(X)$ in (3.1) is fixed to $-\frac{i}{2}$ for later convenience. According to the abelian transformation rules (2.42), the chiral multiplet based on $A' = -\frac{1}{2} i F(X)$ consists of the following components:

$$\begin{aligned} A' &= -\tfrac{1}{2} i F , \\ \Psi' &= -\tfrac{1}{2} i F_I \Omega_i^I , \\ B'_{ij} &= -\tfrac{1}{2} i F_I Y_{ij}^I + \tfrac{1}{4} i F_{IJ} \bar{\Omega}_i^I \Omega_j^J , \\ F'^{-}_{ab} &= -\tfrac{1}{2} i F_I \mathscr{F}_{ab}^{-\,I} + \tfrac{1}{8} i F_{IJ}\, \varepsilon^{kl} \bar{\Omega}_k^I \sigma_{ab} \Omega_l^J , \\ \Lambda'_i &= \tfrac{1}{2} i F_I\, \varepsilon_{ij} \not{D} \Omega^{j\,I} + \tfrac{1}{4} i F_{IJ}\, \sigma_{\mu\nu} \mathscr{F}^{-\,I\,\mu\nu} \Omega_i^J \\ &\quad + \tfrac{1}{4} i F_{IJ}\, \varepsilon^{kl} Y_{ik}^I \Omega_l^J + \tfrac{1}{24} i F_{IJK} (\varepsilon^{kl} \bar{\Omega}_k^I \sigma^{\mu\nu} \Omega_l^J) \sigma_{\mu\nu} \Omega_i^K , \\ C' &= i F_I \Box_C \bar{X}^I + \tfrac{1}{8} i F_I \mathscr{F}_{\mu\nu}^{+\,I} T_{ij}^{\mu\nu} \varepsilon^{ij} \\ &\quad + \tfrac{1}{8} i F_{IJ} Y_{ij}^I Y^{J\,ij} - \tfrac{1}{4} i F_{IJ} \mathscr{F}_{\mu\nu}^{-\,I} \mathscr{F}^{-\,\mu\nu\,J} \\ &\quad - \tfrac{1}{2} i F_{IJ}\, \bar{\Omega}_i^I \not{D} \Omega^{i\,J} + \tfrac{3}{2} i F_I\, \bar{\chi}_i \Omega^{i\,I} \\ &\quad - \tfrac{1}{8} i F_{IJK}\, Y^{ij\,I} \bar{\Omega}_i^J \Omega_j^K + \tfrac{1}{8} i F_{IJK}\, \varepsilon^{ij} \bar{\Omega}_i^I \sigma^{\mu\nu} \Omega_j^J \mathscr{F}_{\mu\nu}^{-\,K} \\ &\quad - \tfrac{1}{48} i F_{IJKL} \varepsilon^{ij} \varepsilon^{kl}\, \bar{\Omega}_i^I \Omega_k^J\, \bar{\Omega}_j^K \Omega_l^L . \end{aligned} \tag{3.5}$$

Substituting these components into the chiral density formula (2.48), we obtain the following superconformally invariant Lagrangian density for abelian vector multiplets:

$$\begin{aligned}
e^{-1}\mathscr{L} \;=\; & iF_I\Box_C\bar{X}^I \\
& -\tfrac{1}{4}iF_{IJ}\mathscr{F}^{-I}_{\mu\nu}\mathscr{F}^{\mu\nu\,-J}+\tfrac{1}{8}iF_I\mathscr{F}^{+I}_{\mu\nu}T^{\mu\nu}_{ij}\varepsilon^{ij}-\tfrac{1}{32}iF\,(T^{\mu\nu}_{ij}\varepsilon^{ij})^2 \\
& -\tfrac{1}{2}iF_{IJ}\,\bar{\Omega}^I_i\not{D}\Omega^{i\,J}+\tfrac{1}{8}iF_{IJ}Y^I_{ij}Y^{J\,ij} \\
& +\tfrac{1}{8}iF_{IJK}\mathscr{F}^{-I}_{\mu\nu}\,\varepsilon^{ij}\bar{\Omega}^J_i\sigma^{\mu\nu}\Omega^K_j-\tfrac{1}{8}iF_{IJK}\,Y^{ij\,I}\bar{\Omega}^J_i\Omega^K_j \\
& -\tfrac{1}{48}iF_{IJKL}\,\varepsilon^{ij}\varepsilon^{kl}\,\bar{\Omega}^I_i\Omega^J_k\,\bar{\Omega}^K_j\Omega^L_l \\
& +\tfrac{3}{2}iF_I\,\bar{\chi}_i\Omega^{i\,I}+\tfrac{1}{2}iF_I\bar{\psi}_{\mu\,i}\gamma^\mu\not{D}\Omega^{i\,I} \\
& +\tfrac{1}{4}iF_{IJ}\mathscr{F}^{-I}_{\rho\sigma}\,\varepsilon^{ij}\bar{\psi}_{\mu\,i}\gamma^\mu\sigma^{\rho\sigma}\Omega^J_j+\tfrac{1}{8}iF_IT_{\rho\sigma\,jk}\,\varepsilon^{ij}\varepsilon^{kl}\bar{\psi}_{\mu\,i}\sigma^{\rho\sigma}\gamma^\mu\Omega^I_l \\
& +\tfrac{1}{4}iF_{IJ}Y^{I\,ij}\bar{\psi}_{\mu\,i}\gamma^\mu\Omega^J_j-\tfrac{1}{12}iF_{IJK}\varepsilon^{ij}\varepsilon^{kl}\bar{\psi}_{\mu\,i}\gamma^\mu\Omega^I_k\,\bar{\Omega}^J_l\Omega^K_j \\
& -\tfrac{1}{2}iF_I\mathscr{F}^{-I\,\mu\nu}\,\varepsilon^{ij}\bar{\psi}_{\mu\,i}\psi_{\nu\,j}+\tfrac{1}{4}iFT^{\mu\nu}_{ij}\varepsilon^{ij}\,\varepsilon^{kl}\bar{\psi}_{\mu\,k}\psi_{\nu\,l}+\tfrac{1}{2}iF_IY^{I\,ij}\bar{\psi}_{\mu\,i}\sigma^{\mu\nu}\psi_{\nu\,j} \\
& -\tfrac{1}{4}iF_{IJ}\varepsilon^{ik}\varepsilon^{jl}\,\bar{\psi}_{\mu\,i}\sigma^{\mu\nu}\psi_{\nu\,j}\,\bar{\Omega}^I_k\Omega^J_l+\tfrac{1}{8}iF_{IJ}\varepsilon^{ij}\varepsilon^{kl}\,\bar{\psi}_{\mu\,i}\psi_{\nu\,j}\,\bar{\Omega}^I_k\sigma^{\mu\nu}\Omega^J_l \\
& +\tfrac{1}{4}iF_I\varepsilon^{ij}\varepsilon^{kl}e^{-1}\varepsilon^{\mu\nu\rho\sigma}\,\bar{\psi}_{\mu\,i}\psi_{\nu\,j}\,\bar{\psi}_{\rho\,k}\gamma_\sigma\Omega^I_l \\
& +\tfrac{1}{4}iF\varepsilon^{ij}\varepsilon^{kl}e^{-1}\varepsilon^{\mu\nu\rho\sigma}\,\bar{\psi}_{\mu\,i}\psi_{\nu\,j}\,\bar{\psi}_{\rho\,k}\psi_{\sigma\,l}\;+\text{h.c.}\,.
\end{aligned}\tag{3.6}$$

Note that the Lagrangian density (2.58) for the minimal field representation is also of the form (3.6). Since, modulo a prefactor, the only possible prepotential based on one vector-multiplet scalar is given by $F(X) = X^2$, (2.58) is unique.

Alternatively, we could have obtained the Lagrangian density (3.6) using the density formula (2.56). In that case we would have constructed a linear multiplet, starting with the $\mathrm{SU}(2)_R$ triplet:

$$L_{ij} = \mathcal{G}_I Y^I_{ij} + \mathcal{C}_{IJ}\bar{\Omega}^I_i\Omega^J_j + \bar{\mathcal{C}}_{IJ}\,\varepsilon_{ik}\varepsilon_{jl}\bar{\Omega}^{I\,k}\Omega^{J\,l}, \tag{3.7}$$

with as yet undetermined functions $\mathcal{G}$ and $\mathcal{C}$ of the vector-multiplet scalars parameterizing the possible terms with the appropriate $\mathrm{SU}(2)_R$ behaviour. The requirement that the above general expression has the appropriate weights and transforms as the first line in (2.51), leads to differential equations relating $\mathcal{G}_I$ and $\mathcal{C}_{IJ}$, which can be solved in terms of a holomorphic, second-degree prepotential $F(X)$. We come back to this alternative construction in chapter 6, when we consider the actions of vector-tensor multiplets.

The construction that employs the chiral multiplet is easily generalized to the case of a non-abelian gauge group. Note that in that case, the scalar fields X^I are no longer gauge-invariant, but transform in the adjoint representation:

$$\delta_{\text{gauge}}(\alpha)\,X^I = g\,f^I_{JK}\alpha^J X^K, \tag{3.8}$$

where α^I are infinitesimal gauge parameters. In order to obtain a gauge-invariant action, we again construct a chiral multiplet based on a prepotential. However, we have to impose that the prepotential is gauge-invariant, which is equivalent to the requirement that:

$$F_I\, f^I_{JK}\, X^K = 0, \tag{3.9}$$

although in certain cases the above condition can be relaxed [32]. A Lagrangian density for the coupling of non-abelian vector multiplets can now be obtained following the same steps as above. The chiral multiplet is constructed using the transformation rules (C.4) and the ensuing Lagrangian density takes the form (3.6) (with non-abelian covariantizations in the covariant derivatives and the field strengths) plus the following terms:

$$\begin{aligned} e^{-1}\mathscr{L}_{\rm non-abelian} &= ig^2\, F_I f^I_{JK}\bar X^K f^J_{MN}\bar X^M X^N - \tfrac{1}{2}ig\, F_{IJ} f^I_{KL}\bar X^K\varepsilon^{ij}\bar\Omega^J_i\Omega^L_j \\ &\quad + \tfrac{1}{2}ig\, F_I f^I_{JK}\varepsilon_{ij}\bar\Omega^{i\,J}\Omega^{j\,K} + \tfrac{1}{2}ig\, F_I f^I_{JK}\bar X^J\varepsilon^{ij}\bar\psi_{\mu\,i}\gamma^\mu\Omega^K_j + \text{h.c.}\,. \end{aligned} \tag{3.10}$$

Note that the non-abelian Lagrangian density contains a scalar potential, which can be written in the form:

$$e^{-1}\mathscr{L} = \ldots - g^2(f^I_{KL}X^K\bar X^L)N_{IJ}(f^J_{MN}X^M\bar X^N), \tag{3.11}$$

where we have used the notation:

$$N_{IJ} = -i(F_{IJ} - \bar F_{IJ}). \tag{3.12}$$

Consequently, suitable vacuum expectation values of the vector-multiplet scalars induce spontaneous symmetry breaking. For instance, any semi-simple gauge group is broken to its maximal abelian subgroup, if the vacuum expectation values are chosen in the Cartan subalgebra. A second scalar potential can arise when the vector multiplets are coupled to other superconformal multiplets, such as hypermultiplets [32]. In that case, integrating out the auxiliary fields Y^I_{ij} leads to a potential for the hypermultiplet scalars.

For future reference we include also the following rewritten version of the purely bosonic part of the abelian vector-multiplet Lagrangian density (3.6):

$$\begin{aligned} e^{-1}\mathscr{L}_{\rm bosonic} &= -i(F_I\bar X^I - \bar F_I X^I)(-\tfrac{1}{6}R + D) \\ &\quad -i(\mathscr{D}_\mu F_I\mathscr{D}^\mu\bar X^I - \mathscr{D}_\mu\bar F_I\mathscr{D}^\mu X^I) - \tfrac{1}{8}N_{IJ}Y^I_{ij}Y^{J\,ij} \\ &\quad -\Big(\tfrac{1}{4}iF_{IJ}F^{-\,I}_{\mu\nu}F^{\mu\nu\,-\,J} + \tfrac{1}{8}i(\bar F_I - F_{IJ}\bar X^J)F^{-\,I}_{\mu\nu}T^{\mu\nu\,ij}\varepsilon_{ij} \\ &\qquad -\tfrac{1}{64}N_{IJ}\bar X^I\bar X^J(T^{\mu\nu\,ij}\varepsilon_{ij})^2 + \text{h.c.}\Big). \end{aligned} \tag{3.13}$$

In going from (3.6) to (3.13), we have split off the curvature term, making exactly the same steps that led to (2.59). The derivatives denoted by $\mathscr{D}_\mu$ contain only the bosonic covariantizations, so in the above abelian case, we have:

$$\mathscr{D}_\mu X^I = (\partial_\mu - b_\mu + iA_\mu)X^I. \tag{3.14}$$

Furthermore, we have rewritten the contributions coming from the covariant field strengths $\mathscr{F}^I_{\mu\nu}$, making the $T^{ij}_{\mu\nu}$-terms explicit and disregarding all fermionic terms. We are left with the abelian bosonic field strengths $F^I_{\mu\nu}$, defined in the standard fashion:

$$F^I_{\mu\nu} = 2\partial_{[\mu} W^I_{\nu]}. \tag{3.15}$$

In a so-called *minimally coupled* vector multiplet model, $F(X)$ is quadratic and its second derivatives determine the coupling constants g_{IJ} and generalized theta angles θ_{IJ} according to:

$$F_{IJ} = \frac{\theta_{IJ}}{2\pi} + i\frac{4\pi}{g^2_{IJ}}\,. \tag{3.16}$$

Note that the Lagrangian density (3.6) changes only by a total derivative if we add to F a quadratic polynomial with real coefficients C_{IJ}, *i.e.* we replace $F \to F + C_{IJ}X^I X^J$. Such a change of the prepotential corresponds to a shift of the theta-angles[1]. In the context of (perturbative) superstring compactifications (see section 5.1 for a discussion of superstring compactifications and [75] for a proof), one often encounters prepotentials of the form [43]:

$$F(X) = \frac{d_{IJK}X^I X^J X^K}{X^0}, \tag{3.17}$$

which are called *very special*. In fact, such a form of the prepotential is found in all models that can be obtained by dimensional reduction from five dimensions [40].

3.2 Symplectic reparameterizations

In section 1.4, we have discussed electric-magnetic duality transformations. Recall that we imposed the Bianchi identity for a vector gauge field through a Lagrange multiplier and we integrated out the field strength. Effectively, the duality transformation amounts to a replacement of the field strength by its dual. Note that the Bianchi identity for the dual field strength is equivalent to the field equation of the original field strength and *vice versa*. Furthermore, the duality transformation induces an inversion of the coupling constant of the form:

$$\tau \mapsto -\frac{1}{\tau}. \tag{3.18}$$

So if we want the dual model to take a manifestly supersymmetric form, we have to make a redefinition of the coupling constant. For this reason the duality transformation is *not an invariance*: it merely states that one model with given coupling constant, or rather its set of field equations and Bianchi identities, can be formulated equivalently as a model with a different coupling constant.

As is well known [42], in many supergravity models the set of all field equations can be transformed to an equivalent set in a way that forms a generalization of the electric-magnetic duality sketched above. However, in a supergravity model, the role of the coupling constant is often taken over by a factor that depends on the scalar fields. So if we want the field equations in the dual model to take the same form as the field equations in the original model, we have to

[1] We come back to the discrete nature of the non-perturbative version of this symmetry in the next section.

redefine the scalar fields along with the redefinition of the field strengths. In a supersymmetric model, the form of the action is prescribed by supersymmetry, so the scalar redefinition is no longer a matter of choice: if, under a duality transformation, we want the model to remain manifestly supersymmetric, the scalar fields must transform as well.

In this section we discuss the duality transformations that play a role in supergravity-coupled $N = 2$ vector-multiplet models, which are called *symplectic reparameterizations* [39]. As was already noted in section 1.4, duality transformations can be made if the gauge fields arise in the action and field equations *only* through their field strengths. If there are minimal couplings, Chern-Simons couplings or the gauge group is non-abelian, duality transformations can not be carried out. Therefore we limit ourselves to a model that describes $n + 1$ *abelian* vector multiplets coupled to the Weyl multiplet only, with the Lagrangian density given by (3.6). Such an abelian vector-multiplet Lagrangian may coincide with the lowest-order effective Lagrangian associated with a supersymmetric non-abelian model in which the gauge group is broken to an abelian subgroup, but for our purposes the origin of the model is not directly relevant.

The order of presentation in the derivation of the symplectic properties of the various quantities involved is as follows: first we introduce the duality transformations as linear transformations among Bianchi identities and field equations for the field strengths. Next we consider the redefinition of the vector-multiplet scalars and the prepotential. Finally, we consider the symplectic properties of the fermion fields and we discuss the symplectic behaviour of the vector-multiplet action.

Let us start by considering the field equations and Bianchi identities for the field strengths of the abelian vector fields, where for the moment we disregard all dependence on fermions:

$$\begin{aligned} \partial_\mu\Big(e(G_I^{+\,\mu\nu} - G_I^{-\,\mu\nu})\Big) &= 0, \\ \partial_\mu\Big(e(F^{+\,\mu\nu\,I} - F^{-\,\mu\nu\,I})\Big) &= 0. \end{aligned} \tag{3.19}$$

Here the antiselfdual component of the field $G_I^{\mu\nu}$ is defined by:

$$G^-_{\mu\nu\,I} = \frac{2i}{e}\frac{\delta\mathscr{L}}{\delta F^{-\,\mu\nu\,I}}, \tag{3.20}$$

and a definition by the complex conjugate for the selfdual component. Note that $G^-_{\mu\nu\,I}$ can be decomposed as follows:

$$G^-_{\mu\nu\,I} = F_{IJ}F^{-\,J}_{\mu\nu} + \mathcal{T}^-_{\mu\nu\,I}. \tag{3.21}$$

The antiselfdual $\mathcal{T}^-_{\mu\nu\,I}$ contains all terms to which $F^{-\,I}_{\mu\nu}$ is coupled linearly in the action. So when we restrict ourselves to the bosonic Lagrangian density, given by (3.13), $\mathcal{T}^-_{\mu\nu\,I}$ contains only a term proportional to $T^{ij}_{\mu\nu}\varepsilon_{ij}$, but when we consider the full Lagrangian density (3.6), also two-fermion terms appear. Note that very often in the discussion of duality transformations among $N = 2$ vector multiplets, the T-fields are eliminated [39]. In our derivation we leave the superconformal background intact, because in this fashion the duality invariance of the higher-spin supergravity sector is demonstrated most clearly.

Clearly, the system of equations (3.19) is transformed into an equivalent system under any invertible, linear redefinition of antiselfdual and selfdual two-tensors of the form:

$$\begin{pmatrix} F^{\pm I} \\ G^{\pm}_I \end{pmatrix} \stackrel{S}{\longmapsto} \begin{pmatrix} U^I{}_J & Z^{IJ} \\ W_{IJ} & V_I{}^J \end{pmatrix} \begin{pmatrix} F^{\pm I} \\ G^{\pm}_I \end{pmatrix}. \tag{3.22}$$

Here $U^I{}_J$, Z^{IJ}, W_{IJ} and $V_I{}^J$ are as yet undetermined constant, real $(n+1)\times(n+1)$ matrices. We denote the $(2n+2)\times(2n+2)$ matrix that represents this so-called *symplectic transformation* S on the field strengths by:

$$\mathcal{O} = \begin{pmatrix} U^I{}_J & Z^{IJ} \\ W_{IJ} & V_I{}^J \end{pmatrix}. \tag{3.23}$$

Of course, we want to maintain the relation (3.20), so the scalar-dependent terms that are present in $G^{\pm}_{\mu\nu\,I}$ must transform as well. At this stage, we concentrate only on the first term on the *r.h.s.* of (3.21), which will turn out to prescribe the symplectic properties of the scalar fields. With these in hand, we return to the second term in (3.21) at a later stage.

In order to reconcile the transformation rule (3.22) with the first term in (3.21), the second derivative F_{IJ} of the prepotential has to transform according to:

$$F_{IJ} \stackrel{S}{\longmapsto} \tilde{F}_{IJ} = (V_I{}^K F_{KL} + W_{IL})[(U+ZF)^{-1}]^L{}_J. \tag{3.24}$$

The required change of F_{IJ} is induced by the following redefinition of the scalar fields:

$$\begin{pmatrix} X^I \\ F_I \end{pmatrix} \stackrel{S}{\longmapsto} \begin{pmatrix} \tilde{X}^I \\ \tilde{F}_I \end{pmatrix} = \begin{pmatrix} U^I{}_J & Z^{IJ} \\ W_{IJ} & V_I{}^J \end{pmatrix} \begin{pmatrix} X^I \\ F_I \end{pmatrix}. \tag{3.25}$$

If we now calculate $\tilde{F}_{IJ}$ as the $\tilde{X}^J$-derivative of $\tilde{F}_I$, we arrive at (3.24). Note that we temporarily consider X^I and F_I as independent of each other and we reinstate their relation in terms of the prepotential *a posteriori*.

The kinetic term of the scalar fields and the X^I-dependent factor multiplying the scalar curvature R in the action (3.13) are invariant under S, only if the transformation (3.25) leaves the product $-i(F_I\bar{X}^I - X^I\bar{F}_I)$ invariant. Hence we require that $\mathcal{O}$ satisfies:

$$\mathcal{O}^T\,\Omega\,\mathcal{O} = \Omega, \quad \text{where} \quad \Omega = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}. \tag{3.26}$$

Hence, we see that $\mathcal{O}$ must be an element of $\mathrm{Sp}(2n+2,\mathbb{R})$. Moreover, since we want the transformed model to describe as many scalar degrees of freedom as the original model, we require that the action of S on the fields X^I is invertible. So if we define the field-dependent transformation-matrix $\mathcal{S}$ by:

$$\mathcal{S}(X)^I{}_J = \frac{\partial \tilde{X}^I}{\partial X^J} = U^I{}_J + Z^{IK}F_{KJ}, \tag{3.27}$$

so that:

$$\tilde{X}^I = \mathcal{S}(X)^I{}_J\, X^J, \tag{3.28}$$

then the $(n+1)\times(n+1)$ matrix $\mathcal{S}(X)$ has to be invertible for all values of X^I. We have already used this property implicitly when we wrote down (3.24). Note that when going from (3.27) to (3.28), we make use of the homogeneity of $F(X)$.

The question now arises which prepotential $\tilde{F}(\tilde{X})$ underlies the couplings of the model after the symplectic transformation. By definition of the new prepotential, we require that the new $\tilde{F}_I$ are given by the $\tilde{X}^I$-derivatives of $\tilde{F}(\tilde{X})$. An integrability condition is found by requiring that $\tilde{F}_{IJ}$ is symmetric in (I,J). According to (3.24), this means that:

$$(U^TW)_{IJ} + (U^TV)_I{}^K F_{KJ} + F_{IK}(Z^TW)^K{}_J + F_{IK}(Z^TV)^{KL}F_{LJ}, \tag{3.29}$$

must be symmetric in (I,J). For generic prepotentials $F(X)$, the above condition implies that the first and the last term are separately symmetric. Furthermore we assume that the identity matrix is the only matrix that commutes with F_{IJ}. The resulting requirements are satisfied as a result of (3.26). Therefore the relation $\tilde{F}_I = \partial/\partial\tilde{X}^I[\tilde{F}(\tilde{X})]$ can be integrated and we find the new prepotential:

$$\tilde{F}(\tilde{X}) = \tfrac{1}{2}(U^TW)_{IJ}X^IX^J + \tfrac{1}{2}(U^TV + W^TZ)^I{}_J F_I X^J + \tfrac{1}{2}(Z^TV)^{IJ}F_IF_J, \tag{3.30}$$

which is again holomorphic and second-degree homogeneous.

In practical situations the expression (3.64) is not always useful, as it requires substituting $\tilde{X}^I$ in terms of X^I, or vice versa. When F remains the same:

$$\tilde{F}(\tilde{X}) = F(\tilde{X}), \tag{3.31}$$

the theory is *invariant* under the corresponding transformation. These invariances are often called *duality invariances* or *duality symmetries* and have been studied extensively [42, 39]. The space of inequivalent couplings of $n+1$ abelian vector supermultiplets is equal to the space of holomorphic functions of $n+1$ variables, divided by the $\mathrm{Sp}(2n+2,\mathbb{R})$ group (or non-perturbatively $\mathrm{Sp}(2n+2,\mathbb{Z})$; we come back to non-perturbative aspects at the end of this section). This group does not act freely on the space of these functions. There are fixed points whenever the field equations exhibit duality symmetries. It is not easy to find solutions of $\tilde{F}(\tilde{X}) = F(\tilde{X})$, unless one considers infinitesimal transformations. In that case the condition reads [39]:

$$C_{IJ}\, X^IX^J - 2B^I{}_J\, X^JF_I + D^{IJ}\, F_IF_J = 0, \tag{3.32}$$

where the constant matrices $B^I{}_J$, C_{IJ} and D^{IJ} parameterize the infinitesimal form of the $\mathrm{Sp}(2n+2,\mathbb{R})$ matrix, according to $U \approx \mathbb{I} + B$, $V \approx \mathbb{I} - B^T$, $W \approx C$ and $Z \approx -D$. For finite transformations, a more convenient method is to verify that the substitution $X^I \mapsto \tilde{X}^I$ into the derivatives $F_I(X)$ correctly induces the symplectic transformations on the the symplectic pair (X^I, F_I).

As far as the second term on the *r.h.s.* of (3.21) is concerned, the antiselfdual component of $\mathcal{T}$ must transform according to:

$$\mathcal{T}^{-}_{\mu\nu\,I} \stackrel{S}{\longmapsto} \tilde{\mathcal{T}}^{-}_{\mu\nu\,I} = \mathcal{T}^{-}_{\mu\nu\,J}[(U+ZF)^{-1}]^{J}{}_{I}, \tag{3.33}$$

in order to reconcile (3.21) with (3.22). Restricting the discussion to the bosonic part of the vector-multiplet action, we find that:

$$\mathcal{T}^{-}_{\mu\nu\,I} = \tfrac{1}{4}(\bar{F}_I - F_{IJ}\bar{X}^J)T^{ij}_{\mu\nu}\varepsilon_{ij}. \tag{3.34}$$

Using (3.25), (3.24) and (3.26), one can indeed show that the combination $\bar{F}_I - F_{IJ}\bar{X}^J$ has the required symplectic behaviour.

When we consider the full Lagrangian density, *c.f.* (3.6), obviously the above tensor $\mathcal{T}_{\mu\nu\,I}$ is going to depend also on the fermion fields Ω^I_i (and the gravitini $\psi_\mu{}^i$). So before we turn to a discussion of the symplectic behaviour of the full action, we consider the properties of the fermion fields. Modulo a total derivative, the kinetic terms for the fermions in the action (3.6) can be written in the form:

$$i\Big(\bar{\Omega}^I_i \overleftrightarrow{\not{D}} (\bar{F}_{IJ}\Omega^{J\,i}) - \bar{\Omega}^{I\,i} \overleftrightarrow{\not{D}} (F_{IJ}\Omega^J_i)\Big), \tag{3.35}$$

which is left invariant by the linear action of $\mathcal{O}$ on the $(2n+2)$-dimensional symplectic vector $(\Omega^I_i, F_{IJ}\Omega^J_i)$ of positive-chiral fermion fields:

$$\begin{pmatrix} \Omega^I_i \\ F_{IJ}\Omega^J_i \end{pmatrix} \stackrel{S}{\longmapsto} \begin{pmatrix} U^I{}_J & Z^{IJ} \\ W_{IJ} & V_I{}^J \end{pmatrix} \begin{pmatrix} \Omega^J_i \\ F_{JK}\Omega^K_i \end{pmatrix}, \tag{3.36}$$

and the complex conjugate for negative-chiral fermions. Writing the action of S for fermions in terms of the X-dependent matrix $\mathcal{S}(X)$, we find:

$$\begin{aligned} \Omega^I_i &\stackrel{S}{\longmapsto} \tilde{\Omega}^I_i = \mathcal{S}(X)^I{}_J\,\Omega^J_i\,, \\ \Omega^{i\,I} &\stackrel{S}{\longmapsto} \tilde{\Omega}^{i\,I} = \bar{\mathcal{S}}(\bar{X})^I{}_J\,\Omega^{i\,J}. \end{aligned} \tag{3.37}$$

Note that, just like the selfdual and antiselfdual field strengths had their own representation of the symplectic transformation and a holomorphic decomposition was present in the representation of S on the scalars, the symplectic behaviour of the fermions decomposes into positive- and negative-chiral.

What remains to show is that the action of a symplectic transformation on the full set of equations of motion and the Bianchi identities leads to an equivalent set. This can be done by identifying the terms in the action that vanish as a result of the field equations of the auxiliary fields and the vector fields. The remainder of terms should then be invariant under symplectic transformations. The proof of the latter is greatly facilitated by the following three observations: firstly, note that the scalar-dependent quantities N_{IJ} and F_{IJK} transform as follows:

$$\begin{aligned} N_{IJ} &\stackrel{S}{\longmapsto} \tilde{N}_{IJ} = N_{KL}[\bar{\mathcal{S}}^{-1}]^K{}_I[\mathcal{S}^{-1}]^L{}_J\,, \\ F_{IJK} &\stackrel{S}{\longmapsto} \tilde{F}_{IJK} = F_{LMN}[\mathcal{S}^{-1}]^L{}_I[\mathcal{S}^{-1}]^M{}_J[\mathcal{S}^{-1}]^N{}_K\,. \end{aligned} \tag{3.38}$$

The symmetry of N_{IJ} and F_{IJK} in their respective indices is guaranteed by the symplectic nature of the transformation. Secondly, we note the following useful construction: given a symplectic $(2n{+}2)$-dimensional vector (A^I, B_I), one can define the quantity $Y_I = B_I - F_{IJ}A^J$, which transforms as:

$$Y_I \xrightarrow{\mathcal{S}} \tilde{Y}_I = Y_J[\,\mathcal{S}^{-1}]^J{}_I. \tag{3.39}$$

And thirdly, note that the Weyl multiplet is symplectically invariant and serves only as a background in which symplectic transformations can be performed consistently. This also implies that symplectic transformations in a rigidly supersymmetric model, to be considered in section 3.4, hardly differ from the above. However, in a model of Poincaré supergravity, the graviphoton partakes in the symplectic transformations and in that case the supergravity couplings complicate matters somewhat, as we shall see in section 3.3.

Because the proof of the symplectic invariance of the field equations in the presence of all the fermionic terms is laborious, we do not make an attempt to discuss this in any detail and leave the actual calculation to the reader. A discussion of the symplectic behavior of the bosonic part of the action can be found in [40].

When non-perturbative effects are taken into account, the group of symplectic transformations becomes $\mathrm{Sp}(2n+2,\mathbb{Z})$. Namely, note that the spin-1 kinetic term in the action can be written in the form:

$$e^{-1}\mathscr{L} = -\tfrac{1}{8}i(F_{IJ} - \bar{F}_{IJ})\,F^I_{\mu\nu}F^{\mu\nu\,J} + \tfrac{1}{8}i(F_{IJ} + \bar{F}_{IJ})\,F^I_{\mu\nu}\tilde{F}^{\mu\nu\,J}. \tag{3.40}$$

In this example we consider an abelian vector-multiplet model with a quadratic prepotential, which leads to an F_{IJ} as given in (3.16). Indeed the coupling constants g_{IJ} appear in front of the F^2-term and the theta-angles θ_{IJ} in front of the $F\tilde{F}$-term. Note that certain symplectic transformations, namely those that have $U = V = \mathbb{I}$, $Z = 0$ and $W_{IJ} = \Delta\theta_{IJ}$, leave the coupling constant the same but result in a constant shift $\Delta\theta_{IJ}$ of the theta-angles. Perturbatively, the above $F\tilde{F}$-term can be ignored, because it leads to a total derivative in the action. However, if for example the above model is the effective U(1) field theory that remains after breaking of an SU(2) Yang-Mills model, the $F\tilde{F}$ is (proportional to) the Pontryagin index that counts the number of SU(2) instantons in the background. So non-perturbatively, the action is *not invariant* under all shifts of the theta-angles, but only under a discrete subset of integer-valued shifts[2]. Alternatively, one can derive the discrete nature of the symplectic group by considering the electric and magnetic charges in the model[3]: the Schwinger-Zwanziger quantization condition implies that the charges form a *lattice*, which must be mapped onto itself by the symplectic group [83]. This means that non-perturbatively, only the discrete subgroup $\mathrm{Sp}(2n+2,\mathbb{Z})$ of the symplectic group is left.

[2]In principle the integers are multiplied by a certain constant determined by the embedding of the U(1) group in the non-abelian gauge group of the underlying field theory. This constant is set to unity.

[3]See the footnote in section 5.5.

3.3 Special geometry

So far, we have only discussed the construction of the vector-multiplet action and symplectic transformations in the superconformal framework. In the current section we consider the vector-multiplet sector in corresponding models of Poincaré supergravity, concentrating on the geometrical aspects of the scalar fields. Namely, we have not yet considered one of the more eye-catching characteristics of the vector-multiplet model; the fact that the scalar kinetic term in the superconformal Lagrangian density, which takes the form:

$$N_{IJ}\, D_\mu X^I D^\mu \bar{X}^J, \tag{3.41}$$

has a field-dependent prefactor $N_{IJ}(X,\bar{X})$, as defined in (3.12). Based on the discussion in section 1.3, we see that the scalar fields form a non-linear sigma-model with metric N_{IJ}. If such scalar fields are part of a supersymmetric model, further requirements can be formulated. In particular, we found that in an $N=1$ supersymmetric sigma-model, the sigma-manifold is a Kähler manifold. Indeed, one easily determines that the Hermitian metric N_{IJ} can be written in terms of a Kähler potential K', given by:

$$K'(X,\bar{X}) = -i(F_I\bar{X}^I - \bar{F}_I X^I). \tag{3.42}$$

If we simply reduce the superconformal model to a rigidly supersymmetric model by taking the flat space-time limit for the Weyl multiplet, the above argument applies, as we shall see in section 3.4.

However, in the case where the vector-multiplet couplings are formulated in a background of conformal supergravity, there is a complication: since the superconformal vector-multiplet model is gauge equivalent to a model of Poincaré supergravity, we can eliminate the compensating degrees of freedom and the corresponding gauge invariances by making appropriate gauge choices, analogous to those given in section 2.4. In particular, the resulting Poincaré vector-multiplet model describes n scalar fields instead of the $n+1$ scalar fields that were present in the superconformal model. Correspondingly, the sigma-manifold of the Poincaré model has a complex dimension that is one less than could naively have been expected on the basis of (3.41).

Let us consider the above reasoning in some more detail. Since we are interested primarily in aspects concerning the scalar fields, fermion fields play only a minor role in the rest of this section and in many cases we shall disregard them. As we have seen, the bosonic part of the superconformal Lagrangian density for abelian vector multiplets is given by (3.13), the form of which is prescribed by a second-degree, holomorphic prepotential $F(X)$.

Analogously to the steps we made at the end of section 2.4, we now break the superconformal symmetry algebra to the Poincaré superalgebra. Again, we impose the gauge condition $b_\mu = 0$ to break K-invariance. In order to decouple spin-0 and spin-2 degrees of freedom, we have to impose a dilatational gauge condition, such that the coefficient of the scalar curvature becomes a constant. A symplectically invariant choice is given by:

$$D\text{-gauge:} \qquad -i(F_I\bar{X}^I - X^I\bar{F}_I) = 1. \tag{3.43}$$

Similarly, to decouple spin-$\frac{3}{2}$ and spin-$\frac{1}{2}$ degrees of freedom, we impose the symplectically invariant condition:

$$S\text{-gauge:} \qquad X^I N_{IJ}\,\Omega^{i\,J} = 0, \tag{3.44}$$

which breaks S-supersymmetry. Of course, since we are only considering the vector-multiplet sector of a superconformal model, we do not have the appropriate compensating fields to fully describe the breaking to Poincaré supergravity. Also, the field equations of some of the auxiliary fields lead to inconsistencies that can only be resolved in a more general model, *e.g.* involving hypermultiplet couplings as well. For the solution to both these problems we refer to [32] (see also the discussions at the end of section 2.4 and in section 4.2). However, for other auxiliary fields the field equations *can* be solved. Of particular importance for the current discussion is the $\mathrm{U}(1)_R$ gauge field A_μ. Solving its (bosonic) field equation, we find that the vector field A_μ is given by:

$$A_\mu = -\tfrac{1}{2}iN_{IJ}(X^I\partial_\mu\bar{X}^J - \bar{X}^I\partial_\mu X^J), \tag{3.45}$$

where we have made use of the dilatational gauge condition (3.43). Substituting this expressions back into the Lagrangian density, we find that the kinetic terms of the scalar fields take the form:

$$e^{-1}\mathscr{L}_{\text{scalar}} = N_{IJ}\,\partial_\mu X^I\partial^\mu\bar{X}^J + \tfrac{1}{4}(N_{IJ}(X^I\partial_\mu\bar{X}^J - \bar{X}^I\partial_\mu X^J))^2. \tag{3.46}$$

Note that this Lagrangian density is still invariant under local $\mathrm{U}(1)_R$ transformations. However, in view of the constraint (3.43), a description in terms of the fields $X^0,\ldots,X^n$ is somewhat unwieldy. Note that the constraint is a projective condition, normalizing the vector formed by the coordinates X^I to unit length. Therefore we decompose the set of scalar fields into X^0 and X^A, $(A=1,\ldots,n)$ and define inhomogeneous coordinates:

$$Z^0 = 1, \qquad Z^A = \frac{X^A}{X^0}, \tag{3.47}$$

which are invariant under $\mathrm{U}(1)_R$. Given a set of inhomogeneous coordinates $Z^1,\ldots,Z^n$, the corresponding homogeneous coordinates $X^0,\ldots,X^n$, subject to the condition (3.43), may be reconstructed, albeit that the phase of X^0 remains unknown. However, this phase is irrelevant in view of local $\mathrm{U}(1)_R$ invariance. In fact, we can eliminate this compensating degree of freedom by imposing the $\mathrm{U}(1)_R$ gauge condition $X^0 = \bar{X}^0$, analogous to (2.62). So modulo a phase, the homogeneous coordinates X^I can be viewed as functions of the inhomogeneous coordinates: $X^I = X^I(Z)$. Consequently, all quantities dependent on X^I, in particular the prepotential F, can now be considered as being dependent on Z^A.

Before writing down the kinetic terms in the action, we solve the field equations for the auxiliary selfdual tensor T^{ij} (and its antiselfdual complex conjugate), which takes the form:

$$X^K N_{KL} X^L\, T^{ij}_{\mu\nu}\varepsilon_{ij} = -4X^I N_{IJ} F^{+\,J}_{\mu\nu}. \tag{3.48}$$

Furthermore, we note that the vector-multiplet auxiliary scalar fields Y_{ij} contribute only to the four-fermion couplings, so they do not play a role for the kinetic part of the action. With a suitable fermion redefinition [32], the kinetic terms of the vector-multiplet components in (3.13) take the form:

$$\begin{aligned} e^{-1}\mathscr{L}_{\text{kin}} &= \frac{1}{(ZN\bar{Z})}\mathcal{M}_{I\bar{J}}\,\partial_\mu Z^I \partial^\mu \bar{Z}^J + \tfrac{1}{4}\mathcal{M}_{I\bar{J}}(\bar{\Omega}^{i\,I}\partial\!\!\!/\Omega_i^J + \bar{\Omega}_i^I \partial\!\!\!/\Omega^{i\,J}) \\ &\quad +\tfrac{1}{4}i\Big(\mathcal{N}_{IJ}F_{\mu\nu}^{+\,I}F^{\mu\nu\,+\,J} - \bar{\mathcal{N}}_{IJ}F_{\mu\nu}^{-\,I}F^{\mu\nu\,-\,J}\Big), \end{aligned} \tag{3.49}$$

where we have used $\mathcal{M}$ and $\mathcal{N}$, defined by:

$$\begin{aligned} \mathcal{M}_{I\bar{J}} &= N_{IJ} - \frac{(N\bar{Z})_I (NZ)_J}{(ZN\bar{Z})}, \\ \mathcal{N}_{IJ} &= \bar{F}_{IJ} - i\frac{(ZN)_I (ZN)_J}{ZNZ}. \end{aligned} \tag{3.50}$$

In both (3.49) and (3.50), we have made use of the notation: $(ZN\bar{Z}) = Z^I N_{IJ}\bar{Z}^J$, $(ZN)_I = Z^J N_{JI}$, etcetera, where I and J run from 0 to n. The Lagrangian density (3.49) describes $n+1$ vector fields, one of which, the graviphoton, belongs in the supergravity multiplet. Note that the matrix $\mathcal{M}$ has two null-directions proportional to Z^I and $\bar{Z}^I$, corresponding to the fact that after elimination of the compensating fields, we are left with only n complex scalars and n fermions. Thus, the above Lagrangian density describes n vector multiplets coupled to Poincaré supergravity.

In the beginning of this section we argued that the scalar fields of the superconformal model parameterize a Kähler manifold, which is required by $N = 1$ supersymmetry. By the same token, the scalars in the Poincaré model must coordinatize a Kähler manifold (of restricted type). Indeed, there is a Kähler potential for the metric $(ZN\bar{Z})^{-1}\mathcal{M}$, given by:

$$K(Z,\bar{Z}) = \log(N_{IJ}Z^I\bar{Z}^J). \tag{3.51}$$

Note that using the Kähler potential, the relation between homogeneous and inhomogeneous special coordinates can be written in the form:

$$Z^I = e^{-\frac{1}{2}K(Z,\bar{Z})}X^I(Z). \tag{3.52}$$

Based on the form of the metric, *c.f.* the first line of (3.50), one easily sees that the Hermitian connection is given by a complicated expression. However, the corresponding curvature tensor takes the surprisingly simple form:

$$R^A{}_{BC}{}^D = -2\delta^A{}_{(B}\delta^D{}_{C)} - e^{2K}Q_{BCE}\bar{Q}^{EAD}, \tag{3.53}$$

where Q is defined by:

$$Q_{ABC} = iF_{IJK}(X(Z))\frac{\partial X^I(Z)}{\partial Z^A}\frac{\partial X^J(Z)}{\partial Z^B}\frac{\partial X^K(Z)}{\partial Z^C}. \tag{3.54}$$

The coordinates Z^I are called *special coordinates* and the corresponding geometry is referred to as *special Kähler geometry.* The above form of the curvature tensor is sometimes (see [45] and references therein) used as the defining property of special Kähler manifolds.

So far, we have only discussed the gauge equivalence of the superconformal model to a model of Poincaré supergravity, concentrating on the geometry of the resulting sigma-model. However, the consequences of the fact that we are dealing with a model that also involves vector fields have not yet received due attention. The presence of the vector gauge fields gave rise to the symplectic transformations discussed in the previous section. Note that the various gauge conditions that lead to the Poincaré model, mix the components of the Weyl multiplet with those that formerly resided in a separate vector multiplet, due to a decomposition rule similar to (2.64). This means that the resulting Poincaré supergravity multiplet is involved in the symplectic transformations, as opposed to the Weyl multiplet that is invariant under symplectic transformations. Needless to say, an analysis of symplectic transformations in the Poincaré supergravity model is complicated [39].

Before we turn to the geometrical consequences of the fact that the model is subject to symplectic transformations, we first note that the parameterization of the sigma-model that we have used so far is suggested by (off-shell) supergravity, but is not covariant with respect to *all* diffeomorphisms, only those that are in the symplectic group. As we have seen in the above, the homogeneous coordinates X^I can be viewed as functions of the inhomogeneous coordinates Z^I. In an arbitrary holomorphic coordinatization z^α, ($\alpha = 1, \ldots, n$), of the sigma-manifold, the inhomogeneous coordinates are given as holomorphic expressions: $Z^I = Z^I(z^\alpha)$ and all Z^I-dependent quantities can now be understood as being z^α-dependent. Note that, as far as (X^I, F_I) is concerned, every recoordinatization $z^\alpha \to (z')^\alpha$ is effected by a symplectic transformation, possibly accompanied by a multiplication with a complex factor corresponding to a dilatation and a $\mathrm{U}(1)_R$ transformation. Namely, irrespective of the particular coordinate frame, the action must have a formulation in terms of X^I and F_I. Furthermore, the X^I and F_I depend only on the Z^I, *i.e.* they too are holomorphic $X^I = X^I(z)$, $F_I = F_I(z)$. Using the relation (3.52), one can show [47, 46] that the pair $(Z^I(z), \partial/\partial Z^I[F(Z(z))])$ transforms under a coordinate transformation as:

$$\begin{pmatrix} Z(z') \\ \partial F(z') \end{pmatrix} = \mathrm{e}^{f(z)} \mathcal{O} \begin{pmatrix} Z(z) \\ \partial F(z) \end{pmatrix}, \tag{3.55}$$

where $\mathcal{O}$ is the symplectic transformation that is also represented on the symplectic pair (X^I, F_I) and $f(z)$ is the holomorphic term of the Kähler transformation, *c.f.* (1.29). This result can be understood in the following way: when the pair (X^I, F_I) is transformed by a symplectic transformation $\mathcal{O}$, the condition $Z^0 = 1$ can only be preserved if the transformation on the pair $(Z^I, \partial_I F)$ has an additional factor to restore the gauge choice. This extra factor is represented in (3.55) by e^f. From (3.55), we conclude that the pair $(Z(z), \partial F(z))$ forms a holomorphic section of a $\mathbb{C} \otimes \mathrm{Sp}(2n+2, \mathbb{R})$ bundle $\mathcal{L} \otimes \mathcal{H}$ over the special Kähler manifold. The line bundle $\mathcal{L}$ corresponds to the line bundle mentioned at the end of section 1.3 in relation to the supergravity coupling of $N = 1$ supersymmetric sigma models. Moreover, we know from that discussion that the cohomology class of the Kähler form must be equal two times the first Chern class of the line bundle, which is of integer cohomology.

Based on the above, Strominger [47] gave a coordinate independent definition of a special Kähler manifold: a manifold $\mathscr{M}$ is special Kähler if $\mathscr{M}$ is Kähler of restricted type and allows a holomorphic $\mathrm{Sp}(2n+2,\mathbb{R})$ bundle $\mathcal{H}$, such that a holomorphic section v of $\mathcal{L} \otimes \mathcal{H}$ can be found, that satisfies:

$$\omega = -i\partial\bar{\partial} \log i\langle v|\bar{v}\rangle, \tag{3.56}$$

where $\langle\,.\,|\,.\,\rangle$ denotes the symplectic bilinear form on $\mathcal{H}$ and ω is the Kähler form. Strominger also proved that this definition is equivalent to the description in terms the fields Z^I, by showing that (3.56) can be used to find a set of special coordinates. However [46], (3.56) must be supplemented by the additional requirement[4]:

$$\langle v|\partial_\alpha v\rangle = 0, \tag{3.57}$$

in order to guarantee that the resulting matrix $\mathcal{N}_{IJ}$ be symmetric.

Yet another definition of special geometry [45] can be given using rheonomic methods [44]. In this context, it was noted [49] that for certain sigma-manifolds, no prepotential F could be found. However, for all such situations there is a symplectic transformation that leads to a frame in which a prepotential *does* exist [46].

3.4 Rigid vector multiplet couplings

In this section we discuss the couplings among on-shell vector multiplets in the context of rigid supersymmetry. The result can be viewed as a reduction from the conformal or Poincaré supergravity couplings discussed in the previous sections, although in one respect the rigid couplings are more general. Namely, the couplings are again encoded by a holomorphic prepotential $F(X)$, but in rigid supersymmetry the requirement that $F(X)$ is second-degree homogeneous is dropped.

This section can be viewed as a preparation for the considerations made in chapter 5, where we discuss the classical mirror map: it is the class of on-shell abelian vector-multiplet models discussed in the current section that will be mapped to hypermultiplet models. Particular emphasis is given to the rigid version of symplectic reparameterizations.

We define n abelian vector multiplets $(X^I, \Omega_i^I, F_{\mu\nu}^I, Y_{ij}^I)$, $I = 1, 2, \ldots, n$, with off-shell rigid transformation rules given by (2.38). The construction of the action proceeds along the same steps as those presented in section 3.1, only now the gravitational background is fixed to the flat space-time limit. So again we construct a chiral multiplet from a scalar A' that depends on the vector multiplet scalars in a holomorphic way. However, the requirement that

[4] This condition is equivalent with a slightly different condition presented earlier in [48], except when the complex dimension of the special Kähler space is 1.

the chiral multiplet has the appropriate weights for a consistent coupling to the conformal symmetries is dropped. Consequently, we no longer require that the prepotential is second-degree homogeneous: in the context of rigid supersymmetry only the holomorphicity of the prepotential is prerequisite to construct a rigidly supersymmetric Lagrangian density. After calculation of the C component of the chiral multiplet based on $-\frac{i}{2}F(X)$ and using the density formula (2.47), we obtain the following Lagrangian density:

$$\begin{aligned}
4\pi\,\mathscr{L} \;=\;& -i\Big(\partial_\mu F_I\,\partial^\mu \bar X^I - \partial_\mu \bar F_I\,\partial^\mu X^I\Big) \\
&-\tfrac{1}{4}i\Big(F_{IJ}F^{-I}_{\mu\nu}F^{-J\,\mu\nu} - \bar F_{IJ}F^{+I}_{\mu\nu}F^{+J\,\mu\nu}\Big) \\
&+\tfrac{1}{4}N_{IJ}\Big(\bar\Omega^{iI}\partial\!\!\!/\Omega_i^J + \bar\Omega_i^I\partial\!\!\!/\Omega^{iJ}\Big) + \tfrac{1}{4}i\Big(\bar\Omega_i^I\partial\!\!\!/ F_{IJ}\Omega^{iJ} - \bar\Omega^{iI}\partial\!\!\!/\bar F_{IJ}\Omega_i^J\Big) \\
&+\tfrac{1}{8}i\Big(F_{IJK}\bar\Omega_i^I\,\sigma^{\mu\nu}F^{-J}_{\mu\nu}\,\Omega_j^K\,\varepsilon^{ij} - \bar F_{IJK}\bar\Omega^{iI}\,\sigma^{\mu\nu}F^{+J}_{\mu\nu}\,\Omega^{jK}\,\varepsilon_{ij}\Big) \\
&-\tfrac{1}{96}i\Big(F_{IJKL} + iN^{MN}(2F_{MIK}F_{JLN} - \tfrac{1}{2}F_{MIJ}F_{KLN})\Big)\,\bar\Omega_i^I\sigma_{\mu\nu}\Omega_j^J\varepsilon^{ij}\,\bar\Omega_k^K\sigma^{\mu\nu}\Omega_l^L\varepsilon^{kl} \\
&+\tfrac{1}{96}i\Big(\bar F_{IJKL} - iN^{MN}(2\bar F_{MIK}\bar F_{JLN} - \tfrac{1}{2}\bar F_{MIJ}\bar F_{KLN})\Big)\,\bar\Omega^{iI}\sigma_{\mu\nu}\Omega^{jJ}\varepsilon_{ij}\,\bar\Omega^{kK}\sigma^{\mu\nu}\Omega^{lL}\varepsilon_{kl} \\
&+\tfrac{1}{16}N^{MN}F_{MIJ}\bar F_{KLN}\,\bar\Omega^{iK}\Omega^{jL}\,\bar\Omega_i^I\Omega_j^J\,,
\end{aligned} \tag{3.58}$$

where we have eliminated the auxiliary fields Y_{ij}^I and performed Fierz re-orderings in the four-fermion terms. Again, $F_{I_1\cdots I_k}$ denotes the k-th derivative of F. Keep in mind, however, that in the rigid case the relations (3.3) do not hold, because they were derived from the homogeneity of the prepotential. Furthermore, we have included an overall normalization factor of $(4\pi)^{-1}$ for later convenience.

The scalar kinetic term in (3.58) again takes the form of a non-linear sigma-model. The complex scalars X^I parameterize a complex n-dimensional target space with metric $g_{I\bar J} = N_{IJ}$, which is Kähler with Kähler potential:

$$K(X,\bar X) = -i(F_I\bar X^I - \bar F_I X^I). \tag{3.59}$$

Note that we have encountered this form of the Kähler potential already in the preliminary discussion in the previous section, *c.f.* (3.42). In this case, the complication in the form of the compensating scalar degrees of freedom no longer plays a role and indeed the sigma-model parameterized by the scalar fields has a Kähler potential of the above form. The resulting geometry is known as *rigid special geometry.* Non-vanishing components of the Hermitian connection and the corresponding curvature tensor are given by:

$$\begin{aligned}
\Gamma^I_{JK} &= g^{I\bar L}\partial_J g_{K\bar L} = -iN^{IL}F_{JKL}\,, \\
R^I{}_{JK}{}^L &= g^{L\bar L}\,\partial_{\bar L}\Gamma^I_{JK} = -N^{IP}N^{LQ}N^{MN}\,\bar F_{PQM}\,F_{NJK}\,,
\end{aligned} \tag{3.60}$$

where we have used N^{IJ} with upper indices to denote the inverse metric. One can choose to formulate the model in a manifestly coordinate-independent fashion, by choosing new coordinates z and considering $(X^I(z), F_I(z))$ as sections of an $\mathrm{Sp}(2n,\mathbb{R})$ bundle over the sigma-model manifold [44], analogous to the definitions made in the previous section. As it is straightforward to cast our results in such a coordinate-independent form, we keep writing them in terms of the X^I.

We also record the on-shell supersymmetry transformation rules for the vector multiplet components, which can be obtained from the off-shell transformation rules (2.38) after elimination of the auxiliary field:

$$\begin{aligned}
\delta_Q(\epsilon)\, X^I &= \bar\epsilon^i \Omega_i^I\,, \\
\delta_Q(\epsilon)\, A_\mu^I &= \varepsilon^{ij}\, \bar\epsilon_i \gamma_\mu \Omega_j^I + \varepsilon_{ij}\, \bar\epsilon^i \gamma_\mu \Omega^{j\,I}\,, \\
\delta_Q(\epsilon)\, \Omega_i^I + \Gamma^I_{JK}\, \delta_Q(\epsilon)\, X^J\, \Omega_i^K &= 2\partial\!\!\!/ X^I \epsilon_i - i\varepsilon_{ij}\sigma^{\mu\nu}\epsilon^j\, N^{IJ} \mathcal{G}^-_{\mu\nu\,J} \\
&\quad + \tfrac{1}{2} i\, N^{IJ} \bar F_{JKL}\, \bar\Omega^{k\,K} \Omega^{l\,L}\, \varepsilon_{ik}\varepsilon_{jl}\, \epsilon^j\,,
\end{aligned} \tag{3.61}$$

where Γ denotes the Hermitian connection and $\mathcal{G}^-_{\mu\nu I}$ is an anti-selfdual tensor defined by:

$$\mathcal{G}^-_{\mu\nu\,I} = iN_{IJ} F^{-J}_{\mu\nu} - \tfrac{1}{4} F_{IJK} \bar\Omega_i^J \sigma_{\mu\nu} \Omega_j^K\, \varepsilon^{ij}\,. \tag{3.62}$$

The significance of the tensor (3.62) and of the particular form of the spinor transformation in (3.61), will be discussed shortly. Observe that both the Lagrangian (3.58) and transformation rules (3.61) are consistent with respect to (rigid) $\mathrm{SU}(2)_R$, but not, in general, with respect to the $\mathrm{U}(1)_R$ subgroup of the automorphism group, due to the fact that the prepotential is no longer second-degree homogeneous.

As in the superconformal model we now proceed to consider the group of transformations that takes the Bianchi identities and field equations of the vector field strength into each other. Since most of the steps that we make are identical to those in section 3.2, we can suffice by referring to the relevant formulae in that section and discuss only the differences with the superconformal case.

Again, we start by considering the Bianchi identities and field equations of the vector field strengths, given by (3.19), where in the rigid case the tensors $G_{\mu\nu\,I}$ take the following form:

$$G^-_{\mu\nu\,I} = F_{IJ} F^{-I}_{\mu\nu} - \tfrac{1}{4} F_{IJK}\, \varepsilon^{ij} \bar\Omega_i^I \sigma_{\mu\nu} \Omega_j^J\,. \tag{3.63}$$

We transform the anti-selfdual tensors $F^{-I}_{\mu\nu}$ and $G^-_{\mu\nu\,I}$ into each other by means of a $2n\times 2n$ matrix $\mathcal{O}$, exactly like the transformation rule (3.22). By analogous reasoning, we find that $\mathcal{O}$ is an element of $\mathrm{Sp}(2n,\mathbb{R})$ and that the scalar fields X^I and the first derivatives F_I transform into each other as in (3.25). Finally, the fermion fields transform as described by (3.36).

However, when we determined the form of the new prepotential $\tilde F(\tilde X)$, we used the homogeneity of $F(X)$, which in the rigid case is no longer valid. However, it is still possible to integrate (3.25) and the new prepotential is given by:

$$\begin{aligned}
\tilde F(\tilde X) &= F(X) - \tfrac{1}{2} X^I F_I(X) \\
&\quad + \tfrac{1}{2}(U^{\mathrm T} W)_{IJ} X^I X^J + \tfrac{1}{2}(U^{\mathrm T} V + W^{\mathrm T} Z)_I{}^J X^I F_J + \tfrac{1}{2}(Z^{\mathrm T} V)^{IJ} F_I F_J\,,
\end{aligned} \tag{3.64}$$

up to a constant and terms linear in the $\tilde X^I$ (which give no contribution to the Lagrangian (3.58)). The terms linear in $\tilde X$ in (3.64) are associated with constant translations in $\tilde F_I$ in addition to the symplectic rotation shown in (3.25). Likewise one may introduce constant shifts in $\tilde{X}^I$. Henceforth we ignore these shifts. Constant contributions to $F(X)$ are always irrelevant. We note also that terms quadratic in the X^I with real coefficients correspond to total divergences in the action. Obviously $F(X)$ does not transform as a function. Such quantities turn out to be rare. Examples are the holomorphic function $F(X) - \frac{1}{2}X^I F_I(X)$ and the Kähler potential (3.59). For a discussion of this, we refer to [40, 41].

When we consider symplectic transformations, it is convenient to employ quantities that transform as tensors under symplectic reparameterization. Before considering some relevant symplectic tensors, we point out that although the definition of the X^I-dependent transformation matrix $\mathcal{S}^I{}_J$ remains the same, *c.f.* (3.27), the transformation of X^I can not be written in the form (3.28) any more. For future reference, we also define here:

$$\mathcal{Z}^{IJ}(X) = [\mathcal{S}^{-1}(X)]^I{}_K \, Z^{KJ} \,. \tag{3.65}$$

The holomorphic quantity $\mathcal{Z}^{IJ}$ is symmetric in I and J, because the identity $Z\,U^T = U\,Z^T$ is one of the implications of (3.26).

After these definitions we note the following transformation rules, given for completeness although they do not differ from the superconformal case:

$$\begin{aligned}
\tilde{F}_{IJ} &= (V_I{}^K F_{KL} + W_{IL})\,[\mathcal{S}^{-1}]^L{}_J \,, \\
\tilde{N}_{IJ} &= N_{KL}\,[\bar{\mathcal{S}}^{-1}]^K{}_I\,[\mathcal{S}^{-1}]^L{}_J \,, \\
\tilde{N}^{IJ} &= N^{KL}\,\bar{\mathcal{S}}^I{}_K\,\mathcal{S}^J{}_L \,, \\
\tilde{F}_{IJK} &= F_{MNP}\,[\mathcal{S}^{-1}]^M{}_I\,[\mathcal{S}^{-1}]^N{}_J\,[\mathcal{S}^{-1}]^P{}_K \,, \\
\tilde{\Omega}_i^I &= \mathcal{S}^I{}_J\,\Omega_i^J \,, \qquad \tilde{\Omega}^{i\,I} = \bar{\mathcal{S}}^I{}_J\,\Omega^{i\,J} \,.
\end{aligned} \tag{3.66}$$

The symmetry properties of the first three quantities are preserved due to the symplectic nature of the transformation. The Hermitian connection transforms as a mixed tensor but also acts as a connection for symplectic reparameterizations, as follows from:

$$\begin{aligned}
\tilde{\Gamma}^I_{JK} &= \bar{\mathcal{S}}^I{}_L\,\Gamma^L_{MN}\,[\mathcal{S}^{-1}]^M{}_J\,[\mathcal{S}^{-1}]^N{}_K \\
&= -\partial_M \mathcal{S}^I{}_N\,[\mathcal{S}^{-1}]^M{}_J\,[\mathcal{S}^{-1}]^N{}_K + \mathcal{S}^I{}_L\,\Gamma^L_{MN}\,[\mathcal{S}^{-1}]^M{}_J\,[\mathcal{S}^{-1}]^N{}_K \,.
\end{aligned} \tag{3.67}$$

Note that the construction indicated before formula (3.39) still holds. An example is the tensor $\mathcal{G}$ that we defined in (3.62), which follows from:

$$\mathcal{G}^-_{\mu\nu\,I} = G^-_{\mu\nu\,I} - \bar{F}_{IJ} F^{-\,J}_{\mu\nu} \,, \tag{3.68}$$

upon substitution of (3.63). This particular combination of field strengths transforms under symplectic reparameterizations as:

$$\tilde{\mathcal{G}}^-_{\mu\nu\,I} = \mathcal{G}^-_{\mu\nu\,J}\,[\bar{\mathcal{S}}^{-1}]^J{}_I \,. \tag{3.69}$$

With this result one can verify that the spinor transformation rule in (3.61) is manifestly covariant under symplectic reparameterizations. The same is true for the supersymmetry variation of the scalar field, but not for the variation of the vector field. This is not surprising, because the symplectic reparameterizations are not defined for the gauge fields. The reader may also verify that the Lagrangian (3.58) is invariant under symplectic reparameterizations, but only up to terms proportional to the equations of motion of the vector fields.

Chapter 4

Hypermultiplet Couplings

The smallest, on-shell representations of $N = 2$ supersymmetry are based on $4 + 4$ degrees of freedom [11]. The $N = 2$ vector multiplet, which has been discussed at length in chapters 2 and 3, is one such representation. In the current chapter, we concentrate on another, the hypermultiplet [51]. Like the vector multiplet can be defined as the $N = 2$ combination of an $N = 1$ scalar and an $N = 1$ vector multiplet, the hypermultiplet is the $N = 2$ combination of two $N = 1$ scalar multiplets and on-shell it contains 4 scalar fields and 2 Majorana fermions. As is well known [52, 53, 54], such a helicity (or spin) content can be realized only, if the multiplet is massless or has a central charge equal to the mass. Off-shell, a central charge is unavoidable. Similar reasoning applies to the vector-tensor multiplet, to be considered in chapter 6, and for that reason we discuss central charges in the context of supergravity in some detail in section 4.1, where the transformation rules of the hypermultiplet are considered, in a background of the Weyl multiplet and vector multiplets.

In section 4.2, we construct off-shell Lagrangian densities for hypermultiplets [53, 61], invariant under the superconformal symmetries and gauge symmetries [32]. The action and on-shell restriction of the resulting models turns out to be insufficient to describe general couplings of hypermultiplets. Bagger and Witten [59] have shown that in on-shell hypermultiplet actions, the manifold parameterized by the scalar fields is given by a quaternionic manifold in the case where the hypermultiplets are coupled to supergravity. In rigid supersymmetry, on-shell hypermultiplets describe a hyperkähler manifold [58]. The off-shell actions constructed in section 4.2 give rise only to a subclass of quaternionic (and hyperkähler) sigma-models in their corresponding on-shell description. Therefore in section 4.3, we abandon the off-shell description and derive the sigma-model underlying rigid on-shell hypermultiplets directly. Although closely related to the work in [59], our results are cast in a somewhat different form in order to facilitate the comparison with the models that emerge from the vector multiplets under the action of the mirror map discussed in chapter 5. Furthermore, we find that one of the geometrical restrictions given in [59] is unnecessary and in fact too restrictive. In section

4.4, the on-shell hypermultiplet model is coupled to vector multiplets, through a minimal coupling of the isometry group of the hyperkähler sigma-manifold. Finally, we discuss some aspects of the coupling of on-shell hypermultiplets to the Weyl multiplet, where the crucial ingredient is the representation of the dilatational and $\mathrm{SU}(2)_R$ symmetry through a coupling to a corresponding set of four isometries that span a quaternionic section of the tangent bundle.

4.1 Off-shell hypermultiplets

Although the primary goal of this chapter is the description of on-shell hypermultiplets, we first give a discussion of off-shell hypermultiplets, for the following two reasons: first of all they form a good introduction to the subject of (off-shell) central charges, which arise also in the discussion of the vector-tensor multiplet. Secondly, a good understanding of the structure of the hypermultiplet is needed for the considerations in the next section, which concerns the construction of off-shell hypermultiplet actions.

As was said in the introduction, hypermultiplets are based on spin-0 and spin-$\frac{1}{2}$ degrees of freedom only. The fermion fields are denoted by Majorana spinors ζ^α and ζ_α, where the label α serves as the index of a representation (and its conjugate) for the coupling to a gauge group, to be introduced shortly. The scalar fields are denoted by $A^i{}_\alpha$ (and their complex conjugates by $A_i{}^\alpha$), where the index i indicates that the scalar fields transform under the $\mathrm{SU}(2)_R$-factor of the automorphism group in the fundamental representation. Of course, they also carry the gauge-representation index α. So instead of considering first one hypermultiplet and then building representations of the gauge group based on multiple copies, we include a gauge-group representation from the start.

The scalar fields satisfy the (pseudo-)reality condition:

$$A^i{}_\alpha = \varepsilon^{ij}\rho_{\alpha\beta}\, A_j{}^\beta. \tag{4.1}$$

Consistency requires that the matrix $\rho_{\alpha\beta}$ satisfies:

$$\rho_{\alpha\beta}\rho^{\beta\gamma} = -\delta_\alpha{}^\gamma, \tag{4.2}$$

where $\rho^{\beta\gamma}$ is by definition the complex conjugate of $\rho_{\beta\gamma}$. Taking the determinant on both sides of (4.2), we find $|\det(\rho)|^2 = \det(-\mathbb{I})$, which implies that the index α must run from 1 to some even number $2r$. Hence, the number of scalar fields is given by $4r$ and the number of fermion fields by $2r$, where r denotes the number of hypermultiplets we consider. Performing field redefinitions, it is possible (see appendix B in reference [32]), to bring ρ in block-diagonal form, where every block is of the skew-symmetric form:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4.3}$$

Note that (4.2) is the defining condition for a complex structure. So besides the complex structure that was defined implicitly when we regarded the hypermultiplet scalars as complex fields, there is an independent complex structure of the form (4.3). One easily shows that these two complex structures anticommute, so that their product represents another (independent) complex structure. Three independent, anticommuting complex structures together with the identity, span the quaternionic division algebra, which we denote by $\mathbb{H}$. The pseudo-reality condition (4.1) implies that the scalars $A^i{}_\alpha$ can be written as an r-dimensional quaternion-valued vector.

Denoting the representation matrices of the Lie algebra of the gauge group by $(t_I)^\alpha{}_\beta$ and the conjugate representation by $(t_I)_\alpha{}^\beta$, the condition (4.1) implies the following condition for t_I:

$$(t_I)_\alpha{}^\beta\, \rho_{\beta\gamma} = \rho_{\alpha\beta}\, (t_I)^\beta{}_\gamma. \tag{4.4}$$

This means that the gauge group is compatible with all three complex structures and, as such, must be contained in the group $\mathrm{GL}(r,\mathbb{H})$, the group of invertible linear quaternionic transformations.

Before we turn to the transformation rules of the massless hypermultiplet, let us consider the counting of degrees of freedom. On-shell counting is balanced, leading to $4r$ bosonic and $4r$ fermionic degrees of freedom. Off-shell counting, in which the number of fermionic degrees of freedom is doubled, reveals the need for $4r$ auxiliary bosonic degrees of freedom.

To determine the nature of these auxiliary fields, let us briefly consider the representation theory of on-shell extended supersymmetry. A detailed discussion of the spins that occur in on-shell representations of N-extended supersymmetry [11, 12, 52, 53, 54], leads to the following proposition. *Any on-shell representation of N-extended supersymmetry contains fields with spins greater then or equal to one, unless the representation is massless or there is a central charge that equals the mass, i.e. there is a saturated BPS-bound.* Based on that, we conclude that an on-shell *massive* hypermultiplet has a central charge equal to the mass. In the case of an on-shell *massless* hypermultiplet the central charge vanishes. Returning to the off-shell hypermultiplet, we see that the central charge re-occurs, because an off-shell representation lacks a mass condition.

Whereas in case of a massive on-shell hypermultiplet the action of the central charge on a field is given simply by multiplication with the mass, such an assumption may not be made in the off-shell case. Hence, we leave the action of the central charge transformation (as yet) undetermined and denote its action by the superscript (z), *i.e.*:

$$\begin{aligned} \delta_{\mathrm{z}}(z)\, A^i{}_\alpha &= z\, A^{(\mathrm{z})\,i}{}_\alpha\,, \\ \delta_{\mathrm{z}}(z)\, \zeta^\alpha &= z\, \zeta^{(\mathrm{z})\,\alpha}. \end{aligned} \tag{4.5}$$

To denote further central-charge transformations, we simply increase the number of z's in the superscript. Since the central charge commutes with supersymmetry, the supersymmetry representation of the central-charge transformed fields is isomorphic with the representation of the original fields. Successive applications of the central charge transformations therefore generate a hierarchy of hypermultiplets, with scalar components:

$$A^i{}_\alpha \xrightarrow{\mathrm{z}} A^{(\mathrm{z})\,i}{}_\alpha \xrightarrow{\mathrm{z}} A^{(\mathrm{zz})\,i}{}_\alpha \xrightarrow{\mathrm{z}} \cdots . \tag{4.6}$$

Hence, the inclusion of central-charge transformations in the symmetry algebra renders the off-shell representation infinite dimensional, unless there are dependencies between the fields in the hierarchy. This implies the possibility to complete the off-shell counting [56] by assigning the central-charge transformed scalars $A^{(\mathrm{z})\,i}{}_\alpha$ to be the auxiliary degrees of freedom. Then off-shell counting is balanced if other central-charge transformed fields, $\zeta^{(\mathrm{z})\,\alpha}$, $A^{(\mathrm{zz})\,i}{}_\alpha$ and fields at higher z-levels, are eliminated as independent degrees of freedom by conditions that relate them to fields at lower z-level. In short, the off-shell hypermultiplet contains scalars $A^i{}_\alpha$, fermions ζ^α and auxiliary fields $A^{(\mathrm{z})\,i}{}_\alpha$ that are related to the scalars by the central charge. The auxiliary fields $A^{(\mathrm{z})\,i}{}_\alpha$ themselves can again serve as scalar components of a hypermultiplet and the infinite hierarchy that is generated in this way is terminated by dependencies of all higher central-charge transformed fields on lower-lying components.

Note that the above truncation is a *choice* that we make to have a representation based on a finite number of off-shell degrees of freedom. Other descriptions of the off-shell hypermultiplet involve an infinite number of fields. Examples thereof are the description in harmonic superspace (see [55], and references therein) and central-charge superspace [53].

With a clear picture of the role of the central charge in hand, we now turn to the hypermultiplet transformation rules, which we formulate in a background of the Weyl multiplet and the vector multiplets associated with the gauge group. At the end of section 2.2 it was argued that in conformal supergravity, the vector-field that gauges the central charge is part of an abelian vector multiplet. Therefore the gauge group consists of two parts: an abelian factor associated with central-charge transformations and another factor that is represented on the hypermultiplets through the index α (a more rigorous argument is given in reference [32], which builds on results presented in [56, 38, 57]). Henceforth, when we talk of 'the gauge group' we refer to the latter and consider the central charge as separate. The central charge is represented in the superconformal transformation rules as a coupling to the vector multiplet $(X^0, \Omega^0_i, W^0_\mu, Y^0_{ij})$. The vector multiplets associated with the gauge group are denoted by $(X^I, \Omega^I_i, W^I_\mu, Y^I_{ij})$, with the index I running from 1 to n. The transformation rules under Q- and S-supersymmetry are given by:

$$\begin{aligned} \delta A_i{}^\alpha &= 2\,\bar\epsilon_i \zeta^\alpha + 2\,\rho^{\alpha\beta}\varepsilon_{ij}\,\bar\epsilon^j\zeta_\beta\,, \\ \delta\zeta^\alpha &= \not{D} A_i{}^\alpha\,\epsilon^i + 2g\,X^I(t_I)^\alpha{}_\beta A_i{}^\beta\,\varepsilon^{ij}\epsilon_j + 2\,X^0 A^{(\mathrm{z})}_i{}^\alpha\,\varepsilon^{ij}\epsilon_j + A_i{}^\alpha\eta^i. \end{aligned} \tag{4.7}$$

The dilatational and $\mathrm{U}(1)_R$ weights are given in table C.II. Needless to say, the transformation rules for higher z-level components are given by adding superscript (z)'s. The covariant derivative in the fermionic transformation rule is covariant with respect to the superconformal symmetries as well as the central charge and the gauge symmetries:

$$\begin{aligned} D_\mu A_i{}^\alpha &= \partial_\mu A_i{}^\alpha + \tfrac{1}{2}\mathcal{V}_\mu{}_i{}^j A_j{}^\alpha - b_\mu A_i{}^\alpha - g W_\mu^I (t_I)^\alpha{}_\beta A_i{}^\beta \\ &\quad - W_\mu^0 A^{(\mathrm{z})}{}_i{}^\alpha - \bar\psi_{\mu i}\zeta^\alpha - \rho^{\alpha\beta}\varepsilon_{ij}\,\bar\psi_\mu{}^j\zeta_\beta. \end{aligned} \tag{4.8}$$

The transformation rules (4.7) satisfy the algebra:

$$[\,\delta_Q(\epsilon_1),\delta_Q(\epsilon_2)\,] = \delta^{\mathrm{cov}}(\xi) + \delta_M(\varepsilon) + \delta_K(\Lambda_K) + \delta_S(\eta) + \delta_{\mathrm{z}}(z) + \delta_{\mathrm{gauge}}(\theta^I)\,, \tag{4.9}$$

where the parameters ξ, ε, Λ_K and η are given by (2.31) and z and θ^I by (2.45). However, closure of the algebra on the fermion field imposes the following two constraints:

$$\begin{aligned} -2\bar X^0\zeta^{(\mathrm{z})\,\alpha} &= \rho^{\alpha\beta}\not{D}\zeta_\beta + g\Omega^{i\,\alpha}{}_\beta A_i{}^\beta + 2g\,\bar X^\alpha{}_\beta\zeta^\beta + \Omega^{i\,0}A^{(\mathrm{z})}{}_i{}^\alpha \\ &\quad + \tfrac{1}{8}\sigma_{\mu\nu}T^{\mu\nu}_{ij}\varepsilon^{ij}\,\zeta^\alpha - \tfrac{3}{2}\varepsilon^{ij}A_j{}^\alpha\,\chi_i\,, \\ -4|X^0|^2A^{(\mathrm{zz})}{}_i{}^\alpha &= (\Box_C + \tfrac{3}{2}D)A_i{}^\alpha + 2g^2\,\{\,\bar X, X\,\}^\alpha{}_\beta A_i{}^\beta + g\,\varepsilon_{ik}Y^{jk\,\alpha}{}_\beta A_j{}^\beta \\ &\quad + 2g(\rho^{\alpha\beta}\bar\Omega^\gamma_{i\,\beta}\zeta_\gamma - \varepsilon_{ij}\bar\Omega^{j\,\alpha}_\beta\zeta^\beta) + 2g(\bar X^0X^\alpha{}_\beta + X^0\bar X^\alpha{}_\beta)A^{(\mathrm{z})}{}_i{}^\beta \\ &\quad + \varepsilon_{ik}Y^{jk\,0}A^{(\mathrm{z})}{}_j{}^\alpha + 2(\rho^{\alpha\beta}\bar\Omega^0_i\zeta^{(\mathrm{z})}_\beta - \varepsilon_{ij}\bar\Omega^{j\,0}\zeta^{(\mathrm{z})\,\alpha}), \end{aligned} \tag{4.10}$$

where we have used an obvious notation to denote contraction of indices between vector-multiplet fields and the corresponding generators. So, as expected, we find constraints, *c.f.* (4.10), that express the higher z-level components $\zeta^{(\mathrm{z})\,\alpha}$ and $A^{(\mathrm{zz})\,i}{}_\alpha$ in terms of the lower z-level components, if $X^0 \neq 0$.

4.2 Off-shell hypermultiplet actions

Given the form of the off-shell hypermultiplet discussed in the previous section, we can now proceed to construct Lagrangian densities, using the methods given in chapter 2. First we construct a superconformally invariant action and next we consider the gauge equivalence to a model of Poincaré supergravity. However, we shall find that the resulting models do *not* represent the most general couplings of hypermultiplets. In the on-shell reduction of the off-shell actions we are about to construct, only specific subclasses of quaternionic sigma-models are found.

A superconformal action for the off-shell hypermultiplets discussed in the previous section can be constructed by means of the density formula (2.56). The components of the linear multiplet involved in (2.56) are defined in terms of the hypermultiplet component fields as follows:

$$\begin{aligned} L_{ij} &= \Big(A_i{}^\alpha A^{(\mathrm{z})}{}_j{}^\beta + A_j{}^\alpha A^{(\mathrm{z})}{}_i{}^\beta\Big)\eta_{\alpha\beta}\,, \\ \varphi_i &= 2\Big(A_i{}^\alpha\zeta^{(\mathrm{z})\,\beta} + A^{(\mathrm{z})}{}_i{}^\beta\zeta^\alpha\Big)\eta_{\alpha\beta}\,, \\ \bar G &= 4\Big(\bar\zeta^\alpha\zeta^{(\mathrm{z})\,\beta} + 2g\,A_i{}^\alpha X^\beta{}_\gamma A^{(\mathrm{z})}{}_j{}^\gamma\varepsilon^{ij} - 2g\,X^\alpha{}_\gamma A_i{}^\gamma A^{(\mathrm{z})}{}_j{}^\beta\varepsilon^{ij}\Big)\eta_{\alpha\beta}\,, \\ E_\mu &= A_i{}^\alpha\overleftrightarrow{D}_\mu\,A^{(\mathrm{z})}{}_j{}^\beta\,\varepsilon^{ij}\eta_{\alpha\beta} - 2\Big(\rho^{\alpha\gamma}\,\bar\zeta_\gamma\gamma_\mu\zeta^{(\mathrm{z})\,\beta}\,\eta_{\alpha\beta} + \mathrm{h.c.}\Big). \end{aligned} \tag{4.11}$$

Note that in order to use the density formula (2.56), the linear multiplet must be invariant under all gauge transformations, except maybe under an abelian factor. Thence we find the restriction:

$$\eta_{\alpha\gamma}(t_I)^{\gamma}{}_{\beta} + (t_I)^{\gamma}{}_{\alpha}\eta_{\gamma\beta} = 0, \tag{4.12}$$

for every $I = 1, \ldots, n$. Since the above linear multiplet has a non-trivial action of the central charge, the abelian vector-multiplet components that are involved in the density formula (2.56) are given by $(X^0, \Omega^0_i, W^0_\mu, Y^0_{ij})$. Substitution of (4.11) into (2.56) results in an expression in which the dependent fields $\zeta^{(\mathrm{z})\,\alpha}$ and $A^{(\mathrm{zz})}{}_i{}^{\alpha}$ are still present. These fields are expressed in the independent fields using (4.10). If, furthermore, we isolate the terms containing the auxiliary fields $A^{(\mathrm{z})}{}_i{}^{\alpha}$, and we write out the fermionic covariantizations, we arrive at the following expression for the superconformal hypermultiplet action [32]:

$$\begin{aligned}
e^{-1}\mathscr{L} \;=\; & \Big(-\mathscr{D}_\mu A^i{}_\beta \mathscr{D}^\mu A_i{}^\alpha + \tfrac{1}{2}(D + \tfrac{1}{3}R)A^i{}_\beta A_i{}^\alpha \\
& + (4\bar X^0 X^0 + W^0_\mu W^{\mu\,0})A^{(\mathrm{z})\,i}{}_\beta A^{(\mathrm{z})}{}_i{}^\alpha \\
& + 4g^2\, A^i{}_\beta \bar X^\alpha_\gamma X^\gamma{}_\delta A_i{}^\delta + g A^i{}_\beta Y^{jk\,\alpha}{}_\gamma A_k{}^\gamma \varepsilon_{ij}\Big) d_\alpha{}^\beta \\
& \Big[\Big(-2\bar\zeta^\alpha \not{\mathscr{D}}\zeta_\beta - \tfrac{1}{6}A^j{}_\beta A_j{}^\alpha e^{-1}\varepsilon^{\mu\nu\rho\sigma}\bar\psi_{\mu\,i}\gamma_\nu \mathscr{D}_\rho\psi_\sigma{}^i \\
& + \tfrac{1}{4}A^j{}_\beta A_j{}^\alpha\, \bar\psi_{\mu\,i}\gamma^\mu\chi^i - \tfrac{1}{24}A_k{}^\alpha A^k{}_\beta\, \bar\psi^i_\mu\psi_\nu{}^j\, T^{\mu\nu}_{ij} \\
& - 2A^i{}_\beta\bar\zeta^\alpha\chi_i + \bar\zeta^\alpha\rho_{\beta\gamma}\sigma_{\mu\nu}T^{\mu\nu}_{ij}\varepsilon^{ij}\zeta^\gamma - 4g\, A^i{}_\beta\bar\Omega^{j\,\alpha}_\gamma\zeta^\gamma\varepsilon_{ij} \\
& + 4g\,\bar\zeta^\alpha\rho_{\beta\gamma}\bar X^\gamma_\delta\zeta^\delta + 2\bar\zeta^\alpha\gamma^\mu\not{\mathscr{D}}A^i{}_\beta\psi_{\mu\,i} \\
& - \tfrac{8}{3}\bar\zeta^\alpha\sigma^{\mu\nu}\mathscr{D}_\mu\psi_{\nu\,i}A^i{}_\beta + \tfrac{1}{6}A^i{}_\beta\bar\zeta^\alpha\sigma_{\mu\nu}T^{\mu\nu}_{ij}\gamma^\rho\psi_\rho{}^j \\
& - 4g\,\bar\psi^i_\mu\gamma^\mu A^j{}_\beta\varepsilon_{ij}\bar X^\alpha_\gamma\zeta^\gamma - g\,\bar\psi^i_\mu\gamma^\mu A^j{}_\beta\varepsilon_{ij}\Omega^{k\,\alpha}{}_\gamma A_k{}^\gamma \\
& - 2g\,\bar\psi^i_\mu\sigma^{\mu\nu}\psi_\nu{}^j A_i{}^\alpha\bar X_\beta{}^\gamma A^k{}_\gamma\varepsilon_{kj} - \tfrac{1}{2}e^{-1}\varepsilon^{\mu\nu\rho\sigma}\,\bar\psi^i_\mu\gamma_\nu\psi_{\rho\,j}A_i{}^\alpha\mathscr{D}_\sigma A^j{}_\beta \\
& - \bar\zeta^\alpha\gamma^\mu\gamma^\nu\psi_{\mu\,i}(\bar\zeta_\beta\psi_\nu{}^i + \rho_{\beta\gamma}\varepsilon^{ij}\bar\zeta^\gamma\psi_{\nu\,j})\Big)d_\alpha{}^\beta + \text{h.c.}\Big],
\end{aligned} \tag{4.13}$$

where we have used the matrix d, defined by:

$$d^\alpha{}_\beta = -\eta^{[\alpha\gamma]}\rho_{\gamma\beta}\,. \tag{4.14}$$

One easily derives that d is gauge invariant, Hermitian and that it satisfies the quaternionic condition $d_\alpha{}^\beta = \rho_{\alpha\gamma}\rho^{\beta\delta}\, d^\gamma{}_\delta$. Note that in the definition of d, only the antisymmetric part of η plays a role. As it turns out [32], the symmetric part leads only to a total derivative in the action and can be discarded.

Note that the central-charge hierarchy discussed in the previous section collapses when we impose the field equations: the auxiliary field $A^{(\mathrm{z})\,i}{}_\alpha$ evaluates to zero and the relations that express higher-z components in terms of the independent fields, *c.f.* (4.10), have a vanishing *r.h.s.* as a result of the field equations of the fermion and scalar fields respectively. Note that in the (non-conformal) case of an on-shell massive hypermultiplet, the mass-term in the field equation would have led to a central charge equal to the mass.

Analogous to the reasoning that followed formula (3.41), we now consider the sigma-models that are parameterized by the scalar kinetic term in (4.13). However, the above hypermultiplet action holds an unpleasant surprise: the metric d is independent of the scalar fields $A^i{}_\alpha$ and consequently the variety of spaces that can be coordinatized by the scalar fields is relatively small. By comparison, the off-shell hypermultiplet action is not characterized by an analog of the prepotential for vector-multiplet models. One may wonder whether the construction of the action was general enough: in particular, the scalar component of the linear multiplet (4.11) contains a *constant* tensor η, which is directly related to the metric d. If η is made scalar-dependent, then possibly d would indeed be the metric of a non-linear sigma-model. However, in the construction of the ensuing action, such a scalar dependence leads to higher-derivative terms in the action, which are non-renormalizable and hence pose a greater problem than the one we started with [32].

The limitations of the superconformal description are best clarified in the comparison of rigidly supersymmetric couplings. From the Lagrangian density (4.13), we can derive an on-shell rigidly supersymmetric model when we fix the Weyl-background to the flat-space limit and integrate out the auxiliary degrees of freedom. The sigma-manifold of the resulting model is still a flat hyperkähler space, whereas it has been shown [58][1], that rigidly supersymmetric, on-shell hypermultiplet models can be formulated for *any* hyperkähler sigma-manifold. Note that a similar construction in the case of vector multiplet models, as discussed in section 3.4, *did* give rise to rigid special geometry.

One aspect of the sigma-manifold geometry has been ignored until this point, namely the fact that the gauge equivalence with a model of Poincaré supergravity implies that the compensating degrees of freedom and corresponding gauge symmetries can be eliminated by suitable gauge choices. This leads to a projective condition on the scalar fields, like we have seen in the case of special geometry in section 3.3. Note that the hypermultiplet scalars transform under dilatations and under chiral $\mathrm{SU}(2)_R$. The $A^i{}_\alpha$-dependent coefficient of the scalar-curvature term in the Lagrangian density (4.13) can be rendered constant by a condition of the form $A_i{}^\alpha d_\alpha{}^\beta A^i{}_\beta = \text{constant}$. In a model that describes both vector and hypermultiplets, the above constant is fixed by the field equation of the auxiliary D field in the Weyl multiplet. Analogous to the fact that integrating out the $\mathrm{U}(1)_R$ gauge field A_μ gave rise to additional terms in the metric for the vector-multiplet scalar fields, in the case at hand, integrating out the $\mathrm{SU}(2)_R$ gauge field $\mathcal{V}_\mu{}^i{}_j$ gives rise to additional terms in the hypermultiplet scalar metric. Again, we can define $\mathrm{SU}(2)_R$-invariant, inhomogeneous coordinates and consider the sigma-manifold coordinatized by those. One finds [32] that the resulting sigma-manifold, describing the coupling of hypermultiplets in Poincaré supergravity, is a so-called quaternionic manifold (for a definition the reader is referred to the next section).

[1]Although the results of Alvarez-Gaumé and Freedman are formulated for two-dimensional $N = 4$ supersymmetric sigma-models, their conclusions can be generalized to four-dimensional $N = 2$ models immediately, because the two-dimensional model can be obtained from the four-dimensional model through dimensional reduction.

With the various choices of the constant tensor d, one can obtain the following quaternionic spaces [32]: depending on the signs of the kinetic terms for the quaternionic scalars, we are dealing with either the (compact) projective quaternionic space, or a non-compact version thereof, *i.e.*:

$$\mathbb{H}\mathrm{P}(r-1) = \frac{\mathrm{Sp}(r)}{\mathrm{Sp}(r-1)\times\mathrm{Sp}(1)}, \quad \text{or} \quad \frac{\mathrm{Sp}(r-1,1)}{\mathrm{Sp}(r-1)\times\mathrm{Sp}(1)}. \tag{4.15}$$

If there are interactions with gauge fields, there is the possibility that some of the vector multiplets lack kinetic terms, due to singularity of the matrix N_{IJ}, *c.f.* (3.13). In that case not only the fields Y^I_{ij}, but also the scalars X^I and the fermions Ω^I_i are auxiliary. Hence, the coupling between vector- and hypermultiplets lets the auxiliary vector multiplets act as Lagrange multipliers, imposing constraints on the hypermultiplet components. Correspondingly, the quaternionic space parameterized by the hypermultiplets is a subspace of the compact or non-compact projective spaces found above. This construction was used in [62, 61] to derive hypermultiplet models on the symmetric spaces:

$$X(n) = \frac{\mathrm{SU}(n+2)}{\mathrm{SU}(n)\times\mathrm{SU}(2)\times\mathrm{U}(1)}, \qquad Y(n) = \frac{\mathrm{SO}(n+4)}{\mathrm{SO}(n)\times\mathrm{SO}(4)}, \tag{4.16}$$

where $n > 1$ and their non-compact versions. If we include the $n = 1$ case, the two four-dimensional self-dual Einstein spaces S^4 and $\mathbb{C}\mathrm{P}^2$ are obtained. In [62], also the rigid limit of these hypermultiplet models and their relation to hyperkähler manifolds was studied. However, the following quaternionic spaces can not be described by a hypermultiplet model of the above form:

$$\frac{G_2}{\mathrm{SO}(4)}, \qquad \frac{F_4}{\mathrm{Sp}(3)\times\mathrm{Sp}(1)}, \qquad \frac{E_6}{\mathrm{SU}(6)\times\mathrm{Sp}(1)},$$
$$\frac{E_7}{\mathrm{SO}(12)\times\mathrm{Sp}(1)}, \qquad \frac{E_8}{E_7\times\mathrm{Sp}(1)}. \tag{4.17}$$

The quaternionic manifolds (4.16) and (4.17) are discussed by Wolf [63]. More classes of quaternionic manifolds are known, for instance the homogeneous spaces discussed in [64] and those in [50], for which no hypermultiplet descriptions have been found within the superconformal framework. It has been shown [59] that on-shell hypermultiplets in a supergravity background, can describe any quaternionic scalar manifold of constant negative scalar curvature.

The conclusion must be that within the superconformal framework, the possible hypermultiplet models do not describe the most general couplings. A marked success of the hypermultiplet description in harmonic superspace is the fact that within that framework, the hypermultiplet action is given as an integral over (analytic) harmonic superspace, where the integrand is a *prepotential* of the hypermultiplet harmonic superfields [55]. Such a construction can be regarded as the analog of the chiral superspace integration over the prepotential for general vector-multiplet actions, and leads to a much larger class of off-shell hypermultiplet couplings, if not to the general coupling.

4.3 Rigid hypermultiplets and hyperkähler geometry

In chapter 5, the classical mirror map is considered, which gives a relation between abelian vector multiplet models and hypermultiplet models. Given the limitations of the off-shell hypermultiplet models discussed in the previous section, a more general hypermultiplet description is needed to formulate this correspondence. In first instance, the mirror map gives a relation between *rigid, on-shell* vector- and hypermultiplet models, so we limit our discussion in this section to rigid, on-shell hypermultiplet couplings. Our analysis, which is self-contained, is closely related to the one presented in [59]. However, our results are cast in a somewhat different form in order to facilitate the comparison with the models that emerge from the vector multiplets under the action of the mirror map. Furthermore, we find that one of the geometrical restrictions given in [59] is unnecessary and in fact too restrictive.

The definitions of the various quantities involved in the formulation of the on-shell hypermultiplet model are slightly different from those used in the off-shell framework. We assume $4n$ real scalars ϕ^A and $2n$ positive-chirality spinors $\zeta^{\bar\alpha}$ and $2n$ negative-chirality spinors ζ^α, which are related by conjugation (so that we have $2n$ Majorana spinors). Therefore, under complex conjugation indices are converted according to $\alpha \leftrightarrow \bar\alpha$, while, just as before, $\mathrm{SU}(2)_R$ indices $i, j, \ldots$ are raised and lowered.

Note that the number of scalar fields is the same as in the off-shell case, but the nature of the index A is as yet undefined. Eventually, we expect to find a supersymmetric sigma-model, where the scalar fields play the role of local coordinates. As is well known [58, 59, 60], the resulting sigma-manifold is a so-called hyperkähler manifold. Before we go into the derivation, we briefly review some aspects of hyperkähler geometry. We combine this with the definition of quaternionic manifolds, which were used in the previous section, and discuss the differences between hyperkähler and quaternionic geometry. We use the definition of hyperkähler and quaternionic manifolds that is given in [65, 66, 67]. In the following, we concentrate on presenting the mathematical context for our discussion, leaving the proofs of most statements [66, 67] and a more comprehensive treatment of the subject [65] to the mathematical literature.

Consider first a manifold $\mathscr{M}$ and a coordinatization with patches U_i on $\mathscr{M}$. If *locally* (for every U_i), $\mathscr{M}$ admits two anticommuting, almost complex structures J^1 and J^2, this implies the existence of a third, independent, anticommuting, almost complex structure $J^3 = J^1 J^2$. By definition, this set of locally defined, almost complex structures satisfies the Clifford-like property ($\Lambda, \Sigma = 1, 2, 3$):

$$J^\Lambda\, J^\Sigma + J^\Sigma\, J^\Lambda = -\delta^{\Lambda\Sigma}\, \mathbb{I}. \tag{4.18}$$

Note that any linear combination of the three local almost complex structures of the form $J = a_\Lambda J^\Lambda$ with real coefficients a^Λ satisfying $a_\Lambda\, \delta^{\Lambda\Sigma}\, a_\Sigma = 1$, again defines a local almost complex structure on $\mathscr{M}$. Given J, two orthogonal linear combinations can be found, such that the Clifford-property (4.18) again holds. We assume that, when going from one coordinatization to another in an overlap region $U_i \cap U_j$, the local almost complex structures are subject to such an SO(3) transformation. This can be rephrased by saying that $\mathscr{M}$ allows an SO(3) subbundle $\mathcal{G}$ of End(T$\mathscr{M}$), with local sections J^1, J^2 and J^3 satisfying (4.18).

Given a Riemannian manifold $\mathscr{M}$ with metric g, we call g *quaternion-Hermitian*, if it is endowed with a bundle $\mathcal{G}$ and in every patch U_i and for every $\Lambda = 1, 2, 3$, $g(J^\Lambda x, J^\Lambda y) = g(x, y)$ for all vector fields x and y, *i.e.* a three-fold Hermiticity condition of the form (1.22). Given a *globally* defined almost complex structure J on a quaternion-Hermitian manifold $\mathscr{M}$, a two-form ω_J can be associated with J through: $\omega^\Lambda(x, y) = g(x, J^\Lambda y)$. Since J^1, J^2 and J^3 are not defined globally, this definition does not necessarily lead to well-defined two-forms on a quaternion-Hermitian manifold $\mathscr{M}$. However, the SO(3)-invariant expression:

$$\omega = \omega^1 \wedge \omega^1 + \omega^2 \wedge \omega^2 + \omega^3 \wedge \omega^3, \tag{4.19}$$

gives rise to a well-defined four-form, called the *fundamental four-form.*

A quaternion-Hermitian manifold of real dimension $4n$, $(n > 1)$ is called a *quaternionic* manifold[2] if the fundamental four-form ω is covariantly constant with respect to the Levi-Civita connection[3], *i.e.*:

$$D\omega = 0. \tag{4.20}$$

This implies that ω is closed.

In the definition of a hyperkähler manifold, we assume that the local almost complex structures can be extended to globally defined almost complex structures, *i.e.* the sections J^1, J^2 and J^3 of $\mathcal{G}$ are global sections. Accordingly, the SO(3) bundle is trivial and the associated two-forms ω^Λ are well-defined. A quaternion-Hermitian manifold $\mathscr{M}$ is called hyperkähler, if the three two-forms ω^Λ are closed separately:

$$\mathrm{d}\omega^1 = \mathrm{d}\omega^2 = \mathrm{d}\omega^3 = 0. \tag{4.21}$$

It was shown by Hitchin [69] that with the above requirement, the almost complex structures J^1, J^2 and J^3 are integrable. Note that, as the name suggests, a hyperkähler manifold satisfies the definition of a Kähler manifold, as given in section 1.3. Requiring that $\mathscr{M}$ is hyperkähler is equivalent with the requirement that $\mathscr{M}$ allows the definition of three anticommuting almost complex structures that are covariantly constant with respect to the Levi-Civita connection.

The difference between quaternionic and hyperkähler manifolds is clearly demonstrated at the level of their holonomy groups. The above definition of a $4n$-dimensional, quaternionic manifold $\mathscr{M}$ is equivalent to the requirement that the holonomy group of $\mathscr{M}$ is contained in $\mathrm{Sp}(n) \times \mathrm{Sp}(1)$ (modulo a $\mathbb{Z}_2$-factor) [65]. In this factorization the Sp(1)-factor (which is isomorphic to SU(2)) acts on $\mathcal{G}$ through the SO(3) representation, transforming the local

[2] The nomenclature here is not unambiguous: Swann [65] refers to the manifolds that we call quaternionic as *quaternionic Kähler*, because he reserves the name quaternionic for a more general class of manifolds that is not necessarily Riemannian [68]. Since we are interested in sigma-models, which inherently describe Riemannian manifolds, we do not make this distinction.

[3] A definition of four-dimensional quaternionic manifolds can be found in [65, 59].

almost complex structures among each other. Note that a quaternionic manifold is Riemannian, but not necessarily complex (*e.g.* S^4 is quaternionic, but not complex [70]), whence we find $\mathrm{Sp}(n) \times \mathrm{Sp}(1)$ as a subgroup of $\mathrm{O}(4n)$. However, $4n$-dimensional hyperkähler manifolds are Kähler manifolds and consequently their holonomy group must be a subgroup of $\mathrm{U}(2n)$. The additional requirement that there are *three* complex structures implies that the holonomy group is a subgroup of $\mathrm{Sp}(n)$. In fact, an alternative (but equivalent) definition of a hyperkähler manifold is given by the requirement that the holonomy group is contained in $\mathrm{Sp}(n)$ [59]. Comparing the holonomy groups of quaternionic and hyperkähler manifolds, we see that the holonomy group of a hyperkähler manifold lacks the Sp(1)-factor that rotates the local almost complex structures, as is to be expected based on the fact that on a hyperkähler manifold, the complex structures are covariantly constant and globally defined.

Since the holonomy group is generated by the Riemann curvature tensor, the above considerations lead to restrictions on the local geometry. More specifically, the factorization of the holonomy group of quaternionic manifolds implies that the Riemann tensor can be decomposed into two terms that generate the $\mathrm{Sp}(n)$- and Sp(1)-factors respectively. Note that the Sp(1) term is missing in the case of a hyperkähler manifold. This corresponds to the well-known fact that hyperkähler manifolds are Ricci-flat, *i.e.* the Ricci tensor is equal to zero. As far as the curvature constraints of quaternionic manifolds are concerned, we suffice to say that all quaternionic manifolds are Einstein manifolds. A relatively simple condition on the Riemann tensor [66, 67] can be formulated to ensure that the scalar curvature is even constant.

A precise formulation of the decomposition of the Riemann tensor requires the definition of an $\mathrm{Sp}(n) \times \mathrm{Sp}(1)$ frame on the tangent bundle. Following [59], we locally define a coordinate-dependent vielbein $V_{A\,i}^{\alpha}(\phi)$, which maps the tangent bundle (with corresponding index A) to an $\mathrm{Sp}(n) \times \mathrm{Sp}(1)$ vector bundle, with indices α and i respectively, such that the action of the holonomy group factorizes. A suitable contraction of the Riemann tensor with vielbeins then gives rise to the decomposition in two terms proportional to the $\mathrm{Sp}(n)$ and Sp(1) curvatures respectively (*c.f.* formula (3) in [59] and its hyperkähler-specific form, (4.39) below).

Returning to the hypermultiplet models, we identify the Sp(1)-factor with the $\mathrm{SU}(2)_R$-factor of the $N = 2$ automorphism group. So the supersymmetry parameters ϵ^i take their values in the Sp(1) bundle. Furthermore, the hypermultiplet fermions ζ^α take their values in the $\mathrm{Sp}(n)$ bundle. The supersymmetry variation $\delta_Q \phi^A$ of the scalar fields, which naturally takes its values in the tangent bundle, is found by the (real part of) the contraction of $\bar{\epsilon}^i \zeta^\alpha$ with the inverse vielbein, which we denote by γ. Note that in the case of rigid supersymmetry, the supersymmetry parameter is constant throughout the sigma-manifold, *i.e.* there are global sections of the Sp(1) bundle, which is equivalent to the triviality of the Sp(1) bundle. In the case of supergravity, this requirement is not made. This clearly demonstrates why rigidly supersymmetric hypermultiplet models must be formulated on a hyperkähler manifold, whereas supergravity coupled hypermultiplets describe a quaternionic manifold.

In the derivation of on-shell hypermultiplet couplings presented below, we do not *assume* the geometrical context we have just discussed. Instead we derive the geometrical properties of the sigma-manifold coordinatized by the scalars of the hypermultiplets. The discussion relies only on supersymmetry of the action and closure of the on-shell rigid supersymmetry algebra.

Using scalar-dependent quantities γ^A and V_A, which we shall identify as the vielbeins mentioned above at a later stage, the supersymmetry transformations are written in the following form:

$$\begin{aligned} \delta_Q(\epsilon)\,\phi^A &= 2\Big(\gamma^A_{i\bar\alpha}\,\bar\epsilon^i\zeta^{\bar\alpha} + \bar\gamma^{Ai}_{\alpha}\,\bar\epsilon_i\zeta^\alpha\Big), \\ \delta_Q(\epsilon)\,\zeta^{\bar\alpha} &= \bar V_A^{i\bar\alpha}\,\partial\!\!\!/\phi^A\epsilon_i - \delta_Q(\epsilon)\phi^A\,\bar\Gamma_A{}^{\bar\alpha}{}_{\bar\beta}\,\zeta^{\bar\beta}, \\ \delta_Q(\epsilon)\,\zeta^\alpha &= V^\alpha_{A\,i}\,\partial\!\!\!/\phi^A\epsilon^i - \delta_Q(\epsilon)\phi^A\,\Gamma_A{}^\alpha{}_\beta\,\zeta^\beta. \end{aligned} \tag{4.22}$$

A connection for the fermionic fields and its complex conjugate are denoted by Γ and $\bar\Gamma$, which we leave unspecified for the moment. As it turns out, with the proper definition, the above ansatz comprises the full supersymmetry transformation laws. Observe that the variations are consistent with a $\mathrm{U}(1)_R$ invariance under which the scalars remain invariant. However, for generic γ^A and V_A, the $\mathrm{SU}(2)_R$ part of the automorphism group cannot be realized consistently. In the above, we only used that ζ^α and $\zeta^{\bar\alpha}$ are related by complex conjugation.

A first condition on the quantities γ^A and V_A follows from the closure of the supersymmetry transformations (4.22) on the scalar fields. This yields the Clifford-like condition:

$$\gamma^A_{i\bar\alpha}\,\bar V_B^{j\bar\alpha} + \bar\gamma^{A\,j}_{\alpha}\,V^\alpha_{B\,i} = \delta^j{}_i\,\delta^A{}_B\,. \tag{4.23}$$

Subsequently let us turn to the action, which we parameterize as:

$$4\pi\,\mathscr{L} = -\tfrac12 g_{AB}\,\partial_\mu\phi^A\partial^\mu\phi^B - G_{\bar\alpha\beta}\Big(\bar\zeta^{\bar\alpha}D\!\!\!\!/\,\zeta^\beta + \bar\zeta^\beta D\!\!\!\!/\,\zeta^{\bar\alpha}\Big) + \mathscr{L}(\zeta^4)\,, \tag{4.24}$$

where $G_{\bar\alpha\beta}$ is a Hermitian metric. A possible anti-hermitian part can be absorbed into the Noether term, modulo a total derivative. In principle, it is possible to absorb G into the definition of the fermion fields, but we refrain from doing so for reasons that will become clear in due course. Furthermore, we use the covariant derivatives:

$$D_\mu\zeta^\alpha = \partial_\mu\zeta^\alpha + \partial_\mu\phi^A\,\Gamma_A{}^\alpha{}_\beta\,\zeta^\beta, \quad D_\mu\zeta^{\bar\alpha} = \partial_\mu\zeta^{\bar\alpha} + \partial_\mu\phi^A\,\bar\Gamma_A{}^{\bar\alpha}{}_{\bar\beta}\,\zeta^{\bar\beta}\,. \tag{4.25}$$

The Noether term thus takes the following form:

$$4\pi\,\mathscr{L}_{\rm N} = \Big(\bar\Gamma_A{}^{\bar\gamma}{}_{\bar\alpha}\,G_{\bar\gamma\beta} - G_{\bar\alpha\gamma}\,\Gamma_A{}^\gamma{}_\beta\Big)\,\bar\zeta^{\bar\alpha}\partial\!\!\!/\phi^A\zeta^\beta. \tag{4.26}$$

Observe that only a linear combination of the two connections appears in the action.

Considering various terms of the supersymmetry variation of the action (4.24) leads to further conditions. Cancelation of the variations proportional to $\partial^2\phi^A$ implies:

$$g_{AB}\,\gamma^B_{i\bar\alpha} = G_{\bar\alpha\beta}\,V^\beta_{A\,i}, \qquad g_{AB}\,\bar\gamma^{B\,i}_{\alpha} = G_{\bar\beta\alpha}\,\bar V_A^{i\bar\beta}\,. \tag{4.27}$$

Then variations proportional to $\partial_\mu \phi^B \, \partial_\nu \phi^C$ require :

$$\begin{aligned} 2G_{\bar\beta\alpha}\, D_B V^\alpha_{A\,i} + D_B G_{\bar\beta\alpha}\, V^\alpha_{A\,i} &= 0\,, \\ 2G_{\bar\beta\alpha}\, D_B \bar V^{i\bar\beta}_A + D_B G_{\bar\beta\alpha}\, \bar V^{i\bar\beta}_A &= 0\,. \end{aligned} \tag{4.28}$$

Note that the first covariant derivative in (4.28) contains also the Levi-Civita connection $\{A;BC\}$. Now redefine the connections according to:

$$\begin{aligned} G_{\bar\beta\gamma}\Gamma_A{}^\gamma{}_\alpha + \tfrac{1}{2} D_A G_{\bar\beta\alpha} &\rightarrow G_{\bar\beta\gamma}\hat\Gamma_A{}^\gamma{}_\alpha\,, \\ G_{\bar\gamma\alpha}\bar\Gamma_A{}^{\bar\gamma}{}_{\bar\beta} + \tfrac{1}{2} D_A G_{\bar\beta\alpha} &\rightarrow G_{\bar\gamma\alpha}\hat{\bar\Gamma}_A{}^{\bar\gamma}{}_{\bar\beta}\,. \end{aligned} \tag{4.29}$$

Taking the difference, one sees that this modification does not modify the Noether term. Furthermore, one can verify that the γ^A tensors are covariantly constant with respect to the connection $\hat\Gamma$, and so is the metric $G_{\bar\alpha\beta}$. Thus we specify the connection that we have used in (4.22) further, replacing the connection Γ everywhere by the new connection $\hat\Gamma$ and dropping the hat. These are then the connections that appear in the variations of the spinor fields in (4.22) and, as it turns out, no additional terms quadratic in the spinor fields are required in these transformation rules.

According to the above results we define four real, antisymmetric covariantly constant tensors:

$$(J^\Lambda)^A{}_B = i\gamma^A_{i\bar\alpha}\,\bar V^{j\bar\alpha}_B\,(\sigma^\Lambda)^i{}_j\,, \qquad (\Lambda = 1,2,3) \tag{4.30}$$

and:

$$C^A{}_B = i(\gamma^A_{i\bar\alpha}\,\bar V^{i\bar\alpha}_B - \delta^A{}_B). \tag{4.31}$$

It follows that C must vanish, so that γ and $\bar V$ are each others inverse:

$$\bar V^{i\bar\alpha}_A\,\gamma^A_{j\bar\beta} = \delta^i{}_j\,\delta^{\bar\alpha}{}_{\bar\beta}\,. \tag{4.32}$$

The precise analysis leading to this result is somewhat subtle, and is based on an extension of the arguments used in [58]. It makes use of the fact that the five covariantly constant two-tensors, the metric, the J^Λ and C, and products thereof, must commute with the curvature tensor and therefore with the holonomy group. The latter can act reducibly, so that the target space factorizes and the model decomposes into the sum of several independent models. If the holonomy group acts irreducibly, then according to Schur's lemma, the algebra generated by the above tensors must be a division algebra. This implies a degeneracy between the tensors (4.30) and (4.31). Combining this fact with the Clifford property leads to (4.32).

From (4.32) it then follows directly that the J^Λ are covariantly constant complex structures, satisfying:

$$J^\Lambda\, J^\Sigma = -\mathbb{I}\,\delta^{\Lambda\Sigma} - \varepsilon^{\Lambda\Sigma\Pi} J^\Pi, \tag{4.33}$$

reflecting the fact that the target space must be hyperkähler.

Furthermore we note the existence of covariantly constant antisymmetric tensors:

$$\Omega_{\bar\alpha\bar\beta} = \tfrac{1}{2}\varepsilon^{ij}\, g_{AB}\,\gamma^A_{i\bar\alpha}\,\gamma^B_{j\bar\beta}\,, \quad \bar\Omega^{\bar\alpha\bar\beta} = \tfrac{1}{2}\varepsilon_{ij}\, g^{AB}\,\bar V^{i\bar\alpha}_A\,\bar V^{j\bar\beta}_B\,, \tag{4.34}$$

satisfying:

$$\varepsilon_{ij}\,\Omega_{\bar\alpha\bar\beta}\,\bar V_A^{j\bar\beta} = g_{AB}\,\gamma^B_{i\bar\alpha}\,. \tag{4.35}$$

According to (4.27) and (4.35) the γ and V tensors are linearly related and pseudo-real. Therefore the tensor Ω is also pseudo-real and it satisfies:

$$\Omega_{\bar\alpha\bar\gamma}\,\bar\Omega^{\bar\gamma\bar\beta} = -\delta^{\bar\beta}{}_{\bar\alpha}\,. \tag{4.36}$$

The existence of covariantly constant tensors implies a variety of integrability conditions for the curvature tensors. From the constancy of $G_{\bar\alpha\beta}$ and $\Omega_{\bar\alpha\bar\beta}$ we obtain:

$$R_{AB}{}^{\bar\beta}{}_{\bar\alpha} = -G_{\bar\alpha\gamma}\,G^{\delta\bar\beta}\,R_{AB}{}^{\gamma}{}_{\delta}\,, \qquad R_{AB}{}^{\bar\gamma}{}_{[\bar\alpha}\,\Omega_{\bar\beta]\bar\gamma} = 0\,. \tag{4.37}$$

These conditions imply that $R_{AB}{}^{\alpha}{}_{\beta}$ takes values in sp(n) so that the holonomy group acts symplectically on the fermions.

Furthermore, constancy of the γ tensor implies:

$$R_{ABD}{}^{C}\,\gamma^D_{i\bar\alpha} - R_{AB}{}^{\bar\gamma}{}_{\bar\alpha}\,\gamma^C_{i\bar\gamma} = 0\,. \tag{4.38}$$

From this result one proves that Riemann curvature and the Sp(n) curvature are related:

$$\begin{aligned} R_{AB}{}^{\bar\beta}{}_{\bar\alpha} &= \tfrac{1}{2} R_{ABE}{}^{C}\,\gamma^E_{i\bar\alpha}\,\bar V_C^{i\bar\beta}\,,\\ R_{ABD}{}^{C} &= R_{AB}{}^{\bar\beta}{}_{\bar\alpha}\,\gamma^C_{i\bar\beta}\,\bar V_D^{i\bar\alpha}. \end{aligned} \tag{4.39}$$

Using the pair-exchange property of the Riemann tensor and contracting with $\gamma^C\,\bar\gamma^D$ one derives:

$$R_{AB}{}^{\bar\beta}{}_{\bar\alpha} = \tfrac{1}{2} W_{\bar\alpha\epsilon\bar\gamma\delta}\,\bar V_A^{i\bar\gamma}\,V^\delta_{Bi}\,G^{\epsilon\bar\beta}\,, \tag{4.40}$$

where:

$$W_{\bar\alpha\beta\bar\gamma\delta} = R_{AB}{}^{\bar\epsilon}{}_{\bar\gamma}\,\gamma^A_{i\bar\alpha}\,\bar\gamma^{iB}_{\beta}\,G_{\bar\epsilon\delta} = \tfrac{1}{2} R_{ABCD}\,\gamma^A_{i\bar\alpha}\,\bar\gamma^{iB}_{\beta}\,\gamma^C_{j\bar\gamma}\,\bar\gamma^{jD}_{\delta}\,. \tag{4.41}$$

The tensor W can be written as $W_{\alpha\beta\gamma\delta}$ by contracting with the metric G and the antisymmetric tensor Ω. It then follows that $W_{\alpha\beta\gamma\delta}$ is symmetric in symmetric index pairs $(\alpha\beta)$ and $(\gamma\delta)$. Using the Bianchi identity for Riemann curvature, which implies $g_{D[A}R_{BC]}{}^{\bar\beta}{}_{\bar\alpha}\,\gamma^D_{i\bar\beta} = 0$, one shows that it is in fact symmetric in all four indices.

Hence all the curvatures are expressed in terms of the fully symmetric tensor $W_{\alpha\beta\gamma\delta}$. From this result many other identities for the curvatures can be derived. In particular we note the identity:

$$R_{AB}{}^{\bar\gamma}{}_{[\bar\alpha}\,\gamma^B_{\bar\beta]i} = 0\,, \tag{4.42}$$

which plays a crucial role in proving the supersymmetry of the action. For that, one needs to include a four-fermion interaction into the Lagrangian, equal to:

$$4\pi\,\mathscr{L}(\zeta^4) = -\tfrac{1}{4} W_{\bar\alpha\beta\bar\gamma\delta}\,\bar\zeta^{\bar\alpha}\gamma_\mu\zeta^\beta\,\bar\zeta^{\bar\gamma}\gamma^\mu\zeta^\delta, \tag{4.43}$$

corresponding to the four-fermion term in (1.23).

The above results are closely related to the ones derived long ago in [59]. One feature that is different is the presence of a fermionic metric, which, as we will demonstrate in the next chapter, is important in exhibiting the effect of symplectic reparameterizations for models in the image of the c-map. Another feature concerns the condition imposed in [59] that $\gamma^B_{i\bar\alpha}\,\bar\gamma^{C\,i}_{\beta} + \gamma^C_{i\bar\alpha}\,\bar\gamma^{B\,i}_{\beta}$ be proportional to the product of g^{BC} and $G_{\bar\alpha\beta}$ and inversely proportional to the number of hypermultiplets n. We found no need for this condition. In fact, an explicit counter example can be constructed using the classical mirror map that is discussed in chapter 5: in the hypermultiplet model that arises as the mirror image of a minimally coupled vector multiplet model, no $1/n$ terms can arise. Unfortunately, this erroneous identity has found its way into a large part of the $N = 2$ literature.

4.4 Gauged isometries

Having analysed the couplings among on-shell hypermultiplets and the resulting geometry in rigid supersymmetry, we are immediately confronted with two obvious questions: how can such a model be coupled to supergravity and what are the possible couplings to a vector-multiplet background? In this section we concentrate on the latter question and in the next section we discuss some aspects of the answer to the former question. Both these issues will be considered in a forthcoming paper [71].

First of all, we note that the sigma-manifold that underlies the scalar sector of a combined vector- and hypermultiplet model factorizes into a special Kähler and a hyperkähler manifold:

$$\mathscr{M} = \mathscr{M}_V \times \mathscr{M}_H. \tag{4.44}$$

If the scalar manifold would not factorize in the above way, then the scalars of the vector- and hypermultiplets would be coordinates on the sigma-manifold, interchangeable through diffeomorphisms of $\mathscr{M}$. Due to the very different nature of the vector- and hypermultiplets, such a coupling would not be supersymmetric. So a coupling of vector- and hypermultiplets in the scalar sector is not possible.

However, a coupling between the vector gauge fields and the hypermultiplet scalars *can* be made. To appreciate this, we concentrate on the hyperkähler manifold $\mathscr{M}_H$. We assume that the isometry group of $\mathscr{M}_H$ is non-trivial, *i.e.* $G = \mathrm{Isom}(\mathscr{M}_H)$ is an d-dimensional Lie group with generators G_I, $I = 1, \ldots, d$. Accordingly, d globally defined Killing vector fields k_I exist on $\mathscr{M}_H$, such that the metric is invariant under an infinitesimal coordinate transformation generated by the vectors k_I:

$$\delta(\theta)\,\phi^A = g\,\theta^I (k_I)^A. \tag{4.45}$$

The commutation relations characterizing the Lie algebra of G are realized on the Killing vectors through their Lie derivatives:

$$\mathcal{L}_{k_I} k_J = [\,k_I, k_J\,] = f_{IJ}^K\,k_K. \tag{4.46}$$

where f_{IJ}^{K} are the structure constants of the Lie algebra of G.

Before we address minimal coupling of the isometry group, we have to consider the implications of the fact that we are dealing with a hyperkähler manifold for the Killing vector fields. Analogous to the holomorphicity of isometries of the Kähler manifolds that play a role in $N = 1$ supersymmetric sigma-models considered in section 1.3, we assume here that the Killing vector fields are *tri-holomorphic*. The covariant constancy of the complex structures J^Λ allows a separate holomorphic/antiholomorphic decomposition of the tangent bundle with respect to every complex structure. Tri-holomorphicity of the isometry means that none of the complex structures is altered by an infinitesimal shift along a Killing vector. Hence, as far as the SO(3) bundle $\mathcal{G}$ defined in the previous section is concerned, tri-holomorphic isometries act as the identity. Using its triviality, the same conclusion can be drawn for the Sp(1)-factor in the tangent bundle. Consequently, tri-holomorphic isometries are represented in the Sp(n)-factor of the tangent bundle only.

Tri-holomorphicity can be expressed by three requirements similar to (1.34), but in this case it is more convenient to rephrase it in a coordinate-independent fashion. Since both the metric and the complex structures are left invariant by a tri-holomorphic isometry, the Lie derivatives of the hyperkähler two-forms with respect to the Killing vectors vanish:

$$\mathcal{L}_{k_I}\omega^\Lambda = (\mathrm{d}i_{k_I} + i_{k_I}\mathrm{d})\omega^\Lambda = 0, \tag{4.47}$$

where i_k denotes contraction of the differential form with the vector field k. Using (4.21), we find that $\mathrm{d}(\omega^\Lambda \cdot k_I) = 0$, *i.e.* the contractions $\omega_{AB}^\Lambda (k_I)^B$ are closed one-forms for every Λ and every I, implying that *locally* they are exact. So on every (simply connected) coordinate patch U_i on $\mathscr{M}_H$, d triplets of functions P_I^Λ can be found, such that:

$$\omega_{AB}^\Lambda (k_I)^B = \partial_A P_I^\Lambda. \tag{4.48}$$

Note that these so-called *Killing potentials* [17, 60] (or *moment maps* [44, 45]) are defined only up to additive constants p_K^Λ. Using (4.48) and the fact that the hyperkähler two-forms are closed, one can prove from (4.46) that:

$$(k_{[I})^A \partial_A P_{J]}^\Lambda = (k_{[I})^A (k_{J]})^B \omega_{AB}^\Lambda = f_{IJ}^K P_K^\Lambda + c_{IJ}^\Lambda, \tag{4.49}$$

where c_{IJ}^Λ are constants. Depending on the particulars of the isometry group, some or all of the constants p_K^Λ can be fixed by the requirement that they cancel the c_{IJ}^Λ [44]. In that case the Killing potentials transform in the adjoint representation of the isometry group.

As was noted in section 1.3, the isometry group is part of the *global* symmetry group of the action of a non-linear sigma-model. The isometries can be realized as *local* symmetries through minimal coupling to vector fields associated with the isometry group G. In the case of a hypermultiplet model with gauged isometries, the vector fields reside in vector multiplets described by an action of the form (3.6) with non-abelian terms, *c.f.* (3.10). Note that we choose to keep the vector multiplets off-shell. In the following, we first extend the on-shell transformation rules for the hypermultiplets, *c.f.* (4.22), to the case of gauged isometries. Subsequently, we determine the additional terms in the action, required by the fermionic field equation as it is read off when we impose on-shell closure of the supersymmetry algebra. Finally, there are additional terms in the action that do not contribute to the fermionic field equations, but are required by supersymmetry of the action. Note that the coupling constant g can be absorbed in the normalization of the isometry vector field k_I and is included here only as an indicator for terms that arise as a result of the vector-multiplet coupling.

The on-shell transformation rule for the bosons ϕ^A remains in the form (4.22) but the fermion transformation rule receives g-proportional contributions:

$$\delta_Q(\epsilon)\,\zeta^{\bar\alpha} = \bar V_A^{i\bar\alpha}\,\not{D}\phi^A\epsilon_i - \delta_Q\phi^A\,\bar\Gamma_A{}^{\bar\alpha}{}_{\bar\beta}\,\zeta^{\bar\beta} + 2g\,\bar X^I(\bar V_A^{i\bar\alpha}(k_I)^A)\,\varepsilon_{ij}\epsilon^j, \tag{4.50}$$

where the covariant derivative $D_\mu\phi^A$ is prescribed by minimal gauge coupling:

$$D_\mu\phi^A = \partial_\mu\phi^A - gA_\mu^I(k_I)^A. \tag{4.51}$$

The last term in (4.50) is the analogon of the g-proportional term in the off-shell fermionic transformation rule (4.7) and leads to the gauge term in the supersymmetry commutator for the bosons ϕ^A:

$$[\,\delta_Q(\epsilon_1),\delta_Q(\epsilon_2)\,]\,\phi^A = 4(\bar\epsilon^i_{[2}\gamma^\mu\epsilon_{1]\,i})D_\mu\phi^A + g\,\theta^I(k_I)^A, \tag{4.52}$$

where $\theta^I(\epsilon_1,\epsilon_2)$ is given by (2.45). Based on (4.50), we can calculate the supersymmetry commutator for the fermion field $\zeta^{\bar\alpha}$. Besides the covariant translation, we find a $\theta^I(\epsilon_1,\epsilon_2)$-proportional gauge transformation, with an action on the fermions of the following form:

$$\delta(\theta)\,\zeta^{\bar\alpha} = g\,\theta^I(t_I)^{\bar\alpha}{}_{\bar\beta}\zeta^{\bar\beta}, \quad \text{where} \quad (t_I)^{\bar\alpha}{}_{\bar\beta} = \gamma^A_{i\bar\beta}\bar V_B^{i\bar\alpha}D_A(k_I)^B - (k_I)^A\,\bar\Gamma_A{}^{\bar\alpha}{}_{\bar\beta}\,. \tag{4.53}$$

Using the tri-holomorphicity of the isometry, one can show that the vielbeins $\bar V_A^{i\bar\alpha}$ transform under the gauge-transformations in an identical way, *i.e.* only the Sp(n) index is affected:

$$\delta(\theta)\bar V_A^{i\bar\alpha} = g\,\theta^I(t_I)^{\bar\alpha}{}_{\bar\beta}\bar V_A^{i\bar\beta}. \tag{4.54}$$

which proves the earlier statement that only the Sp(n) subbundle is rotated when we move along a tri-holomorphic Killing vector field.

Finally, the supersymmetry commutator on the fermion field contains two terms proportional to the fermionic field equation. From the latter we derive several modifications to the action (4.24): firstly, the covariant derivative for the fermion takes the gauge-covariant form:

$$D_\mu\zeta^{\bar\alpha} = \partial_\mu\zeta^{\bar\alpha} + \bar\Gamma_A{}^{\bar\alpha}{}_{\bar\beta}D_\mu\phi^A\zeta^{\bar\beta} - gA_\mu^I(t_I)^{\bar\alpha}{}_{\bar\beta}\zeta^{\bar\beta}, \tag{4.55}$$

which, of course, was already present in the covariant translation on the *r.h.s.* of the supersymmetry commutator on the fermion field. Secondly, there is a new X^I-proportional term in the action, given by:

$$\mathscr{L}_1 = 2g\,\bar X^I\bar\gamma_\beta^{Ai}V_{Bi}^\alpha D_A(k_I)^B\Omega_{\gamma\alpha}(\bar\zeta^\beta\zeta^\gamma) + \text{h.c.}\,, \tag{4.56}$$

and finally, we find that the field equation requires a coupling of $\Omega^{i\,I}$ and ζ^α of the form:

$$\mathscr{L}_2 = 2g\, V^\alpha_{Ai}(k_I)^A \Omega_{\beta\alpha}(\bar{\zeta}^\beta \Omega^{i\,I}) + \text{h.c.}\,. \tag{4.57}$$

However, these additions to the hypermultiplet action do not constitute an invariant of supersymmetry. Specifically, a scalar potential of the form:

$$V = -2g^2\, X^I \bar{X}^J\, g_{AB}\, (k_I)^A (k_J)^B - \frac{1}{2} ig\, P^\Lambda_I Y^I_{ik}\, (\sigma^\Lambda)^i{}_j \varepsilon^{jk}, \tag{4.58}$$

is required to make the action supersymmetric. The resulting action takes the form:

$$\begin{aligned} 4\pi\mathscr{L} \;=\;& 4\pi\mathscr{L}_V \\ & -\tfrac{1}{2} g_{AB}\, D_\mu\phi^A D^\mu\phi^B - G_{\bar{\alpha}\beta}\Big(\bar{\zeta}^{\bar{\alpha}} \not{D}\zeta^\beta + \bar{\zeta}^\beta \not{D}\zeta^{\bar{\alpha}}\Big) - \tfrac{1}{4} W_{\bar{\alpha}\beta\bar{\gamma}\delta}\, \bar{\zeta}^{\bar{\alpha}}\gamma_\mu\zeta^\beta\, \bar{\zeta}^{\bar{\gamma}}\gamma^\mu\zeta^\delta \\ & + \Big(2g\, \bar{X}^I \bar{\gamma}^{Ai}_\beta V^\alpha_{Bi} D_A (k_I)^B \Omega_{\gamma\alpha}(\bar{\zeta}^\beta\zeta^\gamma) + 2g\, V^\alpha_{Ai}(k_I)^A \Omega_{\beta\alpha}(\bar{\zeta}^\beta\Omega^{i\,I}) + \text{h.c.}\Big) \\ & -2g^2\, X^I\bar{X}^J\, g_{AB}\,(k_I)^A(k_J)^B - \frac{1}{2} ig P^\Lambda_I Y^I_{ik}\,(\sigma^\Lambda)^i{}_j\varepsilon^{jk}, \end{aligned} \tag{4.59}$$

where $\mathscr{L}_V$ is the action for off-shell vector multiplets, *c.f.* (3.6) and (3.10).

Note that although the on-shell model is more general, there are numerous similarities between this model and the off-shell model discussed in section 4.2. All of the terms required by gauging of the isometries have analogs in the off-shell hypermultiplet model, *i.e.* when we impose the flat space-time limit for the Weyl multiplet in the action (4.13) every term in (4.59) above has a corresponding term in the resulting rigidly supersymmetric off-shell action. The most important shortcoming of the off-shell formulation seems to be the fact that the scalar fields (must) carry the $\mathrm{Sp}(1)\times\mathrm{Sp}(n)$ indices. Let us therefore examine under which conditions such a situation can arise in the on-shell model. Note that the vielbeins offer the possibility to express any element of *the tangent space* in the $\mathrm{Sp}(1)\times\mathrm{Sp}(n)$ frame. Hence, if we could identify the coordinate space with the tangent space and give a global definition for the $\mathrm{Sp}(1)\times\mathrm{Sp}(n)$ frame, as is the case for *flat* hyperkähler manifolds, the coordinates ϕ^A could be transformed to coordinates with $\mathrm{Sp}(1)\times\mathrm{Sp}(n)$ indices. In retrospect, this gives some insight into the flat nature of the hyperkähler spaces that arise when we go to the rigid limit of (4.13).

Also note that if we solve the field equations for the auxiliary fields Y^I_{ij}, we find that besides the four-fermion couplings that were already present in the (abelian) rigidly supersymmetric, on-shell vector-multiplet action (3.58), the auxiliary fields lead to a scalar potential proportional to $P^\Lambda_I N^{IJ} P^\Lambda_J$. In fact, this is the scalar potential alluded to after formula (3.12).

4.5 Supergravity coupling, an outlook

In the current section, we address the remaining question, that of the coupling between on-shell hypermultiplets to a background described by the Weyl multiplet. Although the work that we report on is still in progress [71], it seems appropriate to present at least some of the preliminary results here, if only for completeness of the chapter.

Besides the lingering question of the supergravity coupling, the reader may also wonder about the role of isometries that are not tri-holomorphic, because such isometries do represent global symmetries of the on-shell rigid action (4.24). It may come as no surprise that these questions are intimately related.

Based on the strategy that has been advocated in the preceding chapters, the coupling to conformal supergravity is to be derived in the following way: we should take the rigid transformation rules, assign appropriate dilatational and chiral $\mathrm{U}(1)_R$ weights, make an educated guess at the covariantizations in the derivatives and verify the superconformal algebra. Furthermore, the geometrical analysis presented in section 4.3 would have to be repeated, whence we would expect to find that the appropriate set of sigma-manifolds for hypermultiplet couplings in a supergravity background is quaternionic, possibly of constant negative curvature.

However, this procedure immediately leads to some awkward surprises. First of all, checking the supersymmetry commutator for the scalar field, we find that the results are exactly the same as before in the rigid case: we find that the vielbeins satisfy the Clifford-like condition (4.23) and are covariantly constant *without the need for an* $\mathrm{Sp}(1)$ *covariantization*. This is, of course, due to the fact that we have not made any assumptions about the $\mathrm{SU}(2)_R$ behaviour of the scalar fields. When one proceeds to compute the supersymmetry commutator for the fermion fields ζ^α, one quickly finds that the problems become only bigger: not only is it impossible to relate the $\mathrm{SU}(2)_R$ connection $\mathcal{V}_\mu{}^i{}_j$ to the $\mathrm{Sp}(1)$ bundle over the sigma-manifold, but problems arise with the coupling of the dilatational connection b_μ as well, due to the assignment of a definite scaling weight to the scalar fields.

Central in the above reasoning is the fact that the representation of the extra local bosonic symmetries $\mathrm{SU}(2)_R$ and D on the scalar fields is not compatible with the fact that they are coordinates on a manifold. From the previous section, we know how to realize local symmetries in a sigma-model, namely by the gauging of isometries. However, this does not change the fact that the vielbeins are covariantly constant without the need for an $\mathrm{Sp}(1)$ connection, *i.e.* the sigma-manifold has a flat $\mathrm{Sp}(1)$ subbundle. Summerizing, we find that the coupling of conformal supergravity must be made on a *hyperkähler* manifold $\mathscr{M}$, with isometry group:

$$\mathrm{Isom}(\mathscr{M}) = \mathrm{SU}(2)_\mathrm{R} \times \mathbb{R} \times G, \tag{4.60}$$

where the first factor is minimally coupled to the $\mathrm{SU}(2)_R$ gauge field $\mathcal{V}_\mu{}^i{}_j$ and the second factor to the dilatational gauge field b_μ. The third factor contains alle other isometries of the hyperkähler manifold, which can be gauged by coupling to vector multiplets through the method described in the previous section. The Killing vector fields corresponding to the $\mathrm{SU}(2)_R$-factor are denoted k_Λ, $\Lambda = 1, 2, 3$, and the dilatational isometry is denoted k_D. Note that these four Killing vector fields form a global section of a quaternionic subbundle of the tangent bundle of $\mathscr{M}$. Further isometries are denoted by k_I, $I = 1, \ldots, \dim(G)$, like in the previous section.

From the arguments given just prior to the introduction of the transformation rules (4.22), we know that the $\mathrm{SU}(2)_R$-factor of the automorphism group is to be identified with the Sp(1)-factor in the holonomy group of the sigma-manifold. Any change in the Sp(1) frame induces a change of the complex structures, so the isometries that are coupled to the gauge field $\mathcal{V}_\mu{}^i{}_j$ are *not* tri-holomorphic, and in fact act on the Sp(1) index of the vielbeins through a **2** or $\bar{\mathbf{2}}$ representation, depending on whether the vielbein has an upper or lower Sp(1) index.

The variations under Q- and S-supersymmetry are now given by:

$$\begin{aligned} \delta\phi^A &= 2\Big(\gamma^A_{i\bar\alpha}\,\bar\epsilon^i\zeta^{\bar\alpha} + \bar\gamma^{Ai}_{\alpha}\,\bar\epsilon_i\zeta^{\alpha}\Big), \\ \delta\zeta^{\bar\alpha} &= \bar V^{i\bar\alpha}_A\,\not{D}\phi^A\epsilon_i - \delta\phi^A\,\bar\Gamma_A{}^{\bar\alpha}{}_{\bar\beta}\,\zeta^{\bar\beta} + 2g\,\bar X^I(\bar V^{i\bar\alpha}_A(k_I)^A)\,\varepsilon_{ij}\epsilon^j \\ &\quad + \tfrac14\bar V^{i\bar\alpha}_A(k_D)^A\eta_i + \tfrac12\bar V^{i\bar\alpha}_A(k^A)^j{}_i\eta_j, \end{aligned} \tag{4.61}$$

where the covariant derivative on the scalar is given by:

$$\begin{aligned} D_\mu\phi^A &= \partial_\mu\phi^A + \tfrac12\mathcal{V}_\mu{}^i{}_j(k^A)^j{}_i - b_\mu(k_D)^A - gA^I_\mu(k_I)^A \\ &\quad -\gamma^A_{i\bar\alpha}\,\bar\psi_\mu{}^i\zeta^{\bar\alpha} - \bar\gamma^{Ai}_{\alpha}\,\bar\psi_{\mu\,i}\zeta^{\alpha}. \end{aligned} \tag{4.62}$$

In the above we have made use of an obvious notation $(k^A)^i{}_j$ to denote the contraction of the SU(2) Killing vector fields with the appropriate $\mathrm{SU}(2)_R$ generators. The variation of the fermion under S-supersymmetry is found by imposing (2.12) on the scalar fields. Furthermore, the commutator (2.9) imposes a relation between the dilatational and SU(2) isometries. Finally, we remark that, analogous to the reasoning that led to formulae (4.56), (4.57) and (4.58), the fermionic field equation that is found when one tries to close the QQ-commutator on ζ^α, can be integrated to fermionic additions to the rigid action (4.59). However, we refrain from giving explicit results, because the calculations involved have not been finished at the moment of writing. Accordingly, the transformation rules (4.61) must be considered preliminary.

The above construction on a hyperkähler manifold seems to contradict the result that (on-shell) supergravity-coupled hypermultiplets coordinatize *quaternionic* manifolds. However, we emphasize that this result holds in Poincaré supergravity and the two can be reconciled through gauge equivalence. Namely, if we choose an $\mathrm{SU}(2)_R\times\mathbb{R}$ gauge in the superconformal model, which can be done locally on $\mathscr{M}$, *i.e.* with a different gauge choice for every coordinate patch, we break $\mathrm{SU}(2)_R$ and dilatational invariance and at the same eliminate time four degrees of freedom from the description. The fact that the gauge choice can be made locally, ensures that the resulting manifold is not (necessarily) without Sp(1) curvature, *i.e.* can be quaternionic. Note that the gauge condition must be formulated for the vielbeins and involves the Killing vector fields.

Chapter 5

The Mirror Map

5.1 Mirror symmetry

When discussing the relevance of superstring models to the physics at relatively low energy scales, one must first understand the way in which the inherently ten-dimensional target-space of the superstring can be reconciled with the obviously four-dimensional nature of its effective low-energy model. It has to be noted that the picture sketched of superstring compactifications in the introductory section 1.1 was highly simplified and not entirely accurate. In the current section we give a more detailed discussion of superstring compactifications to motivate the further developments in this chapter.

In a superstring compactification, the scalar fields in the two-dimensional (super)conformal worldsheet field theory is decomposed in two sectors: one sector of the two-dimensional model is given by a (flat) sigma-model with the $(3 + 1)$-dimensional Minkowski space as its sigma-manifold and coordinates X^μ, $(\mu = 0, 1, 2, 3)$. The remaining, so-called *internal* degrees of freedom are then chosen such that the complete two-dimensional quantum field theory is consistent. The scalar fields described by the latter are denoted by X^i, $i = 1, \ldots, 6$. The bosonic part of the two-dimensional action for the internal sector takes the form:

$$S_\Sigma = \frac{1}{\pi\alpha'} \int_\Sigma d^2z \Big(G_{ij} \partial X^i \bar{\partial} X^j + B_{ij} \partial X^i \bar{\partial} X^j \Big), \tag{5.1}$$

where α' is the string-tension and Σ is the worldsheet. The two-tensors G_{ij} and B_{ij} are symmetric and antisymmetric, respectively.

The physical degrees of freedom in the four-dimensional effective description, are associated with the massless vertex operators in the superstring compactification. Massive vertex-operators typically have a mass of the order of the Planck scale and do not play a role in the effective four-dimensional model [2]. Based on the particular way in which the combination of the fields in both sectors is made, the massless vertex operators fall into a representation of the four-dimensional Lorentz group, thus corresponding to scalars, spinors, vector fields etcetera in the four-dimensional model. The effective action of the four-dimensional model is determined by the requirement that its n-point functions reproduce the corresponding correlators involving n massless vertex operators as they are calculated in string theory. Often, this correspondence is made at tree-level. Alternatively, one can consider the requirement that the background in which the string propagates gives rise to exact two-dimensional conformal invariance. Vanishing of the β-functions then leads to field equations for various fields in the four-dimensional model, which can be integrated to find a (classical) action. More on these issues, can be found, for instance, in [2].

Of more direct relevance to our discussion are the spaces that parameterize the different superconformal systems used as internal sectors in the compactification of superstrings, the so-called *moduli spaces*. Given a set of requirements for the effective four-dimensional model, the choice for the internal sector is usually not completely fixed: for instance, the phenomenologically interesting $D = 4$, $N = 1$ compactifications of the heterotic string, can be formulated with *any* $c = 9$, $(2, 2)$-superconformal internal sector [72, 73]. When using a $c = 9$, $(2, 2)$-superconformal internal sector to compactify type-II string theories, the resulting four-dimensional model displays $N = 2$ space-time supersymmetry [75]. Similarly, if we choose as the internal sector a $c = 6$, $(4, 4)$-superconformal system [74], the resulting four-dimensional low-energy model is $N = 2$ supersymmetric in the case of a heterotic string theory and $N = 4$ supersymmetric for a type-II model.

The question then arises how the moduli space is represented in the four-dimensional effective model. To answer that question, let us consider the action (5.1) in some more detail: due to the superconformal symmetry of the action, the two-tensors G and B play a role in the conformal field theory as operators of definite conformal dimensions and U(1) weights. As a result, infinitesimal deformations of G and B are given by so-called exactly marginal operators, which by definition have the appropriate conformal dimensions and U(1) weights. The space of all exactly marginal operators locally parameterizes the space of all possible internal superconformal systems. In the case of $c = 9$, $(2, 2)$-superconformal systems, the exactly marginal operators are subdivided in two classes, depending on the relative sign of the U(1) weights [74]:

$$(h = \bar{h} = \tfrac{1}{2},\, q = \bar{q} = 1), \qquad \text{and} \qquad (h = \bar{h} = \tfrac{1}{2},\, q = -\bar{q} = 1). \tag{5.2}$$

It should be noted at this point, that the assignment of the relative sign can be changed per convention, *i.e.* if the exactly marginal operators in the first class are given the relative sign in U(1) weights, the superstring model is not changed physically. Zamolodchikov [76] has shown that the space of exactly marginal operators in any (super)conformal system is naturally endowed with a Riemannian structure. Furthermore [2], the exactly marginal operators are vertex operators that correspond to massless scalars in the effective four-dimensional model. From this, one concludes that given a superstring model compactified on a certain internal system, the moduli space of internal superconformal systems that can be obtained from the original one by deformation via marginal operators, arises in the effective four-dimensional theory as a supersymmetric sigma-model. This observation plays a central role in the rest of this section, because it allows us to combine the considerations following from string theory with the methods that were presented in previous chapters.

In this chapter, we concentrate on type-II compactifications on $c = 9$, $(2, 2)$-superconformal systems. Often [45], such a compactification has a geometrical interpretation of the type that was discussed in section 1.1: the ten-dimensional space in which the string is embedded factorizes into the four-dimensional Minkowski space and a six-dimensional, compact manifold $\mathscr{M}$. Consistency of the worldsheet (super)conformal field theory and $N = 2$ supersymmetry in four dimensions[1], require that this manifold is a Calabi-Yau manifold, *i.e.* a compact, Ricci-flat, Kähler manifold with holonomy group SU(3) (for a definition, see section 1.3). In this context, the moduli space is placed in a different light: it is simply the moduli space of the Calabi-Yau space under consideration. As is well-known (see, for instance, [77, 78, 73, 75]), the moduli of Calabi-Yau spaces can be classified in two distinct sets, the $(2, 1)$-moduli which correspond to changes of the complex structure and the $(1, 1)$-moduli that describe the changes of the Kähler class. Note that any deformation of the Hermitian metric on $\mathscr{M}$ is effected through either a change of $\delta G_{\mu\bar{\nu}}$ or $\delta G_{\mu\nu}$. The former can be identified with a real $(1, 1)$-form and the latter with a complex $(2, 1)$-form, as follows:

$$\omega^{(1,1)} = i\,\delta G_{i\bar{\jmath}}\,\mathrm{d}z^i \wedge \mathrm{d}\bar{z}^{\bar{\jmath}}, \qquad \omega^{(2,1)} = \Omega_{jkl}\,G^{l\bar{m}}\delta G_{\bar{\imath}\bar{m}}\,\mathrm{d}\bar{z}^{\bar{\imath}} \wedge \mathrm{d}z^j \wedge \mathrm{d}z^k, \tag{5.3}$$

where Ω is the unique, nowhere-vanishing, covariantly constant, holomorphic $(3, 0)$-form on $\mathscr{M}$ [2, 79, 78]. The condition that after the deformation $\mathscr{M}$ is still Ricci-flat, implies that $\omega^{(1,1)}$ and $\omega^{(2,1)}$ are *harmonic*. Furthermore, there are the changes in the two-form $B_{i\bar{\jmath}}$, which are combined with the deformations of the Kähler class. With the identifications $2\Delta = \Delta_\partial = \Delta_{\bar{\partial}}$, valid for any Kähler manifold, we can then identify spaces on which the moduli take their values with $H^{(1,1)}(\mathscr{M})$ and $H^{(2,1)}(\mathscr{M})$. The number of $(1, 1)$-moduli is the Hodge-number $h^{(1,1)}(\mathscr{M})$ and the number of $(2, 1)$-moduli is the Hodge-number $h^{(2,1)}(\mathscr{M})$. As it turns out [75], the compactification of the type-IIA model on a Calabi-Yau manifold leads to an effective four-dimensional field theory in which there are $h^{(1,1)}(\mathscr{M})$ abelian vector multiplets, the scalars of which parameterize the $(1, 1)$-moduli space, which indeed is special Kähler. To every order in a (string) loop-expansion, the corresponding prepotential is of the *very-special* form (3.17), where the coefficients d_{IJK} are given by the three-point functions involving the vertex operators for the vector multiplet, called Yukawa couplings [80]. Furthermore, the four-dimensional theory describes $h^{(2,1)}(\mathscr{M}) + 1$ hypermultiplets, where the extra hypermultiplet is included to accommodate the dilaton and axion from the universal sector.

[1] The number of residual supersymmetries depends on the number of Killing spinors that exist on the compactification manifold. The fact that Calabi-Yau manifolds have SU(3) holonomy is enough to show [2] that there is exactly *one* Killing spinor, so that each ten-dimensional supersymmetry is realized in the four-dimensional model exactly once. Compactifications of type-II models on Calabi-Yau three-folds thus lead to $D = 4$, $N = 2$ supersymmetric models. Ricci-flatness ensures exact conformal symmetry of the two-dimensional worldsheet action.

In type-IIB compactifications the relation between the moduli and the massless multiplets is exactly the opposite: in the effective four-dimensional model the moduli spaces of the Calabi-Yau manifold are parameterized by $h^{(2,1)}(\mathscr{M})$ vector multiplets (with a prepotential that is not necessarily very-special) and $h^{(1,1)}(\mathscr{M}) + 1$ hypermultiplets.

Recall that if we interchange the two classes of exactly marginal operators in the internal $c = 9$, $(2, 2)$ superconformal system, the superstring model is physically unchanged. For the geometrical interpretation given above, however, this interchange has far-reaching consequences: it interchanges the $(1, 1)$- and $(2, 1)$-moduli and demonstrates the equivalence of type-II string theories compactified on *topologically distinct* Calabi-Yau manifolds. This equivalence is known as *mirror symmetry* (for reviews, see [79, 78]). More specifically, there are so-called mirror-pairs of Calabi-Yau manifolds $\mathscr{M}$ and $\mathscr{M}^*$, for which $h^{(1,1)}(\mathscr{M}) = h^{(2,1)}(\mathscr{M}^*)$ and $h^{(2,1)}(\mathscr{M}) = h^{(1,1)}(\mathscr{M}^*)$. However, there are Calabi-Yau manifolds that do not have a mirror[2].

The question which immediately presents itself is that of the consequences for the four-dimensional effective model: given the fact that for every internal $c = 9$, $(2, 2)$-superconformal model, there is a mirror, the effective four-dimensional models must be related as well. The consequences of this relation between mirror-pairs of compactifications of type-II superstring models have been investigated in the context of $D = 4$, $N = 2$ supergravity models by Cecotti, Ferrara and Girardello in [75]. Their most important conclusion is the fact that there is a map that relates the vector- and hypermultiplet moduli spaces. More specifically, the $2n$-dimensional very-special Kähler manifolds of the type-IIA $(1, 1)$-moduli are taken into a subclass of $4(n + 1)$-dimensional quaternionic manifolds, which they call *dual-quaternionic*. They show that through dimensional reduction to three space-time dimensions of the bosonic part of the vector-multiplet sector, a duality transformation can be used to arrive at a quaternionic sigma-model, from which the metric of the four-dimensional hypermultiplet sector can be read off (see also [101]). Because of its similarity to Calabi's construction of hyperkähler metrics on co-tangent bundles of Kähler manifolds [95], this map was called the c-map. In the rest of the chapter, we also refer to this map as the *classical mirror map*, because it describes the relation between classical vector- and hypermultiplet actions, namely as a tree-level approximation of the full string-perturbative effective action. Furthermore, Cecotti *et al.* give an extensive analysis of the relation between *symmetric* very-special Kähler manifolds and their dual-quaternionic partners. A general classification of dual quaternionic spaces based on the isometry structure has been given in [96, 97].

In the rest of this chapter, we investigate the c-map in the context of *rigid* $N = 2$ supersymmetry [88]. It will be demonstrated that mirror symmetry, or rather the c-map, is not

[2]Since for every compact Kähler manifold, $h^{(1,1)}(\mathscr{M}) \geq 1$, the obvious examples are so-called rigid Calabi-Yau manifolds for which $h^{(2,1)} = 0$. The 'mirror' of such a compactification is to be interpreted as a non-geometrical compactification: the $c = 9$, $(2, 2)$-superconformal system can be defined, but there is no corresponding Calabi-Yau manifold [79, 45].

specific to string theory, but may be formulated at the level of three- and four-dimensional massless multiplets and the corresponding sigma-manifolds, even without a coupling to supergravity. As was discussed in chapter 4, the geometry of rigid hypermultiplet sigma-models is hyperkähler, so we expect to find dual hyperkähler manifolds in the image of the rigid c-map. While the form of the metric in 'special coordinates' for these manifolds has been known for quite some time [75, 92], we cast our results in a form that is compatible with the general hypermultiplet couplings discussed in section 4.3 and we study the behaviour of the various geometrical quantities under symplectic reparameterizations of the rigid, abelian vector-multiplet models. In particular, our interest goes to the $\mathrm{Sp}(n) \times \mathrm{Sp}(1)$ vielbeins, which turn out to transform in symplectically covariant fashion. The symplectic properties of all other geometrical quantities, such as the complex structures, the metric and the Kähler potential can be derived from this fact. In effect, we demonstrate the consequences of special geometry in dual hypermultiplet models. Furthermore, we investigate the isometry structure of the hyperkähler spaces that arise as a result of the rigid classical mirror map.

We also consider the possible central charges that may be generated as surface terms in the anticommutator of the supersymmetry charges. It turns out that the vector multiplets generate the scalar and pseudoscalar charges associated with the holomorphic BPS mass and a vectorial central charge expressed in terms of the integral over the pull-back of the Kähler form. The hypermultiplets on the other hand only exhibit vectorial charges expressed as integrals over the pull-back of the hyperkähler two-forms. In three dimensions the central charges associated with these two multiplets can be related via the classical mirror map. Our hope is that eventually, our considerations may prove useful in the study of perturbative string corrections for the hypermultiplets in type-II compactifications, guided by what we know from the vector-multiplet side.

Coming back to our earlier discussion, one may wonder if the moduli-spaces of Calabi-Yau manifolds, or more generally the moduli spaces of $c = 9$, $(2,2)$-superconformal systems, are in any way related to the sigma-models that play a role in the rigidly supersymmetric version of the c-map. In that respect, two recent developments are of importance: first of all, the moduli spaces of rigid $N = 2$ vector-multiplet models have received considerable attention in the past few years, due to the work of Seiberg and Witten [83, 84]. There, the moduli spaces are identified with the moduli spaces of certain families of Riemann surfaces and this identification is used to give an exact non-perturbative expression for the prepotential. Secondly, progress has been made in the area of string-dualities [81], which relate weak- and strong-coupling regimes in compactifications of the various superstring models. A prominent role [85] is played in the duality between heterotic and type-II models by so-called K3-fibrations [86], which describe the Calabi-Yau manifolds in the type-II compactification as a fibre-bundle of K3-surfaces over a projective sphere. It has been shown [87], that in the point-particle limit of a heterotic compactification on $\mathrm{K3} \times \mathrm{T}^2$, heterotic/type-II duality can be used to identify the projective sphere as a three-fold cover of the Riemann surfaces that play a role in the Seiberg-

Witten approach. Correspondingly, the exact results that are obtained in the vector-multiplet moduli space of a type-II compactification are used to re-derive the non-perturbative Seiberg-Witten result from string duality. Furthermore, an correspondence between the symplectic pair $(X, \partial F)$ and the periods on the Calabi-Yau manifold is given. In the next section, we show that the sigma-manifold that results as the image of the c-map applied to the abelian vector-multiplet model that was used in [83], can be identified as the bundle of Jacobian varieties over the moduli space of auxiliary Riemann surfaces. A similar construction can be made in the context of Calabi-Yau moduli spaces by a generalization of the concept of the Jacobian bundle to Calabi-Yau manifolds.

5.2 Dimensional reduction of vector multiplet models

In this section we reduce the general Lagrangian (3.58) for on-shell, abelian vector multiplets to three space-time dimensions. This is done by compactifying one of the spatial dimensions (say, the one parameterized by x^3) on a circle with radius R and suppressing all the modes that depend non-trivially on x^3. The four-dimensional gauge fields decompose into three-dimensional gauge fields A^I_μ and additional scalar fields $A^I \equiv A^I_3$. If we impose the Bianchi identity in three dimensions through addition of a Lagrange multiplier term proportional to $B_I \varepsilon^{\mu\nu\rho}\partial_\mu F^I_{\nu\rho}$ and integrate out the field strength, the degrees of freedom of the four-dimensional gauge field are captured in the two scalars A^I and B_I, as was discussed in section 1.4.

Before turning to more explicit results we deal with the consequences of the dimensional reduction for the fermions, which, in four space-time dimensions, are four-component Majorana spinors. When reducing to three space-time dimensions, every spinor decomposes into two two-component spinors. In order to discuss this systematically one decomposes the Clifford algebra of the gamma matrices in four dimensions into two mutually *commuting* Clifford algebras: one is the algebra generated by the gamma matrices appropriate to three dimensions and the second one is the algebra generated by γ^3. This is accomplished by defining:

$$\gamma^\mu = \gamma^\mu_{(4)}\tilde{\gamma}, \quad \mu = 0, 1, 2, \tag{5.4}$$

where $\gamma^\mu_{(4)}$ are the four-dimensional gamma-matrices and:

$$\tilde{\gamma} = -i\gamma^3\gamma^5, \tag{5.5}$$

so that $\gamma^1\gamma^2\gamma^0$ is proportional to the identity matrix. This implies that the Clifford algebra generated by these three-dimensional gamma-matrices acts on the two-component spinors in equivalent representations. The gamma-matrices γ^3 and γ^5 coincide with their four-dimensional expressions: $\gamma^3 = \gamma^3_{(4)}$ and $\gamma^5 = \gamma^5_{(4)}$. The matrices $\tilde{\gamma}$, γ^3 and γ^5 *commute* with the three γ^μ. An observation that will be relevant later on, is that the three anticommuting matrices $\hat{\sigma}^1$, $\hat{\sigma}^2$ and $\hat{\sigma}^3$, defined by: $\hat{\sigma}^1 = \gamma^3$, $\hat{\sigma}^2 = \gamma^5$ and $\hat{\sigma}^3 = \tilde{\gamma}$ form an su(2) Lie algebra: $\hat{\sigma}^1\,\hat{\sigma}^2 = i\hat{\sigma}^3$.

Because the Dirac conjugate of a spinor involves the matrix γ^0, it will acquire an extra factor $\tilde{\gamma}$ as compared to the four-dimensional definition. Correspondingly we absorb a factor $\tilde{\gamma}$ into the three-dimensional charge-conjugation matrix C. With this definition we have the following identities:

$$C\,\gamma^\mu C^{-1} = -\gamma^{\mu\,\mathrm{T}}, \quad C\,\gamma^3 C^{-1} = \gamma^{3\,\mathrm{T}}, \quad C\,\tilde{\gamma}\,C^{-1} = \tilde{\gamma}^{\mathrm{T}}, \quad C\,\gamma^5 C^{-1} = -\gamma^{5\,\mathrm{T}}. \tag{5.6}$$

It is possible to choose C such that it commutes with γ^3, $\tilde{\gamma}$ and γ^5.

Now we turn to the Lagrangian of the compactified theory. After converting the three-dimensional gauge field into a scalar field by means of the three-dimensional vector-scalar duality discussed in section 1.4, the terms in the Lagrangian (3.58) that contain the field strengths, are replaced by:

$$\begin{aligned}&-\tfrac{1}{4}i\Big(\bar{F}_{IJ}\,W^I_\mu W^{J\,\mu} - F_{IJ}\,\bar{W}^I_\mu \bar{W}^{\mu\,J}\Big)\\ &-\tfrac{1}{8}i\Big(F_{IJK}\bar{\Omega}^I_i\,\bar{W}^J_\mu\gamma^\mu\gamma_3\Omega^K_j\,\varepsilon^{ij} - \bar{F}_{IJK}\bar{\Omega}^{i\,I}\,W^J_\mu\gamma^\mu\gamma_3\Omega^{j\,K}\varepsilon_{ij}\Big),\end{aligned} \tag{5.7}$$

where W_μ is defined by:

$$\begin{aligned}W^I_\mu \;=\;& 2iN^{IJ}(\partial_\mu B_J - F_{JK}\,\partial_\mu A^K)\\ &+\tfrac{1}{4}iN^{IJ}\Big(\bar{F}_{JKL}\,\bar{\Omega}^{i\,K}\gamma_\mu\gamma_3\Omega^{j\,L}\varepsilon_{ij} + F_{JKL}\,\bar{\Omega}^K_i\gamma_\mu\gamma_3\Omega^L_j\varepsilon^{ij}\Big).\end{aligned} \tag{5.8}$$

Substituting this into (5.7) and combining with the other terms of the Lagrangian (3.58) yields:

$$\begin{aligned}4\pi\,\mathscr{L} \;=\;& i\Big(\partial_\mu F_I\,\partial^\mu\bar{X}^I - \partial_\mu\bar{F}_I\,\partial^\mu X^I\Big) - N^{IJ}(\partial_\mu B_I - F_{IK}\partial_\mu A^K)(\partial^\mu B_J - \bar{F}_{JM}\partial^\mu A^M)\\ &-\tfrac{1}{4}N_{IJ}\Big(\bar{\Omega}^{i\,I}\partial\!\!\!/\Omega^J_i + \bar{\Omega}^I_i\partial\!\!\!/\Omega^{i\,J}\Big)\\ &-\tfrac{1}{4}iF_{IJK}\Big(\bar{\Omega}^I_i\partial\!\!\!/ X^J\Omega^{i\,K} - i\bar{\Omega}^I_i N^{JL}(\partial\!\!\!/ B_L - \bar{F}_{LM}\partial\!\!\!/ A^M)\gamma_3\Omega^K_j\varepsilon^{ij}\Big)\\ &+\tfrac{1}{4}i\bar{F}_{IJK}\Big(\bar{\Omega}^{i\,I}\partial\!\!\!/\bar{X}^J\Omega^K_i + i\bar{\Omega}^{i\,I}N^{JL}(\partial\!\!\!/ B_L - F_{LM}\partial\!\!\!/ A^M)\gamma_3\Omega^{j\,K}\varepsilon_{ij}\Big)\\ &+\tfrac{1}{96}i\Big(F_{IJKL} + 3iN^{MN}F_{M(IJ}F_{KL)N}\Big)\bar{\Omega}^I_i\,\gamma_3\gamma_\mu\Omega^J_j\varepsilon^{ij}\bar{\Omega}^K_k\,\gamma_3\gamma^\mu\Omega^L_l\varepsilon^{kl}\\ &-\tfrac{1}{96}i\Big(\bar{F}_{IJKL} - 3iN^{MN}\bar{F}_{M(IK}\bar{F}_{JL)N}\Big)\bar{\Omega}^{Ii}\gamma_3\gamma_\mu\Omega^{j\,J}\varepsilon_{ij}\,\bar{\Omega}^{k\,K}\gamma_3\gamma^\mu\Omega^{l\,L}\varepsilon_{kl}\\ &-\tfrac{1}{48}N^{MN}F_{MIJ}\bar{F}_{KLN}\,\Big(2\bar{\Omega}^I_i\gamma_\mu\Omega^{i\,K}\,\bar{\Omega}^J_j\gamma^\mu\Omega^{j\,L} + \bar{\Omega}^I_i\gamma_\mu\gamma_3\Omega^J_j\varepsilon^{ij}\,\bar{\Omega}^{k\,K}\gamma_\mu\gamma_3\Omega^{l\,L}\varepsilon_{kl}\Big),\end{aligned} \tag{5.9}$$

where we have suppressed a factor $2\pi R$ corresponding to the integration over the compactified coordinate x^3. Observe that the Lagrangian remains manifestly invariant under $\mathrm{SU}(2)_R$. Note also that we keep the fermion fields in their original four-dimensional form, *i.e.* they are doublets of $\frac{1}{2}(\mathbb{I}\pm\gamma^5)$ projections of four-dimensional Majorana spinors. Only the definition of the Dirac conjugate has been changed in accord with the rules obtained above.

The above Lagrangian is invariant under the following supersymmetry transformations:

$$\delta_Q(\epsilon)X^I \;=\; -i\bar{\epsilon}^i\gamma_3\Omega^I_i\,,$$

$$
\begin{aligned}
\delta_Q(\epsilon)A^I &= i\varepsilon^{ij}\bar\epsilon_i\Omega^I_j - i\varepsilon_{ij}\bar\epsilon^i\Omega^{j\,I}\,,\\
\delta_Q(\epsilon)B_I &= iF_{IJ}\varepsilon^{ij}\bar\epsilon_i\Omega^J_j - i\bar F_{IJ}\varepsilon_{ij}\bar\epsilon^i\Omega^{j\,J}\,,\\
\delta_Q(\epsilon)\Omega^I_i &= 2i\partial\!\!\!/ X^I\gamma_3\epsilon_i + 2N^{IJ}\Big(\partial\!\!\!/ B_J - \bar F_{JK}\partial\!\!\!/ A^K\Big)\varepsilon_{ij}\epsilon^j\\
&\quad + iN^{IJ}\Big[\delta_Q(\epsilon)F_{JK}\Big]\Omega^K_i - N^{IJ}\bar F_{JKL}N^{KM}\Big[\delta_Q(\epsilon)B_M - F_{MN}\delta_Q(\epsilon)A^N\Big]\varepsilon_{ij}\gamma_3\Omega^{j\,L}\,,\\
\delta_Q(\epsilon)\Omega^{i\,I} &= -2i\partial\!\!\!/ \bar X^I\gamma_3\epsilon^i + 2N^{IJ}\Big(\partial\!\!\!/ B_J - F_{JK}\partial\!\!\!/ A^K\Big)\varepsilon^{ij}\epsilon_j\\
&\quad - iN^{IJ}\Big[\delta_Q(\epsilon)\bar F_{JK}\Big]\Omega^{i\,K} - N^{IJ}F_{JKL}N^{KM}\Big[\delta_Q(\epsilon)B_M - \bar F_{MN}\delta_Q(\epsilon)A^N\Big]\varepsilon^{ij}\gamma_3\Omega^L_j\,.
\end{aligned}
\tag{5.10}
$$

Under symplectic reparameterizations (A, B) transform as a symplectic pair, just as the field strengths in (3.22). From (A^I, B_I) we can construct a complex scalar:

$$
Y_I = B_I - F_{IJ}A^J, \tag{5.11}
$$

which transforms as a (co)vector under symplectic reparameterizations, *c.f.* (3.39). The supersymmetry transformation rule for Y_I is given by:

$$
\delta_Q(\epsilon)\,Y_I - \Gamma^K_{IJ}\Big[\delta_Q(\epsilon)\,X^J\Big]Y_K = -N_{IJ}\varepsilon_{ij}\bar\epsilon^i\Omega^{j\,J} + iF_{IJK}\,N^{JL}\,\bar Y_L\,\bar\epsilon^i\gamma_3\Omega^K_i\,, \tag{5.12}
$$

where the left-hand side takes the form of a symplectically covariant variation, while the right-hand side is explicitly symplectically covariant, due to the symplectic connection Γ, as defined in (3.60). Observe that X^I and Y_I all transform holomorphically, *i.e.* their supersymmetry variations are proportional to $\bar\epsilon^i$ and not to $\bar\epsilon_i$. All supersymmetry variations take a symplectically covariant form, as follows from using the transformations properties given in section 3.4.

After the dualization of the vector to scalar fields, the symplectic reparameterizations can be applied to the equations of motion *or* directly to the Lagrangian. As we have seen in section 3.2, these reparameterizations express the fact that the theory retains its form under certain diffeomorphisms, provided that we simultaneously change the function $F(X)$. As with general diffeomorphisms, this is not an invariance statement, but it characterizes the equivalence classes of the theory as encoded in functions $F(X)$. Henceforth we will use the term 'symplectically invariant' to indicate that quantities retain their form under the combined effect of a certain diffeomorphism and a change of the function $F(X)$. Note that the Lagrangian (5.9) is symplectically invariant. In particular, we note that the four-fermion terms are proportional to either the special Kähler curvature, given in (3.60), or to the symmetric tensor C, defined by:

$$
C_{IJKL} = F_{IJKL} + 3iN^{MN}\,F_{M(IJ}\,F_{KL)N}\,. \tag{5.13}
$$

Both tensors are symplectically covariant. The latter tensor vanishes for a symmetric Kähler space (defined by the condition that the curvature tensor is covariantly constant).

If the function $F(X)$ describes an effective four-dimensional gauge theory, based on charged fields which have been integrated out, then the θ-angles are defined up to shifts by 2π at the non-perturbative level (see the example at the end of section 3.2). Consequently, the quantity F_{IJ} is only defined up to an additive integer-valued matrix. From this observation it follows that, after compactifying on a circle, we must identify B_I with B_I plus an integer matrix times A^I. Furthermore, the fields A^I are only defined up to an integer times R^{-1}, as a consequence of four-dimensional gauge transformations with non-trivial winding around the compactified direction.[3] Therefore, consistency requires that also B_I is defined up to an integer times R^{-1}. At the perturbative level the corresponding invariance is realized by continuous transformations as can be seen from (5.9), which is invariant under $F_{IJ} \to F_{IJ}+c_{IJ}$ and $B_I \to B_I + c_{IJ}A^J$, where the constants c_{IJ} constitute an arbitrary real symmetric tensor. These transformations correspond to the continuous Peccei-Quinn symmetries and are consistent with the transformations induced by the symplectic reparameterizations of the underlying vector-multiplet theory. Note that these transformations do not presume invariance under continuous shifts of the fields A_I, which, at finite R, do not represent a symmetry at the perturbative level. It is here that our approach fails to capture the dynamical effects associated with the compactification, just because we take $F(X)$ from a four-dimensional setting [89]. This has no direct bearing on the fact that the target space parameterized by the (A^I, B_I) fields constitutes a torus T^{2n}, whose periodicity lattice is in fact directly related to the lattice of dyonic charges. The full space is a fibre bundle over a special Kähler manifold with fibre T^{2n}. In the limit $R \to 0$, the torus decompactifies to $\mathbb{R}^{2n}$.

Let us discuss some properties of the torus at a given point X in the special Kähler moduli space. First we determine the volume of T^{2n}, which turns out to be independent of X. To see this one integrates the square root of the determinant of the (A^I, B_I) metric given in (5.9) over the torus. Including the factor 4π from the left-hand side of (5.9) and the factor $2\pi R$ from the integration over the compactified coordinate x^3, we find the volume is given by:

$$V(\mathrm{T}^{2n}) = (4R)^{-n}. \tag{5.14}$$

Secondly, consider the invariant lengths of cycles $\gamma(X) : t \mapsto (X^I; A^I(t), B_I(t))$, which depend on the point X in the special-Kähler moduli space. For the cycles γ_{A^I} and γ_{B_I} in the A^I and B_I directions, these lengths are equal to:

$$\ell_{A^I}(X) = \frac{1}{R}\sqrt{(FN^{-1}\bar{F})_{II}}\,, \qquad \ell_{B_I}(X) = \frac{1}{R}\sqrt{(N^{-1})^{II}}\,. \tag{5.15}$$

When X^I approaches a point where the Kähler metric becomes singular, one of the cycles $(\gamma_{A^I}, \gamma_{B_I})$ shrinks to zero while the other one grows to infinite length. So the singularities in the moduli space correspond to singular T^2-fibres.

At this point it is tempting to identify the torus at X with the Jacobian variety [90] of an auxiliary Riemann surface $\mathcal{M}_X$ that underlies the four-dimensional non-perturbative dynamics of a gauge theory in the Coulomb phase [83]. Its effective action takes the form of (3.58) and the abelian vector multiplets are associated with the Cartan subalgebra of the

[3] For simplicity, we have set the I-th elementary charge equal to unity.

underlying gauge group. Singularities in the effective action associated with the emergence of massless states correspond to a pinching of the auxiliary Riemann surface $\mathcal{M}_X$ which in turn leads to a degeneration of its Jacobian variety.[4] According to these arguments one may conclude that the complex scalars Y_I take their values in the (rescaled) Jacobian:

$$J(\mathcal{M}_X) = \mathbb{C}^n/L_X, \qquad L_X = \left\{ \frac{1}{R}\Big(m_I - F_{IJ}(X)\, n^J\Big) \middle| \, m_I, n^I \in \mathbb{Z} \right\}, \tag{5.16}$$

where we identify the second derivative of $F(X)$ with the intersection matrix τ of the Riemann surface $\mathcal{M}_X$. The target space parameterized by all the scalar fields thus coincides with the holomorphic $\mathrm{Sp}(2n, \mathbb{Z})$ bundle of Jacobian varieties over the moduli space of auxiliary Riemann surfaces, with metric as given in (5.9) and transitions functions prescribed by the monodromies in the moduli space.

5.3 Geometric features, the symplectic group and isometries

A number of geometric features of the target space associated with the metric defined in the Lagrangian (5.9) deserves further attention. Note that, as a three-dimensional model, we are dealing with four independent supersymmetries. Therefore the target space must be a hyperkähler manifold, which, in the case at hand, is completely determined by the holomorphic function $F(X)$. Some of the properties of the special Kähler space are inherited by the ensuing hyperkähler space. In particular, when the Kähler space is symmetric or homogeneous, then the hyperkähler space is also symmetric or homogeneous, respectively. The material of this subsection covers some of the results presented in [75] and the relation with the work of [92]. Furthermore we discuss the behavior under symplectic reparameterizations of the special hyperkähler manifold and we discuss the isometries of the hyperkähler sigma-manifold.

The bosonic Lagrangian follows from (5.9). It can be rewritten as:

$$\begin{aligned} 4\pi\,\mathscr{L} \;=\; & -N_{IJ}\Big(\partial_\mu X^I\,\partial^\mu \bar{X}^I + \tfrac{1}{4}\partial_\mu A^I\,\partial^\mu A^J\Big) \\ & -N^{IJ}\Big(\partial_\mu B_I - \tfrac{1}{2}(F_{IK}+\bar{F}_{IK})\partial_\mu A^K\Big)\Big(\partial^\mu B_J - \tfrac{1}{2}(F_{JM}+\bar{F}_{JM})\partial^\mu A^M\Big). \end{aligned} \tag{5.17}$$

When the coordinates A and B are frozen to constant values, we have a special Kähler space parameterized by the coordinates X^I. Alternatively, freezing the special Kähler coordinates yields the torus T^{2n}. To describe the resulting $4n$-dimensional hyperkähler space, one specifies the metric and three covariantly constant complex structures, from which three closed two-forms can be defined (see section 4.3).

The metrics (5.17) form a subclass of hyperkähler metrics constructed in [92] using the Legendre-transform method. The latter are characterized by the presence of at least n abelian isometries, which are tri-holomorphic so that they leave the metric as well as the closed two-forms invariant. In (5.17), the tri-holomorphic isometries are generated by constant shifts in

[4] In a full three-dimensional treatment, it is possible that non-perturbative effects associated with monopoles wrapping around the circle smooth out some of these singularities; see the first reference of [91].

A^I and B_I. Hyperkähler metrics with at least n tri-holomorphic abelian isometries can be written in the general form [93]:

$$\mathrm{d}s^2 = U_{IJ}(x)\,\mathrm{d}\vec{x}^I\cdot\mathrm{d}\vec{x}^J + (U^{-1}(x))^{IJ}\Big(\mathrm{d}\varphi_I + \vec{W}_{IK}(x)\cdot\mathrm{d}\vec{x}^K\Big)\Big(\mathrm{d}\varphi_J + \vec{W}_{JL}(x)\cdot\mathrm{d}\vec{x}^L\Big). \quad (5.18)$$

Here, the coordinates are split according to $\{\vec{x}^I,\varphi_I\}$. The n vectors $\vec{x}^I$ comprise $3n$ real components $\vec{x}^{I\,\Lambda}$, where $\Lambda = 1,2,3$; the remaining n real coordinates φ_I are subject to the shift isometries. The tensors U_{IJ} and $\vec{W}_{IJ}$ are independent of φ_I and satisfy the hyperkähler equations:

$$\partial^\Lambda_J W^\Sigma_{KI} - \partial^\Sigma_K W^\Lambda_{JI} = \varepsilon^{\Lambda\Sigma\Pi}\,\partial^\Pi_J U_{KI}, \quad (5.19)$$

where $\partial^\Lambda_I = \partial/\partial x^{\Lambda\,I}$. From this it follows that $\partial^\Lambda_I U_{JK} = \partial^\Lambda_J U_{IK}$. The three hyperkähler two-forms, given in [92], can be rewritten as follows (see *e.g.* [94]):

$$\omega^\Lambda = (\mathrm{d}\varphi_I + \vec{W}_{IJ}\cdot\mathrm{d}\vec{x}^J)\wedge\mathrm{d}x^{\Lambda\,I} + U_{IJ}\,\varepsilon^{\Lambda\Sigma\Pi}\,\mathrm{d}x^{\Sigma\,I}\wedge\mathrm{d}x^{\Pi\,J}. \quad (5.20)$$

Clearly, they are invariant under constant shifts of φ_I, so that these isometries are indeed tri-holomorphic. In the case of (5.17), we have coordinates $\vec{x}^I = (\mathrm{Re}\,X^I, \mathrm{Im}\,X^I, -\frac{1}{2}A^I)$, $\varphi_I = B_I$ and:

$$U_{IJ} = N_{IJ}\,, \qquad \vec{W}_{IJ} = (0,0,F_{IJ}+\bar{F}_{IJ}). \quad (5.21)$$

For this solution both U and $\vec{W}$ are determined by a single holomorphic function F, independent of A^I. It can be shown that F is proportional to the holomorphic function that appears in the contour-integral representation (*c.f.* [92]) of the solution (5.18). Note also that $\vec{W}_{IJ}$ is symmetric. Other examples of hyperkähler metrics of the type (5.18) are Taub-NUT and the asymptotic metric on the moduli space of N SU(2) BPS monopoles. These metrics appear in the effective actions of three-dimensional $N=4$, SU(N) gauge theories [91]. They are not in the class (5.21).

To write down the hyperkähler two-forms and discuss symplectic transformations, it is convenient to use the complex coordinates Y_I (*c.f.* (5.11)). In terms of the fields X^I and Y_I the bosonic Lagrangian reads:

$$\begin{aligned} 4\pi\,\mathscr{L} \;=\;& -N_{IJ}\,\partial_\mu X^I\partial^\mu\bar{X}^J \quad (5.22)\\ & -N^{IJ}\Big(\partial_\mu Y_I + iN^{KL}(Y_L-\bar{Y}_L)\partial_\mu F_{IK}\Big)\Big(\partial^\mu\bar{Y}_J + iN^{MN}(Y_N-\bar{Y}_N)\partial^\mu\bar{F}_{JM}\Big). \end{aligned}$$

At this point we note the identity:

$$\partial_\mu Y_I + iN^{JK}(Y_K-\bar{Y}_K)\,\partial_\mu F_{IJ} = (\partial_\mu Y_I - \Gamma^K_{IJ}\,\partial_\mu X^J\,Y_K) - iF_{IJK}\,\partial_\mu X^J\,N^{KL}\bar{Y}_L\,, \quad (5.23)$$

where the first term is just the Kähler covariant derivative of special geometry with the connection given in (3.60); the second term is separately covariant with respect to symplectic reparameterizations, as can easily be verified from (3.66). Therefore the above Lagrangian is invariant under the symplectic reparameterizations, as was already claimed in the previous subsection.

The combined (X, Y) space is a hyperkähler space with Kähler potential [75]:

$$K(X,Y,\bar{X},\bar{Y}) = iX^I\,\bar{F}_I(\bar{X}) - i\bar{X}^I\,F_I(X) - \tfrac{1}{2}(Y_I - \bar{Y}_I)\,N^{IJ}(X,\bar{X})\,(Y_J - \bar{Y}_J). \tag{5.24}$$

Under symplectic reparameterizations the Kähler potential K changes by a Kähler transformation:

$$K \overset{S}{\longmapsto} \tilde{K}(\tilde{X},\tilde{Y},\tilde{\bar{X}},\tilde{\bar{Y}}) = K(X,Y,\bar{X},\bar{Y}) + \tfrac{1}{2}i\mathcal{Z}^{IJ}(X)\,Y_IY_J - \tfrac{1}{2}i\,\bar{\mathcal{Z}}^{IJ}(\bar{X})\,\bar{Y}_I\bar{Y}_J\,, \tag{5.25}$$

where $\tilde{K}$ is evaluated on the basis of the new function $\tilde{F}$ and $\mathcal{Z}(X)$ is the symmetric holomorphic tensor defined in (3.65). This does not imply that the Kähler metric takes the form of a symplectically covariant tensor, because the coordinates Y_I, unlike the special Kähler coordinates X^I, do not transform as coordinates but as symplectic vectors.[5] To see this, one first computes the metric from the derivatives of the Kähler potential. In the coordinates $z^a = (X^I, Y_J)$ we find:

$$g_{a\bar{b}} = \begin{pmatrix} (N + P\,N^{-1}\bar{P})_{IK} & (P\,N^{-1})_I{}^L \\ (N^{-1}\bar{P})^J{}_K & (N^{-1})^{JL} \end{pmatrix}, \tag{5.26}$$

where we have used the symmetric tensor $P_{IJ} = iF_{IJK}N^{KL}(Y_L - \bar{Y}_L)$. Under a symplectic reparameterization, P_{IJ} transforms as:

$$P_{IJ} \overset{S}{\longmapsto} \Big[P_{KL} + F_{KLM}\,\mathcal{Z}^{MN}Y_N\Big]\,[\mathcal{S}^{-1}]^K{}_I\,[\mathcal{S}^{-1}]^L{}_J\,, \tag{5.27}$$

which implies that the metric is not symplectically covariant.

The inverse metric satisfies the relation:

$$\Omega_{ac}\,g^{c\bar{d}}\,\Omega_{\bar{d}\bar{b}} = -g_{a\bar{b}}\,, \tag{5.28}$$

where Ω_{ab} is a covariantly constant antisymmetric tensor:

$$\Omega_{ab} = \begin{pmatrix} 0 & \delta_I{}^L \\ -\delta^J{}_K & 0 \end{pmatrix}. \tag{5.29}$$

The covariant constancy follows from (5.28). As a result Ω commutes with the holonomy group. Complex structures are then defined by:

$$J^3 = \begin{pmatrix} -i\delta^a{}_b & 0 \\ 0 & i\delta^{\bar{a}}{}_{\bar{b}} \end{pmatrix}, \qquad J^\alpha = \begin{pmatrix} 0 & \alpha\Omega_a{}^{\bar{b}} \\ \bar{\alpha}\Omega_{\bar{a}}{}^b & 0 \end{pmatrix}, \tag{5.30}$$

with α a phase factor and $\Omega_a{}^{\bar{b}} = \Omega_{ac}g^{c\bar{b}}$. Choosing $\alpha = 1, -i$ corresponding to, respectively, J^2, J^1, the matrices represent the product for hyperkähler complex structures (4.33). Finally,

[5] A similar situation is present in Calabi's construction of hyperkähler spaces on cotangent bundles with coordinates (X^I, Y_I) [95]. The corresponding Kähler potential is $K = i(X^I\bar{F}_I - \bar{X}^I F_I) - Y_I N^{IJ}\bar{Y}_J$, and is invariant under symplectic transformations. An essential difference, however, is that Calabi's metric does not possess the same (tri-holomorphic) isometries as the metric described above.

the corresponding two-forms can be computed from (5.20) or, equivalently, from the complex structures. One finds:

$$\begin{aligned}\omega^3 &= -iK_{X\bar{X}}\,\mathrm{d}X\wedge\mathrm{d}\bar{X} - iK_{X\bar{Y}}\,\mathrm{d}X\wedge\mathrm{d}\bar{Y} - iK_{Y\bar{X}}\,\mathrm{d}Y\wedge\mathrm{d}\bar{X} - iK_{Y\bar{Y}}\,\mathrm{d}Y\wedge\mathrm{d}\bar{Y}\,,\\ \omega^+ &= \mathrm{d}X^I\wedge\mathrm{d}Y_I\,,\qquad \omega^- = \mathrm{d}\bar{X}^I\wedge\mathrm{d}\bar{Y}_I\,.\end{aligned}\tag{5.31}$$

Observe that $\omega^\pm$ is purely (anti-)holomorphic, as already mentioned in [95, 92]. This will be important when we discuss the central charges in section 5.5. These two-forms are closed so that locally they can be written as exterior derivatives of the following one-forms:

$$\begin{aligned}A^3 &= \tfrac{1}{2}iK_X\,\mathrm{d}X + \tfrac{1}{2}iK_Y\,\mathrm{d}Y - \tfrac{1}{2}iK_{\bar{X}}\,\mathrm{d}\bar{X} - \tfrac{1}{2}iK_{\bar{Y}}\,\mathrm{d}\bar{Y}\,,\\ A^+ &= \tfrac{1}{2}X^I\,\mathrm{d}Y_I - \tfrac{1}{2}Y_I\,\mathrm{d}X^I,\qquad A^- = \tfrac{1}{2}\bar{X}^I\,\mathrm{d}\bar{Y}_I - \tfrac{1}{2}\bar{Y}_I\,\mathrm{d}\bar{X}^I.\end{aligned}\tag{5.32}$$

Under symplectic reparameterizations, these local one-forms are invariant up to an exact form. For A^3 this follows from (5.25) and for $A^\pm$ this can be seen from noting that the second term is manifestly symplectically invariant, whereas the first term equals the second up to an exact form. Therefore the corresponding two-forms are symplectically invariant. For $\omega^\pm$ this can also be seen directly by observing that replacing the one-forms $\mathrm{d}Y$ by the symplectically covariant forms $\mathrm{d}Y_I + iN^{JK}Y_K\mathrm{d}F_{IJ}$, does not change $\omega^\pm$. Note, however, that the corresponding tensors J^Λ are *not* symplectically covariant, since the metric is not a covariant tensor.

As explained in detail in [96] the isometry group of a special Kähler manifold extends in a characteristic way when performing the *c*-map. The additional isometries are called *extra* symmetries when their origin can be understood directly from the four-dimensional gauge transformations, or *hidden* symmetries when their existence is not generic and depends on special properties of the manifold. In [96] this was discussed for special quaternionic manifolds (*i.e.* in the case of local supersymmetry). In this subsection, we give a similar discussion for the special hyperkähler manifolds. Here the extra symmetries follow directly from the gauge symmetry in four dimensions and correspond to constant shifts in A^I and in B_I, as we discussed previously. In the complex basis the extra isometries take the form:

$$\delta Y_I = \beta_I - F_{IJ}\alpha^J\,,\tag{5.33}$$

with real parameters α^I and β_I and they are tri-holomorphic.

Apart from these, there can be isometries corresponding to duality invariances, *c.f.* (3.31), of the original four-dimensional action of the vector multiplets. Because the two-forms ω^Λ are symplectically invariant, such isometries are also tri-holomorphic. There can be additional isometries of the special Kähler manifold that do not leave the full action invariant [96, 97]. Those isometries do not take the form of symplectic reparameterizations and will in principle not correspond to isometries of the hyperkähler manifold.

Just as for the special quaternionic manifolds, we find that hidden symmetries for the hyperkähler manifolds are subject to certain non-trivial conditions. But unlike the quaternionic case, the conditions seem impossible to satisfy unless one makes a rather simple choice for the function $F(X)$. Before proceeding to derive the conditions for general isometries, we make the following observation. Obviously the commutator of an infinitesimal isometry and a supersymmetry variation defines a fermionic symmetry. However, we know that the fields X^I and Y_I transform only under supersymmetries with positive-chirality parameters. Unless the isometries are holomorphic, we will thus generate new supersymmetries of negative chirality. These can not be accommodated by the standard supersymmetry algebra and the theory can only be invariant under them if it contains non-interacting sectors, *i.e.* if the model is reducible and the target space is a local product space (this argument is identical to the one used in [98] for two-dimensional sigma-models with torsion). So without loss of generality, we may assume that δX^I and δY_I depend only on X^I and Y_I.

With this in mind we first study the variations of the action under an arbitrary infinitesimal isometry that are quadratic in the derivatives of the fields A^I and B_I. This leads to the result that the variation of F_{IJ} must take the form:

$$\delta F_{IJ} = N_{IK}\, N_{JL}\, \frac{\partial^2 f}{\partial \bar{Y}_K \partial \bar{Y}_L}, \tag{5.34}$$

where f is some real function of $Y, \bar{Y}, X, \bar{X}$. Furthermore the transformation rule for Y_I can be written as:

$$\delta Y_I = -iN_{IJ}\, \frac{\partial}{\partial \bar{Y}_J} \left[2f + (Y_K - \bar{Y}_K) \frac{\partial f}{\partial \bar{Y}_K} \right] + iN_{IJ} \Lambda^{JK} \bar{Y}_K\,, \tag{5.35}$$

where the quantity $\Lambda^{IJ}(X,\bar{X})$ is independent of Y and $\bar{Y}$ and antisymmetric in I,J so that it cannot be incorporated into the first term for δY_I. From the fact that the right-hand side of (5.34) must be independent of $\bar{Y}$, it follows that the function f depends at most quadratically on $\bar{Y}$ and obviously the same conclusion can be drawn for the Y-dependence. Therefore the first term in (5.35) is $\bar{Y}$-independent. Because δY itself must also be independent of $\bar{Y}$, it follows that $\Lambda^{IJ} = 0$.

Subsequently, consider the mixed variations in the Lagrangian, proportional to a derivative of A or B and $\bar{X}$. This leads to conditions for the derivatives of δX^I with respect to A^I and B_I, which can be integrated. Specifically, we find two restrictions:

$$\delta X^I \pm iN^{IJ} \frac{\partial f}{\partial \bar{X}^J}\bigg|_{A,B} = \tfrac{1}{2} P^I_\pm\,, \tag{5.36}$$

where P^I_+ depends on $X, \bar{X}, \bar{Y}$ and P^I_- depends on $X, \bar{X}, Y$. The holomorphicity of δX^I implies that $(P_+ + P_-)^I$ depends only on X and Y. Therefore it follows that $N^{IJ} \partial f / \partial \bar{X}^J|_{A,B}$ must be independent of $\bar{Y}$ and $\bar{X}$, up to terms that depend exclusively on $X, \bar{X}$. The holomorphicity in Y restricts f to the following form:

$$\begin{aligned} f(X,\bar{X},Y,\bar{Y}) \;&=\; [N(Y-\bar{Y})]^I [N(Y-\bar{Y})]^J O_{IJ}(X,Y) \\ &\quad + i[N(Y-\bar{Y})]^I \Lambda_I(X,Y) + \tilde{f}(X,\bar{X},Y). \end{aligned} \tag{5.37}$$

The holomorphic functions O_{IJ} and Λ_I can now be expanded in powers of Y. Note that the first one is at most quadratic and the second one at most cubic in Y. Also the non-holomorphic function $\tilde{f}$ can be expanded in Y, up to fourth order.

The reality of f yields a large number of restrictions. For instance, the Y-expansion coefficients of $\tilde{f}$ and Λ are related:

$$-i[\Lambda_I(X) - \bar{\Lambda}_I(\bar{X})]^{J_1\cdots J_n} = nN_{IJ}\,\tilde{f}^{JJ_1\cdots J_n}(X,\bar{X}), \tag{5.38}$$

where the $\tilde{f}^{IJK\cdots}$ must be real. On the other hand, holomorphicity in X restricts the $\tilde{f}^{IJK\cdots}$ to the form:

$$\tilde{f}^{KL\cdots}(X,\bar{X}) = (\bar{X}^I F_{IJ} - \bar{F}_J)\,g^{J,KL\cdots}(X) + h^{KL\cdots}(X). \tag{5.39}$$

Combining these constraints seems to lead to the inevitable conclusion that, at least for non-trivial functions $F(X)$, the $\tilde{f}^{IJ\cdots}$ must be constant. In that case we may rewrite (5.37) in terms of A and B, and observe that there is no $\bar{X}$-dependence anymore. However, the function f must be real, so that we conclude that it is a function of A and B (plus a function of X and $\bar{X}$, which can be ignored). The independence of X and $\bar{X}$ now implies that (5.37) is a real polynomial in A and B that is at most of order two. The terms linear in A and B characterize the shift symmetries (5.33) and the quadratic terms correspond to the isometries embedded in the symplectic reparameterizations of the special Kähler space. The latter can be verified by showing that (5.34) and (5.35) take the form of an infinitesimal symplectic reparameterization as follows from the first equation of (3.66) and (3.39), respectively.

5.4 The classical mirror map

From the material of the previous sections we will explicitly extract the vielbeins and other geometrical quantities of the hyperkähler space that emerges from the four-dimensional $N = 2$ vector multiplets under the action of the c-map. Before doing so, it is important that we first discuss the extension of the chiral $\mathrm{SU}(2)_R \times \mathrm{U}(1)_R$ automorphism group of the supersymmetry algebra in four space-time dimensions to SO(4). Of course, it is well known that the automorphism group in three dimensions contains SO(4), but we are interested in the way this extension is realized, namely by promoting the $\mathrm{U}(1)_R$ group to SU(2). With the aforementioned $\mathrm{SU}(2)_R$ one thus obtains the group $(\mathrm{SU}(2) \times \mathrm{SU}(2))/\mathbb{Z}_2 \cong \mathrm{SO}(4)$.

In section 5.2 we already made reference to the fact that the independent combinations of four-dimensional gamma-matrices that commute with the three-dimensional ones, constitute an su(2) Lie algebra. Therefore spinors ϵ^i in a four-dimensional space-time, which transform under the chiral $\mathrm{SU}(2)_R \times \mathrm{U}(1)_R$ group, can in principle transform under a bigger group after descending to three dimensions. However, we are not interested in any such extension, but only in those that constitute a subgroup of the automorphism group of the supersymmetry algebra in three space-time dimensions.

To understand the fate of the su(2) let us momentarily consider $N = 1$ supersymmetry in four space-time dimensions. The four-dimensional automorphism group contains a chiral $\mathrm{U}(1)_R$. According to the above arguments this group can be extended to SU(2) in the reduction to three space-time dimensions; its generators are just proportional to the three Hermitian matrices $\hat{\sigma}^1, \hat{\sigma}^2, \hat{\sigma}^3$ that were defined in section 5.2. This SU(2) group is consistent with the supersymmetry algebra, but it cannot be realized on Majorana spinors. The Majorana constraint requires the phases appropriate to the group $\mathrm{SL}(2,\mathbb{C})$, which, in turn, is not consistent with the supersymmetry algebra. So, unless one doubles the spinors, the automorphism group $\mathrm{U}(1)_R$ remains unextended when descending to three space-time dimensions.

Starting from $N=2$ in four dimensions, on the other hand, naturally incorporates such a doubling of spinors. The spinor doublets then transform under chiral $\mathrm{SU}(2)_R \times \mathrm{U}(1)_R$ and the extension of the U(1) group to SU(2) is automatic. Starting with a (non-chiral) Majorana doublet ϵ^i (which comprises eight real independent components), the $\mathrm{SU}(2)_R$ transformations act according to:

$$\epsilon^i \to \Big[U^i{}_j\Big(\frac{\mathbb{I}+\hat{\sigma}^2}{2}\Big) + \overline{U^i{}_j}\Big(\frac{\mathbb{I}-\hat{\sigma}^2}{2}\Big)\Big]\epsilon^j. \tag{5.40}$$

In three space-time dimensions the group $\mathrm{U}(1)_R$ is extended to SU(2), with matrices $\hat{U}$ that are associated with the generators $\hat{\sigma}^a$. Their action is given by:

$$\epsilon^i \to \Big[\hat{U}\Big(\frac{\mathbb{I}+\sigma^2}{2}\Big)^i{}_j + \bar{\hat{U}}\Big(\frac{\mathbb{I}-\sigma^2}{2}\Big)^i{}_j\Big]\epsilon^j, \tag{5.41}$$

where $(\sigma^2)^i{}_j$ equals the skew-symmetric imaginary σ-matrix. This extra SU(2) commutes with $\mathrm{SU}(2)_R$ by virtue of the fact that they both have a skew-symmetric invariant tensor, $(\sigma^2)^i{}_j$ and $\hat{\sigma}^2 = \gamma^5$, satisfying $\bar{U} = \sigma^2 U\,\sigma^2$ and likewise for $\hat{U}$ and $\hat{\sigma}^2$. It is convenient to write the above transformations in infinitesimal form employing chiral spinor components. Defining $\hat{U} \approx \mathbb{I} + \frac{1}{2} i\hat{\alpha}_a\,\hat{\sigma}^a$, one obtains:

$$\begin{aligned}\delta\epsilon^i &= \tfrac{1}{2} i\hat{\alpha}_2\,\epsilon^i + \tfrac{1}{2}\varepsilon^{ij}(\hat{\alpha}_1 + i\hat{\alpha}_3)\gamma^3\epsilon_j\,,\\ \delta\epsilon_i &= -\tfrac{1}{2} i\hat{\alpha}_2\,\epsilon_i + \tfrac{1}{2}\varepsilon_{ij}(\hat{\alpha}_1 - i\hat{\alpha}_3)\gamma^3\epsilon^j\,.\end{aligned} \tag{5.42}$$

The above results show that a proper basis for the extra SU(2) transformations is obtained by choosing:

$$\begin{aligned}\epsilon^+ &= \tfrac{1}{2}\sqrt{2}\,\gamma^3(\epsilon_1 - i\epsilon_2), \qquad & \epsilon^- &= \tfrac{1}{2}\sqrt{2}\,(\epsilon^1 - i\epsilon^2),\\ \epsilon_+ &= \tfrac{1}{2}\sqrt{2}\,\gamma^3(\epsilon^1 + i\epsilon^2), \qquad & \epsilon_- &= \tfrac{1}{2}\sqrt{2}\,(\epsilon_1 + i\epsilon_2).\end{aligned} \tag{5.43}$$

These spinors are eigenstates under both σ^2 and $\hat{\sigma}^2$ and transform under phase transformations as the SO(2) subgroup of $\mathrm{SU}(2)_R$. Upper- and lower-index spinors are related by conjugation.

Now let us consider the reduction to three dimensions of the actions presented in sections 3.4 and 4.3 for vector multiplets and hypermultiplets. As pointed out previously, the vector multiplet Lagrangian and supersymmetry transformations are manifestly covariant with respect to the $\mathrm{SU}(2)_R$ group, but not to the group $\mathrm{U}(1)_R$ (at least, not in the general case). Consequently, when descending to three dimensions, the symmetry group is not enhanced and we are left with the $\mathrm{SU}(2)_R$ transformations and the symplectic reparameterizations. On the other hand, the hypermultiplet Lagrangian and supersymmetry transformations are generically only covariant with respect to the group $\mathrm{U}(1)_R$ and when descending to three dimensions, this group is enhanced to a full SU(2) group, with elements $\hat{U}$. However, consistency requires that this extra SU(2) group commutes with the holonomy group and therefore its action incorporates the antisymmetric tensor $\Omega_{\bar\alpha\bar\beta}$ constructed in section 4.3. Infinitesimally, the SU(2) transformations act on the hypermultiplet fermions according to:

$$\begin{aligned}\delta\zeta^{\alpha} &= \tfrac{1}{2}i\hat\alpha_2\,\zeta^{\alpha} - \tfrac{1}{2}G^{\alpha\bar\gamma}\Omega_{\bar\gamma\bar\beta}\,(\hat\alpha_1 + i\hat\alpha_3)\gamma^3\zeta^{\bar\beta},\\ \delta\zeta^{\bar\alpha} &= -\tfrac{1}{2}i\hat\alpha_2\,\zeta^{\bar\alpha} - \tfrac{1}{2}\bar\Omega^{\bar\alpha\bar\gamma}G_{\bar\gamma\beta}\,(\hat\alpha_1 - i\hat\alpha_3)\gamma^3\zeta^{\beta}.\end{aligned} \tag{5.44}$$

In other words, when systems based on both vector multiplets and hypermultiplets are reduced to a three-dimensional , the target space factorizes into two hyperkähler manifolds which will both possess an independent SU(2) invariance group, corresponding to different factors of the SO(4) automorphism group of the supersymmetry algebra. This reflects the general situation in $N = 4$ supersymmetric sigma-models in three dimensions, even when coupled to supergravity. In the latter case the sigma-manifold factorizes into two quaternionic spaces, whose Sp(1) holonomy groups constitute the two different factors of the SO(4) group [99].

The above observations are essential to reconcile the fermionic supersymmetry transformations (5.11) with those of the hypermultiplet (4.22), after dimensional reduction. The SO(2) subgroup of $\mathrm{SU}(2)_R$ will play the role of $\mathrm{U}(1)_R$ after applying the mirror map and returning to four space-time dimensions. Consequently, we must identify the fields ζ^{α} and $\zeta^{\bar\alpha}$ with combinations of the vector multiplet spinor fields, Ω_i^I and Ω^{iI}, that transform as eigen-spinors under the SO(2) group with the proper phase transformations. For the spinor parameters, this means that we must convert to the previously introduced spinor parameters $\epsilon^{\pm}$ and $\epsilon_{\pm}$ (*c.f.* (5.43)). These requirements motivate us to make the following identification:

$$\begin{aligned}\zeta^{\alpha} &= \Big(-\tfrac{1}{2}\sqrt{2}\,\gamma^3(\Omega_1^I - i\Omega_2^I),\, \tfrac{1}{2}\sqrt{2}\,(\Omega^{1\,I} - i\Omega^{2\,I})\Big),\\ \zeta^{\bar\alpha} &= \Big(-\tfrac{1}{2}\sqrt{2}\,\gamma^3(\Omega^{1\,I} + i\Omega^{2\,I}),\, \tfrac{1}{2}\sqrt{2}\,(\Omega_1^I + i\Omega_2^I)\Big),\end{aligned} \tag{5.45}$$

where the relation between $\zeta^{\bar\alpha}$ and $\zeta^{\bar\alpha}$ proceeds via Dirac conjugation and the Majorana condition.

Let us first comment on the various factors in (5.45). As explained above, the identification is such that the ζ^{α} transform under the SO(2) subgroup of $\mathrm{SU}(2)_R$ with a uniform phase. The $\zeta^{\bar\alpha}$ then transform with the opposite phase. The relative factors γ^3 follow from the requirement that the fermions on the *r.h.s.*, whose supersymmetry transformations follow from (5.11), will take a form similar to the transformations of the hypermultiplet fermions, as given in (4.22), when descending to three dimensions. Both the overall and relative factors of γ^3 are required to match the chirality of both sides of the equations. The phase factors adopted for the various components in (5.45), are somewhat arbitrary. They can be changed *a posteriori* by performing certain redefinitions. The same comment applies to the phase factors adopted in the definitions of the spinors (5.43).

In three dimensions, (5.45) and (5.43) represent simply a different basis for the spinors that play a role in the vector multiplet. However, from the point of view of the four-dimensional

Lorentz group, this choice of basis has non-trivial implications. When assuming that the newly defined spinor fields transform in the conventional way under the four-dimensional Lorentz transformations, one implicitly exchanges the $\mathrm{SU}(2)_R$ and the extra SU(2) group that contains $\mathrm{U}(1)_R$. More precisely, taking the vector multiplet to three dimensions, the four-dimensional gamma-matrices are related to the three-dimensional ones, properly combined with the SU(2) generators denoted by $\hat{\sigma}^a$. Returning to four dimensions in the same way as before, but on the basis of the newly defined spinors, implies that the four-dimensional gamma-matrices are now formed from the three-dimensional gamma-matrices combined with the $\mathrm{SU}(2)_R$ generators σ^a. Thus the mere switch in the spinor basis suffices to correctly implement the mirror map.

The fermion basis (5.45) shows an obvious decomposition of the index α according to $\alpha = (I, r)$ with the index r taking values $r = 1, 2$; a similar decomposition holds for $\bar{\alpha}$. This decomposition will be used below. Using (5.45) we can now identify these local one-forms as well as the $\mathrm{Sp}(n)$ connections for a hypermultiplet theory that originates from a four-dimensional vector multiplet theory by comparing the fermion supersymmetry transformations on vector and hypermultiplet sides. We thus find (strictly speaking the indices i now run over $+, -$):

$$V_{A\,i}^{\alpha}\,\mathrm{d}\phi^A = \Big(V_A^I\,\mathrm{d}\phi^A\Big)^r{}_i = 2\begin{pmatrix} \mathrm{d}X^I & N^{IK}\bar{\mathcal{W}}_K \\ N^{IK}\mathcal{W}_K & \mathrm{d}\bar{X}^I \end{pmatrix}, \tag{5.46}$$

where $\mathcal{W}_I = \mathrm{d}B_I - F_{IJ}\mathrm{d}A^J$ and $\alpha = (I, r)$, and:

$$\Gamma_A{}^{\alpha}{}_{\beta}\,\mathrm{d}\phi^A = \Big(\Gamma_A\,\mathrm{d}\phi^A\Big)^{I\,r}{}_{J\,s} = \begin{pmatrix} -iN^{IK}\,\mathrm{d}F_{KJ} & -iN^{IK}\bar{F}_{KJL}N^{LM}\mathcal{W}_M \\ -iN^{IK}F_{KJL}N^{LM}\bar{\mathcal{W}}_M & iN^{IK}\,\mathrm{d}\bar{F}_{KJ} \end{pmatrix}, \tag{5.47}$$

with $\alpha = (I, r)$ and $\beta = (J, s)$. Observe that the above quantities all take their values in the quaternions, *i.e.* they can be written in the form: $a\mathbb{I} + ia_\Lambda\sigma^\Lambda$ with real coefficients a, a_Λ.

From the transformation rules and the action we can now determine all the relevant quantities in the hypermultiplet sector, such as the metric, the complex structures and the antisymmetric tensor Ω. They are all consistent with the general results for hypermultiplets, derived in section 4.3. Let us first give the expressions for the fermionic metric $G_{\bar{\alpha}\beta}$:

$$G_{\bar{\alpha}\beta} = \tfrac{1}{4}N_{IJ}\,\delta_{rs}\,, \tag{5.48}$$

and the antisymmetric tensor $\Omega_{\bar{\alpha}\bar{\beta}}$:

$$\Omega_{\bar{\alpha}\bar{\beta}} = \tfrac{1}{4}N_{IJ}\,\varepsilon_{rs}\,. \tag{5.49}$$

Next we present the local one-forms γ^A, which take the form:

$$\gamma_{i\bar{\alpha}\,A}\,\mathrm{d}\phi^A = \Big(\gamma_{AI}\,\mathrm{d}\phi^A\Big)_{ri} = \tfrac{1}{2}\begin{pmatrix} N_{IK}\mathrm{d}X^K & \bar{\mathcal{W}}_I \\ -\mathcal{W}_I & N_{IK}\mathrm{d}\bar{X}^K \end{pmatrix}, \tag{5.50}$$

where $\bar{\alpha} = (r, I)$. Furthermore, we give the fermionic Lagrangian that follows from (5.9) and (5.45), which exhibits most of the geometric quantities, such as the tensor W defined in (4.41):

$$4\pi\,\mathscr{L}_{\mathrm{ferm}} = -\tfrac{1}{4}N_{IJ}\Big(\bar{\zeta}^{\overline{I1}}\partial\!\!\!/\zeta^{J1} + \bar{\zeta}^{\overline{I2}}\partial\!\!\!/\zeta^{J2} + \mathrm{h.c.}\Big)$$

$$
\begin{aligned}
&+\tfrac{1}{4}iF_{IJK}\Big(\bar\zeta^{\overline{I1}}\partial\!\!\!/ X^J\zeta^{K1}-\bar\zeta^{\overline{I2}}\partial\!\!\!/ X^J\zeta^{K2}\Big)+\text{h.c.}\\
&+\tfrac{1}{2}iF_{IJK}\Big(\bar\zeta^{\overline{I2}}N^{JL}(\partial\!\!\!/ B_L-F_{LM}\partial\!\!\!/ A^M)\zeta^{K1}\Big)+\text{h.c.}\\
&-\tfrac{1}{24}i\Big(F_{IJKL}+3iN^{MN}F_{M(IJ}F_{KL)N}\Big)\,\bar\zeta^{\overline{I2}}\gamma_\mu\zeta^{J1}\,\bar\zeta^{\overline{K2}}\gamma^\mu\zeta^{L1}+\text{h.c.}\\
&-\tfrac{1}{24}N^{MN}F_{MIJ}\bar F_{NKL}\Big(\bar\zeta^{\overline{K1}}\gamma_\mu\zeta^{I1}-\bar\zeta^{\overline{I2}}\gamma_\mu\zeta^{K2}\Big)\Big(\bar\zeta^{\overline{L1}}\gamma^\mu\zeta^{J1}-\bar\zeta^{\overline{J2}}\gamma^\mu\zeta^{L2}\Big)\\
&-\tfrac{1}{12}N^{MN}F_{MIJ}\bar F_{NKL}\,\bar\zeta^{\overline{I1}}\gamma_\mu\zeta^{J2}\,\bar\zeta^{\overline{K1}}\gamma^\mu\zeta^{L2}\,. \qquad (5.51)
\end{aligned}
$$

The tensor W defined in (4.41), is thus expressed in terms of the tensor C_{IJKL}, defined in (5.13), and the curvature tensor of the special Kähler space given in (3.60). Both these tensors, and therefore the tensor W, are covariant with respect to symplectic reparameterizations of the underlying special Kähler manifold. The tensor W fully encodes the curvature tensor of the special hyperkähler manifold. We refrain from giving explicit formulae, but wish to point out that these expressions allow for a coordinate-independent characterization of the special hyperkähler manifolds. We have also verified that the tensor W becomes fully symmetric when written in purely (anti)holomorphic indices, employing the result for the tensors $G_{\bar\alpha\beta}$ and $\Omega_{\bar\alpha\bar\beta}$ given above.

It is clear from their index structure that the local one-forms (5.46) transform covariantly under the symplectic reparameterizations of the underlying vector multiplet by multiplication from the left with matrices

$$
S^{Ir}{}_{Js}=\begin{pmatrix}\mathcal{S}^I{}_J & 0\\ 0 & \bar{\mathcal{S}}^I{}_J\end{pmatrix}, \qquad (5.52)
$$

while the local one-forms (5.50) transform from the left with $[\bar S^{-1}]^J{}_I$. In general these transformations are not contained in the holonomy group $\mathrm{Sp}(n)$, although they have a common subgroup, as can be seen from (5.52).

The above thus constitutes the full construction of a hypermultiplet model in four space-time dimensions associated with a specific theory based on vector multiplets. The detour through three dimensions only serves as a means to arrive at these results. Unlike the corresponding theory of vector multiplets, the hypermultiplet theory does not exhibit an $\mathrm{SU}(2)_R$ invariance, at least not in the generic case. Only a manifest $\mathrm{U}(1)_R$ invariance remains. All the isometries of the vector-multiplet target space that represent invariances of the full set of equations of motion, remain present as isometries of the hypermultiplet target space. The symplectic reparameterizations of the vector multiplets induce corresponding transformations on the hyperkähler side. In this way we deal with a large class of hyperkähler spaces. They can be expressed in terms of certain restrictions on the curvature tensor.

We should stress that the general hypermultiplet action is encoded in the local one-forms V_i^α, but one has to provide one extra ingredient, such as the fermionic metric $G_{\bar\alpha\beta}$, or the antisymmetric tensor $\Omega_{\bar\alpha\bar\beta}$. The expressions given above for these quantities concern the special hyperkähler spaces and are given in special coordinates.

5.5 Central charges

As a last application of the mirror map we turn to the central charges that can emerge in the supersymmetry algebra for a theory based on vector multiplets or hypermultiplets. As the symplectic reparameterizations can be performed in a supergravity background [39], the algebra and therefore the expressions for the central charges should be invariant under these reparameterizations. Likewise, the charges should be consistent with the underlying Kähler or hyperkähler geometry. We will determine the central charges by evaluating the possible surface terms on the right-hand side of the anticommutator of two supercharges. To determine this anticommutator we use canonical quantization. This approach is the same as the one followed in [54] for the elementary super-Yang-Mills system. Here we apply it for an arbitrary function $F(X)$ and arbitrary hyperkähler metrics.

Let us first present the supercurrent for the vector multiplet and hypermultiplet theories,

$$\begin{aligned} J_{\mu\, i} &= \frac{1}{8\pi}\Big\{N_{IJ}\,\partial\!\!\!/\bar X^I\gamma_\mu\Omega_i^J + \tfrac{1}{2}i\varepsilon_{ij}\,\mathcal{G}^-_{\rho\sigma\, I}\,\sigma^{\rho\sigma}\gamma_\mu\Omega^{j\,I} + \tfrac{1}{12}i\bar F_{IJK}\,\gamma_\mu\Omega^{k\,I}\,\bar\Omega^{l\,J}\Omega^{j\,K}\,\varepsilon_{ij}\varepsilon_{kl}\Big\}, \\ J_{\mu\, i} &= \frac{1}{4\pi}\,g_{AB}\,\gamma^A_{i\bar\alpha}\,\partial\!\!\!/\phi^B\gamma_\mu\zeta^{\bar\alpha}. \end{aligned} \tag{5.53}$$

where $\mathcal{G}^-$ was defined in (3.62). The other chirality components follow by complex conjugation. Observe that the first one is invariant under symplectic reparameterizations. Obviously the second expression for the hypermultiplet current is invariant under the hyperkähler holonomy group. The reader may be surprised that the vector-multiplet current contains terms cubic in the fermion fields, whereas the hypermultiplet current is linear in the fermion fields. Still one can verify, by performing the duality transformation in the presence of the gravitino field coupling to the supercurrent, that the expressions for the two currents become compatible upon reduction to three dimensions.

To determine the central charges one needs only the Dirac brackets for the fermions, as the bosonic brackets lead to terms at least quadratic in the fermion fields, representing supersymmetric completions of bosonic terms that are already present in the algebra. In this way [100], we find the following commutators for the vector multiplet,

$$\begin{aligned} \{Q_i,\bar Q^j\} &= i\hbar\,\frac{1-\gamma^5}{2}\,\delta_i{}^j\,\Big\{\gamma_\mu\,P^\mu + \gamma_a\,Z^a\Big\}, \\ \{Q_i,\bar Q_j\} &= -i\hbar\,(1-\gamma^5)\,\varepsilon_{ij}\,\Big\{\bar X^I\,q_{\mathrm{e}\,I} - \bar F_I\,q^I_{\mathrm{m}}\Big\}, \end{aligned} \tag{5.54}$$

where the vector central charge, Z^a, is defined by (a, b, c denote spatial indices),

$$Z^a = \frac{i}{8\pi}\varepsilon^{abc}\int \mathrm{d}^3x\, N_{IJ}\,\partial_b X^I\,\partial_c\bar X^J, \tag{5.55}$$

which is an integral over the Kähler two-form; the second anticommutator yields the anti-holomorphic BPS mass expressed in terms of the values of $\bar X^I$ and $\bar F_I$ taken at spatial infinity (to obtain this result we used the field equations for the vector fields) and the electric and magnetic charges[6]. Obviously the central charges are invariant under symplectic reparameterizations, as predicted above. For the case of a quadratic function F our result for the second commutator coincides with that in [54]. The Kähler form contribution was presented in [100].

For the hypermultiplets we find a similar result for the anticommutators,

$$\begin{aligned}\{Q_i, \bar{Q}^j\} &= i\hbar\,\frac{1-\gamma^5}{2}\,\Big\{\delta_i{}^j\,\gamma_\mu\,P^\mu + (\sigma^\Lambda)_i{}^j\,\gamma_a\,Z^{\Lambda\,a}\Big\},\\ \{Q_i, \bar{Q}_j\} &= 0\,,\end{aligned}\tag{5.57}$$

where we now have three vector central charges defined by

$$Z^{\Lambda\,a} = -\frac{1}{16\pi}\varepsilon^{abc}\int \mathrm{d}^3x\, J^\Lambda_{AB}\,\partial_b\phi^A\partial_c\phi^B.\tag{5.58}$$

The J^Λ are the three complex structures of the hyperkähler space defined in (4.30).

There is a clear systematics in the above results. Note that the central charges for the vector multiplet are singlets under $\mathrm{SU}(2)_R$, whereas those for the hypermultiplets transform as a triplet under this group. In addition to the BPS mass, we find certain integrals over the pull-back of the Kähler form (for the vector multiplet) and the hyperkähler forms (for the hypermultiplet). Naively, all these integrals vanish, as we can write (locally in the target space) these two-forms as the exterior derivative of corresponding local one-forms. This then allows us to write the integrands as total derivatives in the base space, which can be dropped subject to certain reasonable assumptions on the asymptotic values of the scalar fields. Hence the question whether these charges are actually realized depends on the kind of boundary conditions that one wishes to impose. For instance, in $3+1$ dimensions, if one imposes boundary conditions at spatial infinity such that the fields converge in all directions to the same value, with the derivatives vanishing sufficiently fast so as to ensure finite energy, then the central charges associated with the two-forms will vanish. In 2+1 dimensions, the situation is different. In that case the central charges are expressed as integrals of the (hyper-)Kähler two-forms over the image of ϕ. Topologically this image is S^2, so that the central charges are enumerated by the second homology group of the target-space manifold. Obviously the central charges set a BPS bound in the usual fashion.

From the perspective of the preceding sections it is of interest to see how the central charges of the vector multiplet sector and the hypermultiplet sector are related by mirror symmetry. When suppressing the dependence on the compactified coordinate x^3, the central charges Z^3 and $Z^{\Lambda\,3}$ can be finite. It is then straightforward to write down the supersymmetry algebra corresponding to (5.54) in three dimensions. One subtlety is that the momentum in the third

[6] The charges $q_{\mathrm{e}\,I}$ and q_{m}^I are related to electric and magnetic charges and are defined in terms of flux integrals over closed spatial surfaces that surround the charged objects (quantized on a lattice with elementary area equal to $2\hbar$),

$$2\pi\,q_{\mathrm{m}}^I = \oint_{\partial V}(F^+ + F^-)^I,\qquad 2\pi\,q_{\mathrm{e}\,I} = \oint_{\partial V}(G^+ + G^-)_I\,.\tag{5.56}$$

This definition shows that the charges $(q_{\mathrm{m}}^I, q_{\mathrm{e}\,I})$ transform under symplectic reparameterizations precisely as the field strengths (F^I, G_I).

direction is also a surface integral, which should be added to the central charge associated with the Kähler form. As it turns out, the resulting two-form then corresponds precisely with the Kähler form ω^3 defined in (5.30) for the hyperkähler space.

In order to apply the mirror map, we write the charges in an alternative basis in correspondence with the new basis (5.43) for the supersymmetry parameters,

$$\begin{aligned} Q^+ &= \tfrac{1}{2}\sqrt{2}\,\gamma^3(Q_1 - iQ_2), \qquad Q^- = \tfrac{1}{2}\sqrt{2}\,(Q^1 - iQ^2), \\ Q_+ &= \tfrac{1}{2}\sqrt{2}\,\gamma^3(Q^1 + iQ^2)\,, \qquad Q_- = \tfrac{1}{2}\sqrt{2}\,(Q_1 + iQ_2). \end{aligned} \tag{5.59}$$

With these definitions, the three-dimensional version of (5.54) reads

$$\begin{aligned} \{Q_\pm, \bar{Q}^\pm\} &= i\hbar\frac{1-\gamma^5}{2}\Big\{\gamma_\mu P^\mu \mp iZ'\Big\}, \\ \{Q_+, \bar{Q}^-\} &= -2i\hbar\frac{1-\gamma^5}{2}\Big\{X^I\, q_{\mathrm{e}\,I} - F_I\, q_{\mathrm{m}}^I\Big\}, \\ \{Q_-, \bar{Q}^+\} &= 2i\hbar\frac{1-\gamma^5}{2}\Big\{\bar{X}^I\, q_{\mathrm{e}\,I} - \bar{F}_I\, q_{\mathrm{m}}^I\Big\}, \end{aligned} \tag{5.60}$$

where Z' is now defined in terms of the hyperkähler two-form ω^3. This result coincides with the algebra relevant to the hypermultiplets upon reduction to three space-time dimensions, which reads,

$$\begin{aligned} \{Q_i, \bar{Q}^j\} &= i\hbar\,\frac{1-\gamma^5}{2}\Big\{\delta_i{}^j\,\gamma_\mu\, P^\mu + i(\sigma^\Lambda)_i{}^j\, Z^{\Lambda\,3}\Big\}\,, \\ \{Q_i, \bar{Q}_j\} &= 0. \end{aligned} \tag{5.61}$$

This demonstrates that the supersymmetry algebra remains consistent with the mirror map in the presence of the central charge configurations. A gratifying feature of this result is that the holomorphic BPS mass of the vector multiplets is mapped to the holomorphic hyperkähler two-forms, $\omega^\pm$, defined in (5.30).

Although the above results do not capture the full dynamics of the four-dimensional gauge theories in a circle compactification, they are consistent with the results derived in the context of three-dimensional gauge dynamics [91]. There the two sets of central charges are associated with explicit mass terms and Fayet-Iliopoulos terms, which are interchanged under the quantum mirror symmetry. The relation of the central charges with integrals of the hyperkähler two-forms also arose in that context.

Chapter 6

Vector-Tensor Multiplets

6.1 Introduction

So far, we have mainly considered the 'well-known' multiplets in four-dimensional $N = 2$ supersymmetric field theory. Besides the Weyl multiplet, we have given a detailed discussion of the vector and hypermultiplets, emphasizing the relation between the two via the mirror map. Other multiplets, such as the linear multiplet and the chiral multiplet, have been considered, but they have only played an auxiliary role, aiding the construction of models for vector and hypermultiplets. In this chapter we consider a more complicated field representation of supersymmetry, namely the vector-tensor multiplet [102]. Like the vector and hypermultiplet, it describes $8 + 8$ off-shell degrees of freedom. As its name suggests, the multiplet contains both a vector and a tensor gauge field. Recalling the construction made in section 2.1, we note that the vector-tensor multiplet can be considered as the $N = 2$ combination of an $N = 1$ vector and an $N = 1$ tensor multiplet. A similar multiplet containing two tensor gauge fields, called the double-tensor multiplet, arises as the combination of two $N = 1$ tensor multiplets. As we have seen in section 1.4, a tensor gauge field can be dualized into a scalar field with a Peccei-Quinn shift symmetry. Such a duality transformation for the tensor field in the vector-tensor multiplet turns it into a vector multiplet. Likewise, a tensor multiplet is dual to a hypermultiplet and a double-tensor multiplet is converted into a hypermultiplet after dualization of both tensor fields.

Two-tensor gauge fields arise naturally in the four-dimensional $N = 2$ compactifications of the three ten-dimensional supergravity theories. One might therefore expect that the multiplets mentioned above play a role in the effective four-dimensional low-energy models associated with superstring compactifications. In particular, the vector-tensor multiplet was identified as the multiplet of massless vertex operators associated with the dilaton, antisymmetric tensor and dilatini, together with an abelian vector gauge field, in the $\mathrm{K3} \times \mathrm{T}^2$ compactification of heterotic string theory in [103]. The $N = 2$ tensor multiplet contains the dilaton and its superpartners in type-IIA string compactifications [104] and similarly the double-tensor multiplet could serve as the supermultiplet that contains the vertex operators for the dilaton and its superpartners in the context of type-IIB compactifications. Because of the possibility to perform a duality transformation, four-dimensional effective string theories are usually formulated in terms of vector and hypermultiplet actions, which at least in string perturbation theory, yields an equivalent description. We should stress that this conversion rests on a purely on-shell equivalence. The question whether certain off-shell configurations are preferred by string theory has a long history (for a discussion, see [104]). Recent experience in dual systems, for instance in the context of three space-time dimensions [91], has taught us that answers to such questions involves non-perturbative issues. In fact, we argue in section 6.6 that a simple duality transformation does not suffice to relate the off-shell vector-tensor models that are constructed in this chapter, to the vector-multiplet couplings that are used in perturbative heterotic string compactifications to describe the dilaton-axion complex [121].

Meanwhile, it turned out that vector-tensor multiplets have a different role to play and emerge in heterotic compactifications at the non-perturbative level. This phenomenon was initially described in the context of six-dimensional heterotic string compactifications, where certain singularities in the effective action are associated with non-critical strings becoming tensionless [106]. In six-dimensions this is related to the presence of tensor multiplets. In four dimensions, vector-tensor multiplets play a similar role [107].

In this chapter we give an off-shell formulation of the vector-tensor multiplet and its couplings to vector multiplets [110, 111] and to conformal supergravity [112]. A summary of the most important concepts and arguments that are used can be found in [113]. As it turns out, the vector-multiplet couplings are represented by Chern-Simons terms in the field strength of the tensor gauge field. If there is a Chern-Simons term involving only the vector field of the vector-tensor multiplet, the resulting model describes a non-linear self-interaction [114] of the vector-tensor multiplet [110]. When such a Chern-Simons self-coupling is absent, the multiplet takes a less complicated, linear form [111]. At any rate, given a certain set of Chern-Simons coupling constants, the vector-tensor multiplet couplings to vector multiplets and conformal supergravity are fixed [112]. This implies that the possible holomorphic prepotentials for the vector multiplet couplings that one encounters after dualization, are parameterized by the Chern-Simons coupling constants. Hence only a definite class of vector multiplet models can be interpreted as dual vector-tensor models [105]. For this reason, the dual vector-multiplet couplings do not take the form that is predicted by perturbative heterotic string theory.

Note that the discussion in the context of conformal supergravity entails an infinite central-charge hierarchy like we have seen in the hypermultiplet in section 4.1, that is constrained by relations that express higher z-level components in terms of lower lying ones. In this way, we obtain a description with a finite number of independent component fields. Other approaches employ an infinite number of component fields through the central-charge superspace formalism [115, 116, 117] or the harmonic superspace formalism [118]. In view of the complexity of the results in [110, 111, 112, 113] a reformulation in superspace would be very welcome, for instance to analyze the non-renormalization properties of vector-tensor multiplets and their couplings. However, most of this work concerns the linearized version of the rigidly supersymmetric vector-tensor multiplet with its corresponding Chern-Simons couplings, which can be obtained by dimensional reduction from six dimensions [108, 109]. However, using harmonic superspace, the non-linear version of the vector-tensor multiplet was found in [119] and also a rather mysterious non-linear version of the multiplet with an exponential prepotential for the dual (rigid) vector-multiplet Lagrangian density. To date, no supergravity coupling has been constructed in superspace.

6.2 Setting the stage

In section 2.3, a general strategy was outlined for the construction of $N = 2$ superconformal multiplets, the first two steps of which entail the formulation of off-shell transformation rules under (rigid) supersymmetry. In the current section we first make some introductory remarks concerning the on-shell vector-tensor multiplet and the central charge in the off-shell formulation. Anticipating the coupling to conformal supergravity, we consider local central-charge transformations and the balancing of chiral weights in the supersymmetry variations. This necessitates a Chern-Simons coupling, which we generalize by consideration of a background of n vector multiplets, with one vector multiplet to gauge the central charge.

The on-shell vector-tensor multiplet is the $N = 2$ combination of an $N = 1$ vector (see (1.16)) and an $N = 1$ tensor multiplet (see (1.20)). It contains the following components: a real scalar field ϕ, a vector gauge field V_μ, a two-form gauge field $B_{\mu\nu}$ and an $\mathrm{SU}(2)_R$ doublet of Majorana fermions λ_i. So the multiplet describes $4 + 4$ on-shell degrees of freedom, for which we can write down the following free Lagrangian density:

$$\mathscr{L}_{\text{on-shell}} = -\tfrac{1}{2}(\partial_\mu\phi)^2 - \bar\lambda^i\partial\!\!\!/\lambda_i + \tfrac{1}{2}H_\mu H^\mu - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{6.1}$$

which is the ($\mathrm{SU}(2)_R$-invariant) sum of the free $N = 1$ actions (1.15) and (1.18). Here $H_\mu = \frac{i}{2}\varepsilon_{\mu\nu\rho\sigma}\partial^\nu B^{\rho\sigma}$ and $F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$. The action is invariant under the $N = 2$ supersymmetry variations:

$$\begin{aligned}
\delta_Q(\epsilon)\,\phi &= \bar\epsilon^i\lambda_i + \bar\epsilon_i\lambda^i\,,\\
\delta_Q(\epsilon)\,V_\mu &= i\varepsilon^{ij}\,\bar\epsilon_i\gamma_\mu\lambda_j - i\varepsilon_{ij}\,\bar\epsilon^i\gamma_\mu\lambda^j,\\
\delta_Q(\epsilon)\,B_{\mu\nu} &= 2\,\bar\epsilon^i\sigma_{\mu\nu}\lambda_i + 2\,\bar\epsilon_i\sigma_{\mu\nu}\lambda^i,\\
\delta_Q(\epsilon)\,\lambda_i &= (\partial\!\!\!/\phi - iH\!\!\!\!/\,)\,\epsilon_i - i\varepsilon_{ij}\,\sigma\cdot F^-\epsilon^j.
\end{aligned}\tag{6.2}$$

Note that the helicity content of the resulting $N = 2$ multiplet[1] gives rise to $16+16$ off-shell degrees of freedom, unless there is a 'shortening condition' [54] in the form of a central charge.

[1] Based on the fact that both V and B are spin-1 fields, one would in principle expect to find two distinct massive spin-1 representations, each of which gives rise to $8 + 8$ off-shell degrees of freedom

Given the fact that we are looking for an off-shell multiplet with $8+8$ degrees of freedom, we expect to find a non-trivial action of the central charge.

Off-shell counting $(7+8)$ reveals the need for a single auxiliary bosonic degree of freedom, which we denote by $\phi^{(\mathrm{z})}$. A $\phi^{(\mathrm{z})}$ term has to be added to the transformation rule for λ_i in order to cancel contributions proportional to the fermionic field equation that arise upon a second supersymmetry variation. So we extend the transformation rules (6.2) to an off-shell representation as follows:

$$\begin{aligned} \delta_Q(\epsilon)\,\lambda_i &= (\partial\!\!\!/\phi - iH\!\!\!\!/\,)\,\epsilon_i - i\varepsilon_{ij}\,\sigma\cdot F^-\epsilon^j + 2\,\varepsilon_{ij}\phi^{(\mathrm{z})}\epsilon^j\,, \\ \delta_Q(\epsilon)\,\phi^{(\mathrm{z})} &= -\tfrac{1}{2}\varepsilon^{ij}\,\bar\epsilon_i\partial\!\!\!/\lambda_j - \tfrac{1}{2}\varepsilon_{ij}\,\bar\epsilon^i\partial\!\!\!/\lambda^j\,, \end{aligned} \tag{6.3}$$

leaving $\delta\phi$, δV_μ and $\delta B_{\mu\nu}$ unchanged. However, the auxiliary field arises as a central-charge term in the supersymmetry commutator for ϕ. In fact, all the components of the off-shell vector-tensor multiplet have a non-zero action of the central charge: terms on the *r.h.s.* of the on-shell supersymmetry commutator that vanish or can be interpreted as gauge transformations as a consequence of the field equations, generate central-charge contributions in the off-shell multiplet. Denoting the action of the central charge on a component field with a superscript (z), we find that $\delta_{\mathrm{z}}(z')\phi = z'\phi^{(\mathrm{z})}$, $\delta_{\mathrm{z}}(z')V_\mu = z'V_\mu^{(\mathrm{z})}$, etcetera, where:

$$V_\mu^{(\mathrm{z})} = \tfrac{1}{2}H_\mu\,, \qquad B_{\mu\nu}^{(\mathrm{z})} = -i\tilde F_{\mu\nu}\,, \qquad \lambda_i^{(\mathrm{z})} = -\tfrac{1}{2}\varepsilon_{ij}\partial\!\!\!/\lambda^j\,, \qquad \phi^{(\mathrm{zz})} = -\tfrac{1}{4}\Box\phi. \tag{6.4}$$

So the vector-tensor multiplet has an *off-shell central charge*, like the off-shell massless hypermultiplet, discussed in section 4.1. Since the central charge commutes with supersymmetry transformations, the central-charge transformed representation $(\phi^{(\mathrm{z})}, V_\mu^{(\mathrm{z})}, B_{\mu\nu}^{(\mathrm{z})}, \lambda_i^{(\mathrm{z})}, \phi^{(\mathrm{zz})})$ is isomorphic to the original representation $(\phi, V_\mu, B_{\mu\nu}, \lambda_i, \phi^{(\mathrm{z})})$. Successive applications of central-charge transformations thus generate a hierarchy of vector-tensor multiplets, starting with scalars:

$$\phi \xrightarrow{\ \mathrm{z}\ } \phi^{(\mathrm{z})} \xrightarrow{\ \mathrm{z}\ } \phi^{(\mathrm{zz})} \xrightarrow{\ \mathrm{z}\ } \ldots\,. \tag{6.5}$$

Note that the full field representation is infinite dimensional, allowing for a non-trivial and possibly non-linear action of the central charge instead of the constant, multiplicative action discussed in section 2.1. As was discussed in section 4.1, a formulation of the multiplet in terms of a finite number of off-shell degrees of freedom requires constraints that relate the higher z-level components to lower lying components. In the case at hand these constraints are given by the relations (6.4).

In preparation of the coupling to conformal supergravity, we consider the matching of chiral $\mathrm{U}(1)_R$ weights. Note that every real field, such as ϕ, V_μ and $B_{\mu\nu}$, must have chiral weight $c=0$. However, assignment of zero chiral weight for the vector and tensor gauge fields leads to a contradiction between the second and third line of the transformation rules (6.2): the transformation rule for the vector field V_μ requires that the fermionic field λ_i has chiral weight $-\frac{1}{2}$, whereas similar reasoning for the two-form gauge field requires that the chiral weight of λ_i is $\frac{1}{2}$. Such a conflict can only be resolved if we include a field X^0 with non-zero chiral weight, to be inserted at the right places in the vector-tensor transformation rules. Recall also that we need a gauge field W_μ^0 in order to gauge central-charge transformations, which become local when we couple the vector-tensor multiplet to supergravity. Therefore, we prepare the vector-tensor multiplet for coupling to conformal supergravity through the inclusion of a vector multiplet $(X^0, \Omega_i^0, W_\mu^0, Y_{ij}^0)$: the scalar X^0 can be used to match the chiral weights and we associate the vector field W_μ^0 with local central-charge transformations.

Let us briefly shed some light on one of the more important consequences of a gauged central charge in the vector-tensor multiplet. The parameter z' of the central-charge transformation on the *r.h.s.* of a supersymmetry commutator in the off-shell vector-tensor multiplet is given by:

$$z' = 4\,\varepsilon_{ij}\bar{\epsilon}_2^i\epsilon_1^j + \text{h.c.}\,. \tag{6.6}$$

However, according to (2.45), the gauge parameter in the supersymmetry commutator of the vector multiplet $(X^0, \Omega_i^0, W_\mu^0, Y_{ij}^0)$, is X^0-dependent:

$$z = 4\,\bar{X}^0\,\varepsilon_{ij}\bar{\epsilon}_2^i\epsilon_1^j + \text{h.c.}\,. \tag{6.7}$$

In vector-tensor multiplets with a *gauged* central charge, the insertions of factors X^0 therefore play a dual role: besides balancing the weights in the transformations rules, they make the gauge parameter z field-dependent, as is required *c.f.* (2.45). Of course, the insertion of the vector multiplet to gauge the central charge can always be discarded by fixing the vector-multiplet components [112, 113], thus relating (6.6) and (6.7).

In the calculation of the off-shell supersymmetry commutator of the field $B_{\mu\nu}$, we interpreted a term $4i(\varepsilon_{ij}\bar{\epsilon}_2^i\epsilon_1^j - \varepsilon^{ij}\bar{\epsilon}_{2\,i}\epsilon_{1\,j})F_{\mu\nu}$ as a two-form gauge transformation. In the case of a gauged central charge, the factor in front of the field strength becomes space-time dependent due to the insertion of the factor X^0. So in that case, the $F_{\mu\nu}$-proportional term can no longer be interpreted as a tensor gauge transformation. In order to close the transformation rules, we are forced to interpret this term as a Chern-Simons gauge transformation [110] of the two-form gauge field, *i.e.* $\delta_\theta B_{\mu\nu} = \theta F_{\mu\nu}$. For this reason, we include Chern-Simons couplings in the discussion of gauge transformations that follows in the remainder of this section.

To make the discussion of gauge couplings completely general, we assume a background of n vector multiplets instead of just one [111]. Of course, still one of these multiplets, namely $(X^0, \Omega_i^0, W_\mu^0, Y_{ij}^0)$, is abelian and gauges the central charge. Occasionally, we shall denote the parameter z of local central-charge transformations by θ^0. The remaining $n-1$ vector multiplets need not be abelian and we denote them as $(X^A, \Omega_i^A, W_\mu^A, Y_{ij}^A)$. The gauge transformations then act as follows on the background gauge fields:

$$\delta_{\text{gauge}}\, W_\mu^0 = \partial_\mu z\,, \qquad \delta_{\text{gauge}}\, W_\mu^A = \partial_\mu\theta^A + f_{BC}^A\theta^B W_\mu^C\,, \tag{6.8}$$

where f_{BC}^A are structure constants. After the tensor-scalar duality transformation, our model is going to contain the dual vector multiplet for which we reserve the index 1. Therefore the indices $A, B, C\ldots$ run from 2 to n.

In addition to the central charge, the vector-tensor multiplet has its own gauge transformations. We denote the parameter associated with vector gauge transformations by θ^1 and the parameter for tensor gauge transformations by Λ_μ. If we naively treat the central-charge transformations as ordinary gauge transformations, the gauge variations of V_μ and $B_{\mu\nu}$ are in general given by:

$$\delta_{\rm gauge}\, V_\mu = \partial_\mu \theta^1 \,, \tag{6.9}$$

and

$$\delta_{\rm gauge}\, B_{\mu\nu} = 2\partial_{[\mu}\Lambda_{\nu]} + \eta_{IJ}\, \theta^I \partial_{[\mu} W_{\nu]}^J, \tag{6.10}$$

where the indices I, J run from 0 to n. Note the appearance of the η_{IJ}-proportional term in $\delta B_{\mu\nu}$, which is characteristic for Chern-Simons couplings mentioned earlier.

Closure of the combined vector and tensor gauge transformations requires that η_{IJ} be a constant tensor invariant under the gauge group. There is an ambiguity in the structure of η_{IJ}, which derives from the possibility of performing field redefinitions. Without loss of generality, η_{IJ} can be modified by absorbing a term proportional to $W_\mu^I W_\nu^J$ times some group-invariant antisymmetric tensor into the definition of the tensor field $B_{\mu\nu}$. Thus, we remove all components of η_{IJ} except for η_{11}, η_{1A} and η_{AB}, and also we render η_{AB} symmetric. Also note that, since η_{1A} is invariant under the gauge group, it follows that $\eta_{1A} W_\mu^A$ is an abelian gauge field.

The situation is actually more complicated, since V_μ and $B_{\mu\nu}$ are also subject to the central-charge transformation. As described above, under this transformation these fields transform into complicated expressions, denoted $V_\mu^{(\rm z)}$ and $B_{\mu\nu}^{(\rm z)}$, respectively, which involve other fields of the theory. Accordingly, we deform the transformation rule (6.9) to:

$$\delta_{\rm gauge}\, V_\mu = \partial_\mu \theta^1 + z V_\mu^{(\rm z)} \,, \tag{6.11}$$

and, at the same time, (6.10) to:

$$\delta_{\rm gauge}\, B_{\mu\nu} = 2\partial_{[\mu}\Lambda_{\nu]} + \eta_{11}\, \theta^1 \partial_{[\mu} V_{\nu]} + \eta_{1A}\, \theta^1 \partial_{[\mu} W_{\nu]}^A + \eta_{AB}\, \theta^A \partial_{[\mu} W_{\nu]}^B + z B_{\mu\nu}^{(\rm z)} \,. \tag{6.12}$$

All θ^0-dependent terms, including any such Chern-Simons contributions, are now contained in $V_\mu^{(\rm z)}$ and $B_{\mu\nu}^{(\rm z)}$, which are determined by closure of the full algebra, including supersymmetry. The deformed transformation rules must still lead to a closed gauge algebra. In particular one finds that:

$$[\, \delta_z(z), \delta_{\rm vector}(\theta^1)\,] = \delta_{\rm tensor}(\tfrac{1}{2} z\, \eta_{11}\, \theta^1\, V_\mu^{(\rm z)}) \,. \tag{6.13}$$

This implies that $V_\mu^{(\rm z)}$ and the combination $\hat B_{\mu\nu}^{(\rm z)} = B_{\mu\nu}^{(\rm z)} + \eta_{11} V_{[\mu} V_{\nu]}^{(\rm z)}$ both transform covariantly under the central charge, but are invariant under all other gauge symmetries. However, under local supersymmetry, they do not transform covariantly, as we will see below (*c.f.* (6.35)). The resulting gauge algebra consists of the standard gauge algebra for the vector fields augmented by a tensor gauge transformation. Observe that we have neither specified $V_\mu^{(\rm z)}$ nor $B_{\mu\nu}^{(\rm z)}$, which are determined by closure of the supersymmetry algebra and will be discussed in the next section.

Given the form of the gauge transformations for the vector and tensor gauge fields, we can now define field strengths $F(V)$ and H that transform covariantly:

$$\begin{aligned} F_{\mu\nu}(V) &= 2\partial_{[\mu}V_{\nu]} - 2W^0_{[\mu}V^{(z)}_{\nu]}\,, \\ H^\mu &= \tfrac{i}{2}\varepsilon^{\mu\nu\rho\sigma}\Big(\partial_\nu B_{\rho\sigma} - \eta_{11}V_\nu\partial_\rho V_\sigma - \eta_{1A}V_\nu\partial_\rho W^A_\sigma \\ &\qquad -\eta_{AB}W^A_\nu(\partial_\rho W_\sigma - \tfrac{1}{3}f^B_{CD}W^C_\rho W^D_\sigma) - W^0_\nu \hat{B}^{(z)}_{\rho\sigma}\Big). \end{aligned} \tag{6.14}$$

The corresponding Bianchi identities are straightforward to determine and are given by the following expressions:

$$\begin{aligned} D_\mu \tilde{F}^{\mu\nu}(V) &= -V^{(z)}_\mu \tilde{F}^{0\,\mu\nu}, \qquad (6.15) \\ D_\mu H^\mu &= -\tfrac{1}{4}i\Big(\eta_{11}\,\tilde{F}_{\mu\nu}(V)\,F^{\mu\nu}(V) + \eta_{1A}\,\tilde{F}_{\mu\nu}(V)\,F^{\mu\nu\,A} + \eta_{AB}\,\tilde{F}^A_{\mu\nu}\,F^{\mu\nu\,B} + 2\tilde{F}^0_{\mu\nu}\hat{B}^{(z)\,\mu\nu}\Big) \end{aligned}$$

Observe that the Bianchi identity for H_μ is not linear in the vector-tensor fields. On the right-hand side there are non-linear terms that are either of second-order (the term proportional to η_{11}) or of zeroth-order (the term proportional to η_{AB}) in the vector-tensor fields. Such η_{11}-proportional non-linearities will occur throughout the remainder of this chapter and are characteristic of one of the classes of vector-tensor multiplets that we shall distinguish in the next section.

6.3 Rigid vector-tensor multiplets

Having considered the (bosonic) gauge part of the symmetry algebra, we now turn to the supersymmetry transformation rules of the vector-tensor multiplet. As was argued in the preliminary discussion following (6.2), it is beneficial to couple the off-shell vector-tensor multiplet to a background of at least one vector multiplet to gauge the central charge and balance the Weyl and chiral weights in the supersymmetry variations. We emphasize that one can always choose to fix the value of the vector multiplet scalar, thus breaking global scale and chiral invariance. Note that balancing the Weyl and chiral weights forms a first and necessary step in the coupling to conformal supergravity (see section 2.3). In the next section we shall find that once the coupling to a background of vector multiplets is established, coupling to conformal supergravity becomes relatively simple.

Gauging of the central charge and weight balancing require one vector multiplet and it is only when we are interested in general vector-tensor multiplet couplings that we need more than one vector multiplet. For that reason and for the sake of simplicity we choose to first analyze the case of a vector-tensor multiplet coupled to one vector multiplet [110] and then later to generalize to an n vector-multiplet background [111]. In terms of the Chern-Simons coupling constants, this amounts to the choice $\eta_{11} \neq 0$ and $\eta_{1A} = \eta_{AB} = 0$. Note that we can set $\eta_{11} = 1$ consistently by a rescaling of the tensor gauge field. The confident reader may choose to skip the explanatory discussion that follows and immediately go to the general multiplet (6.24).

Let us consider the balancing of weights in some more detail: imposing $w = c = 0$ for both V_μ and $B_{\mu\nu}$, we see that insertions of powers of X^0 are needed to reconcile the second and third lines of (6.2). By the same token we have to include powers of X^0 in the last two terms of the first line of (6.3). Without loss of generality we can assume that the supersymmetry variation of ϕ remains of the form $\bar\epsilon^i\lambda_i + \bar\epsilon_i\lambda^i$, because we can always make a background-dependent field redefinition of ϕ and λ_i to absorb modifications. This also implies that we can choose the Weyl weight of ϕ by rescaling with a power of $|X^0|$ and a suitable redefinition of λ^i. These considerations fix the Weyl and chiral weights of the vector-tensor component fields to those given in table C.III.

Our second assumption, namely that W^0_μ gauges local central-charge transformations, has its consequences for the definitions of covariant quantities. First of all we have to adept our definition of the covariant field strengths $F(V)$ and H, as we have seen at the end of the previous section. Since λ_i is a covariant quantity, we have to ensure that its supersymmetry variation is covariant as well. Hence we replace the field strengths as defined after (6.1) by their central-charge covariant counterparts, *c.f.* (6.14). Furthermore, we covariantize the $\partial_\mu\phi$ term by addition of $-W^0_\mu\phi^{(z)}$. The above considerations lead to the following 'trial' set of supersymmetry variations:

$$
\begin{aligned}
\delta_Q(\epsilon)\,\phi &\approx \bar\epsilon^i\lambda_i + \text{h.c.}\,,\\
\delta_Q(\epsilon)\,V_\mu &\approx iX^0\varepsilon^{ij}\bar\epsilon_i\gamma_\mu\lambda_j + \text{h.c.}\,,\\
\delta_Q(\epsilon)\,B_{\mu\nu} &\approx 2|X^0|^2\bar\epsilon^i\sigma_{\mu\nu}\lambda_i + \text{h.c.}\,,\\
\delta_Q(\epsilon)\,\lambda_i &\approx \Big((\partial\!\!\!/\phi - W\!\!\!\!/^0\phi^{(z)}) - iH\!\!\!\!/\Big)\epsilon_i - \frac{i}{X^0}\varepsilon_{ij}\sigma\cdot F^-(V)\epsilon^j + 2\bar X^0\varepsilon_{ij}\phi^{(z)}\epsilon_j\,.
\end{aligned}
\qquad (6.16)
$$

Further modifications of the supersymmetry transformation rules have to be derived from the supersymmetry algebra. In the next few paragraphs we clarify some points in the calculations of supersymmetry commutators, to give the reader a feel for the the origins and role of the various terms in the final version of the supersymmetry transformation rules, given in (6.24).

First we consider the supersymmetry commutator on ϕ. The reader can easily verify that it results in:

$$[\,\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)\,]\,\phi = (\xi\cdot\partial\,\phi - \xi\cdot W^0\phi^{(z)}) + z\phi^{(z)}. \qquad (6.17)$$

where $\xi_\mu = 4\,\bar\epsilon^i_{[2}\gamma_\mu\epsilon_{1]i}$ and $z = 4\bar X^0\,\varepsilon_{ij}\bar\epsilon^i_2\epsilon^j_1 + \text{h.c.}$. This fixes the background dependent parameter z of the local central-charge transformation on the *r.h.s.* in the algebra.

Next we look at the supersymmetry commutator on V_μ, which should take the following form:

$$[\,\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)\,]\,V_\mu = \delta^{\rm cov}(\xi)\,V_\mu + zV^{(z)}_\mu + \partial_\mu\theta^1. \qquad (6.18)$$

Note that we have left room for a (field-dependent) vector gauge transformation, analogous to (1.17) and (2.45). Substituting the fermionic variation in the supersymmetry transformation rule of the vector field, we find that only the the H_μ-proportional term leads to a contribution to the central-charge term on the *r.h.s.* of (6.18). Therefore, we replace H_μ in $\delta\lambda_i$ by $V^{(z)}_\mu$, keeping in mind that eventually $V_\mu^{(\mathrm{z})}$ will turn out to be dependent on the vector-tensor and background component fields. Furthermore, we note that there is a term proportional to:

$$i\varepsilon^{ij}\bar{\epsilon}_{2\,i}\epsilon_{1\,j}\,X^0\partial_\mu\phi + \text{h.c.}\,, \tag{6.19}$$

coming from the $\partial\!\!\!/\phi\,\epsilon_i$ term in $\delta\lambda_i$. Such a term can only be interpreted as being part of the gauge term on the *r.h.s.* of (6.18). However, we need a total derivative for a gauge transformation, so we have to complete the above term with a term proportional to $\phi\,\partial_\mu X^0$, which necessitates inclusion of an Ω_i-proportional term in $\delta_Q V_\mu$. A fermionic combination that leads to a total derivative as far as the scalars are concerned is given by:

$$(2X^0\lambda_i + \phi\Omega_i). \tag{6.20}$$

Also note that the addition of a term $-iW_\mu^0\bar{\epsilon}^i\lambda_i + \text{h.c.}$ is needed to cancel the central-charge covariantization that accompanies (6.19). Since Ω_i transforms into $F_{\mu\nu}^0$ and Y_{ij}^0 as well, we have to include correction terms in $\delta_Q\lambda_i$ proportional to these fields. One can check that these extra terms do not lead to new contributions in the supersymmetry commutator of ϕ. Apart from the higher-order fermion terms, the supersymmetry variation of λ_i is now fixed completely.

For $B_{\mu\nu}$ the supersymmetry commutator takes the form:

$$[\,\delta_Q(\epsilon_1),\delta_Q(\epsilon_2)\,]\,B_{\mu\nu} = \delta^{\rm cov}(\xi)\,B_{\mu\nu} + zB_{\mu\nu}^{(\mathrm{z})} + \theta^1 F_{\mu\nu}(V) + 2\partial_{[\mu}\Lambda_{\nu]}\,. \tag{6.21}$$

Note that in the multiplet (6.2) the $-iH\!\!\!/\epsilon_i$ term in $\delta\lambda_i$ gave rise to the covariant translation $\delta^{\rm cov}\,B_{\mu\nu}$. Since we have replaced H_μ by $V_\mu^{(\mathrm{z})}$ in $\delta\lambda_i$, closure of (6.21) imposes dependency of $V_\mu^{(\mathrm{z})}$ on the other component fields. Furthermore, the second term leads to a similar dependent expression for $B_{\mu\nu}^{(\mathrm{z})}$. The exact form of these expressions is given in the general background by formula (6.26). Summarizing the above, we arrive at the following supersymmetry transformation rules for an off-shell vector-tensor multiplet with a gauged central charge:

$$\begin{aligned}
\delta\,\phi &= \bar{\epsilon}^i\lambda_i + \bar{\epsilon}_i\lambda^i\,,\\
\delta\,V_\mu &= i\,\varepsilon^{ij}\bar{\epsilon}_i\gamma_\mu(2X^0\lambda_j + \phi\Omega_j^0) - iW_\mu^0\,\bar{\epsilon}^i\lambda_i + \text{h.c.}\,,\\
\delta\,B_{\mu\nu} &= -4\phi\bar{X}^0\,\bar{\epsilon}^i\sigma_{\mu\nu}(2X^0\lambda_i + \phi\Omega_i) + 2iX^0\,\varepsilon^{ij}\bar{\epsilon}_i\gamma_{[\mu}V_{\nu]}\lambda_j\\
&\quad +\varepsilon^{ij}\bar{\epsilon}_i\gamma_{[\mu}W_{\nu]}^0(4X^0\phi\lambda_j + \phi^2\Omega_j^0) - iW_{[\mu}^0V_{\nu]}\bar{\epsilon}^i\lambda_i + \text{h.c.}\,,\\
\delta\,\lambda_i &= \Big(\partial\!\!\!/\phi - W\!\!\!\!/^0\phi^{(\mathrm{z})} - iV\!\!\!/^{(\mathrm{z})}\Big)\epsilon_i - \frac{i}{2X^0}\varepsilon_{ij}\sigma\cdot\Big(F^-(V) - i\phi F^{-0}\Big)\epsilon^j + 2\varepsilon_{ij}\bar{X}^0\phi^{(\mathrm{z})}\epsilon^j\\
&\quad -\frac{1}{X^0}(\bar{\epsilon}^j\lambda_j)\Omega_i^0 - \frac{1}{X^0}(\bar{\epsilon}^j\Omega_j^0)\lambda_i\\
&\quad -\frac{1}{4X^0\phi}\epsilon^j\Big(2\phi^2Y_{ij}^0 - 2(X^0\bar{\lambda}_i\lambda_j - \bar{X}^0\varepsilon_{ik}\varepsilon_{jl}\bar{\lambda}^k\lambda^l)\Big).
\end{aligned} \tag{6.22}$$

The above multiplet is called the *non-linear vector-tensor multiplet* and it was first studied in [110]. The non-linearity can be seen at the level of the supersymmetry transformation rules if we look at the transformation rule for the tensor gauge field, which is second order in the other vector-tensor components.

Although the above case of a single vector multiplet provides valuable insights into the couplings of a vector-tensor multiplet, it is possible that the coupling to a background of several vector multiplets brings about conceptual changes, because the choice for the field that restores the balance of chiral weights is no longer limited to just X^0 and the gauge fields are no longer associated solely with local central charge transformations. Hence, we extend the supersymmetry transformation rules (6.22) to the case of general coupling to n vector multiplets. Such couplings were derived in [111], where it was shown that given the Chern-Simons coupling constants defined in section 6.2, the vector-tensor transformation rules are fixed completely:

$$
\begin{aligned}
\delta_Q(\epsilon)\,\phi &= \bar\epsilon^i\lambda_i + \bar\epsilon_i\lambda^i\,,\\
\delta_Q(\epsilon)\,V_\mu &= i\varepsilon^{ij}\bar\epsilon_i\gamma_\mu(2X^0\lambda_j + \phi\Omega_j^0) - iW_\mu^0\,\bar\epsilon^i\lambda_i + \text{h.c.}\,,\\
\delta_Q(\epsilon)\,B_{\mu\nu} &= -2\,\bar\epsilon^i\sigma_{\mu\nu}|X^0|^2\Big(4\eta_{11}\phi - 2\,\mathrm{Re}\,[g]\Big)\lambda_i\\
&\quad -2\,\bar\epsilon^i\sigma_{\mu\nu}\bar X^0\Big(2\eta_{11}\phi^2\Omega_i^0 + \phi\bar X^0\partial_{\bar I}\bar g\,\Omega_i^I - 4i\mathrm{Re}\,[\partial_I(X^0b)]\Omega_i^I\Big)\\
&\quad +i\varepsilon^{ij}\bar\epsilon_i\gamma_{[\mu}V_{\nu]}\Big(\eta_{11}(2X^0\lambda_j + \phi\Omega_j^0) - i\eta_{1A}\Omega_j^A\Big)\\
&\quad +\varepsilon^{ij}\bar\epsilon_i\gamma_{[\mu}W_{\nu]}^0 m\Big(2X^0(2\eta_{11}\phi - g)\lambda_j + \eta_{11}\phi^2\Omega_j^0 - i\eta_{1A}\phi\Omega_j^A - 4i\partial_I(X^0b)\Omega_j^I\Big)\\
&\quad +\varepsilon^{ij}\bar\epsilon_i\gamma_{[\mu}W_{\nu]}^A\eta_{AB}\Omega_j^B - i\eta_{11}W_{[\mu}^0V_{\nu]}\bar\epsilon^i\lambda_i + \text{h.c.}\,,\\
\delta_Q(\epsilon)\,\lambda_i &= \Big(\partial\!\!\!/\phi - W\!\!\!\!/^{\,0}\phi^{(z)} - iV\!\!\!\!/^{(z)}\Big)\epsilon_i - \frac{i}{2X^0}\varepsilon_{ij}\sigma\cdot\Big(F^-(V) - i\phi F^{-0}\Big)\epsilon^j + 2\varepsilon_{ij}\bar X^0\phi^{(z)}\epsilon^j\\
&\quad -\frac{1}{X^0}(\bar\epsilon^j\lambda_j)\Omega_i^0 - \frac{1}{X^0}(\bar\epsilon^j\Omega_j^0)\lambda_i\\
&\quad -\frac{1}{2X^0(2\eta_{11}\phi - \mathrm{Re}\ g)}\epsilon^j\Big[\,2\eta_{11}\phi^2Y_{ij}^0 + \phi\bar X^0\partial_{\bar I}\bar g\,Y_{ij}^I - 4i\mathrm{Re}\,[\partial_I(X^0b)]Y_{ij}^I\\
&\qquad -2\eta_{11}\Big(X^0\bar\lambda_i\lambda_j - \bar X^0\varepsilon_{ik}\varepsilon_{jl}\bar\lambda^k\lambda^l\Big)\\
&\qquad +X^0\Big(X^0\partial_I g\,\bar\Omega_{(i}^I\lambda_{j)} - \bar X^0\varepsilon_{ik}\varepsilon_{jl}\partial_{\bar I}\bar g\,\bar\Omega^{I(k}\lambda^{l)}\Big)\\
&\qquad +i\Big(\partial_I\partial_J(X^0b)\,\bar\Omega_i^I\Omega_j^J + \varepsilon_{ik}\varepsilon_{jl}\,\partial_{\bar I}\partial_{\bar J}(\bar X^0\bar b)\bar\Omega^{Ik}\Omega^{Jl}\Big)\Big]\,,
\end{aligned}
\tag{6.23}
$$

with homogeneous, holomorphic functions g and b of zero degree given by:

$$
g = i\eta_{1A}\frac{X^A}{X^0}\,,\qquad b = -\tfrac{1}{4}i\eta_{AB}\frac{X^AX^B}{(X^0)^2}\,. \tag{6.24}
$$

The supersymmetry algebra takes the familiar form:

$$
[\,\delta_Q(\epsilon_1),\delta_Q(\epsilon_2)\,] = \delta^{\text{cov}}(\xi) + \delta_z(z) + \delta_{\text{gauge}}(\theta^1) + \delta_{\text{gauge}}(\theta^A) + \delta_{\text{gauge}}(\Lambda_\mu)\,, \tag{6.25}
$$

where the gauge parameters take the same form as the ones given in (2.45) and (6.31). The dependent expressions for $V_\mu^{(\mathrm{z})}$ and $B_{\mu\nu}^{(\mathrm{z})}$ are given by:

$$
V_\mu^{(\mathrm{z})} = -\frac{1}{2|X^0|^2(2\eta_{11}\phi - \mathrm{Re}\,[g])}
$$

$$
\begin{aligned}
&\times\Big(H_\mu - \Big(iX^0 D_\mu \bar X^I\Big(2\eta_{11}\phi^2\delta_I{}^0 + \phi\bar X^0\partial_{\bar I}\bar g - 4i\mathrm{Re}\,[\partial_I(X^0 b)]\Big) + \mathrm{h.c.}\Big)\Big) + \dots\,, \\
\hat B^{(z)}_{\mu\nu} \;=\;& -\tfrac12\mathrm{Im}\,[g]\,F_{\mu\nu}(V) + \tfrac12 i(2\eta_{11}\phi - \mathrm{Re}\,[g])\tilde F_{\mu\nu}(V) - \tfrac12\phi(\eta_{11}\phi - \mathrm{Re}\,[g])F^0_{\mu\nu} \\
&+\tfrac12 i\phi\mathrm{Im}\,[X^0\partial_I g]\tilde F^I_{\mu\nu} + 4\mathrm{Im}\Big[\partial_I(X^0 b)F^{-I}_{\mu\nu}\Big] + \dots\,,
\end{aligned}
\tag{6.26}
$$

where the dots indicate fermionic terms. Note that in this case we have both non-linear and linear terms in the transformation rule of the tensor gauge field, weighed by the Chern-Simons coupling constants η_{11} and (η_{1A}, η_{AB}) respectively.

Before giving specific results on the local supersymmetry transformations, we discuss a crucial feature of the results. Note that the coefficients η_{IJ} that encode the Chern-Simons terms cannot all be set to zero, as otherwise the supersymmetry variations would become singular. In fact, one can show that there are just two inequivalent representations of the vector-tensor multiplet, namely the non-linear and the linear vector-tensor multiplets.

The non-linear vector-tensor multiplet:
If $\eta_{11} \neq 0$, signifying the presence of a $V \wedge \mathrm{d}V$ Chern-Simons self-interaction in the tensor couplings, we can redefine fields in such a way that the $V \wedge \mathrm{d}W^A$ Chern-Simons coupling disappears completely from the field strength H. Namely, if we begin with the transformation rules (6.24), and if $\eta_{11} \neq 0$, then we may perform the following redefinition:

$$
\begin{aligned}
\phi &\longrightarrow \phi - \tfrac14 i\frac{\eta_{1A}}{\eta_{11}}\Big(\frac{X^A}{X^0} - \frac{\bar X^A}{\bar X^0}\Big)\,, \\
V_\mu &\longrightarrow V_\mu - \tfrac14\frac{\eta_{1A}}{\eta_{11}}\Big(\frac{X^A}{X^0} + \frac{\bar X^A}{\bar X^0}\Big)W^0_\mu + \tfrac12\frac{\eta_{1A}}{\eta_{11}}W^A_\mu\,, \\
B_{\mu\nu} &\longrightarrow B_{\mu\nu} + \tfrac14\eta_{1A}\Big(\frac{X^A}{X^0} + \frac{\bar X^A}{\bar X^0}\Big)V_{[\mu}W^0_{\nu]} + \tfrac12\eta_{1A}V_{[\mu}W^A_{\nu]} \\
&\qquad - \tfrac{1}{16}\frac{\eta_{1A}\eta_{1B}}{\eta_{11}}\Big(\frac{X^B}{X^0} - \frac{\bar X^B}{\bar X^0}\Big)W^0_{[\mu}W^A_{\nu]}\,.
\end{aligned}
\tag{6.27}
$$

In terms of the shifted fields, we then obtain precisely the rules (6.24), but without the η_{1A} terms. This version of the vector-tensor multiplet is a straightforward extension of the coupling (6.22), with the background extended to several vector multiplets. A characteristic feature of this version is that the transformation rules are non-linear in the vector-tensor components, as a result of the Chern-Simons coupling between V_μ and $B_{\mu\nu}$. Observe that this version contains at least two abelian vector gauge fields, W^0_μ and V_μ.

The linear vector-tensor multiplet:
If $\eta_{11} = 0$, signifying the absence of a $V \wedge \mathrm{d}V$ Chern-Simons coupling, thereby avoiding non-linearity of the transformation rules in the vector-tensor component fields, we arrive at a distinct formulation. Since not all the η_{1A} Chern-Simons coefficients can vanish simultaneously, the linear vector-tensor multiplet is formulated with at least three abelian vector fields, namely, W^0_μ, $\eta_{1A}W^A_\mu$ and V_μ. The linear class seems to coincide with the theories one obtains by reducing (1,0) tensor multiplets in six space-time dimensions to four dimensions. The

tensor multiplet comprises a scalar, a self-dual tensor gauge field and a symplectic Majorana spinor. The self-dual tensor field decomposes in four dimensions into the vector and tensor gauge fields of the vector-tensor multiplet. To have also a vector field that couples to the central charge presumably requires the dimensional reduction of a theory of tensor multiplets coupled to supergravity. A recent study of various Chern-Simons terms in six dimensions was carried out in [109].

Hence in practical situations the Chern-Simons coefficients can be restricted to satisfy either $\eta_{11} = 0$ or $\eta_{1A} = 0$. In the following we will not pay much attention to this fact, but simply evaluate the transformation rules and the action for general values of the coefficients η_{11}, η_{1A}, η_{AB}. Note also that in the case of rigid supersymmetry, one can freeze some or all of the vector multiplets to a constant, but this will not alter the structure of the couplings.

6.4 Coupling to conformal supergravity

In this section we aim to extend the vector-tensor multiplet and its vector-multiplet couplings to a background of conformal supergravity. We shall find that the structure of the multiplet is not changed in an essential way, *i.e.* coupling of the Weyl multiplet leads only to numerous superconformal covariantizations, but these do not give rise to conceptual changes. Needless to say, the procedure followed in sections 6.2 and 6.3 is tailor-made for an extension to local supersymmetry. First of all, we already insisted on rigid scale and chiral invariance. Because of that, the scalar fields of the vector multiplets play the role of compensating fields to balance possible differences in scaling weigths of the various terms. Secondly, one of the vector multiplets was required to realize the central charge in a local fashion. Of course, local dilations, chiral $\mathrm{U}(1)_R$ and central-charge transformations are necessary prerequisites for the coupling to conformal supergravity.

In this section and following sections, we write X instead of X^0 for the scalar field in the vector multiplet that gauges the central charge for notational convenience, unless specification is necessary.

Because the transformation rules for the superconformal fields are completely known, the supersymmetry algebra is determined up to the gauge and central-charge transformations associated with the vector-tensor multiplet itself and consequently takes the form:

$$\begin{aligned} [\,\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)\,] \;&=\; \delta^{\mathrm{cov}}(\xi) + \delta_M(\varepsilon) + \delta_K(\Lambda_K) + \delta_S(\eta) \\ &\quad + \delta_z(z) + \delta_{\mathrm{gauge}}(\theta^A) + \delta_{\mathrm{gauge}}(\theta^1) + \delta_{\mathrm{gauge}}(\Lambda_\mu)\,, \end{aligned} \tag{6.28}$$

where the parameters of the various transformations on the first line on the *r.h.s.* are given by (2.31) and z and θ^A by (2.45). The parameters θ^1 and Λ_μ of vector and tensor gauge transformations are to be determined from the supersymmetry transformation laws.

In order to define the vector-tensor multiplet as a superconformal multiplet, we must choose the assignments under the special S-supersymmetry transformations (which in turn determine the behaviour under special conformal transformations, gnerated by K). We have assumed that the scalar ϕ is S- and K-invariant, which leads to consistent results. While this is a natural assignment for the lowest-dimensional component of a supermultiplet, we found no rigorous arguments to rule out other assignments. The choice we made is the simplest one and, as it turns out, implies that all the vector-tensor fields remain S- and K-invariant. The latter follows from the commutator of Q- with S-supersymmetry, and subsequently, by using the $[S,S]$ commutation relation, which yields a K transformation.

For arbitrary Chern-Simons coefficients η_{IJ}, the Q-supersymmetry transformation rules (we emphasize that in the remainder of this section and in the next section, the index I does not take the value $I=1$) take the form (6.24), where the variation of the gauge fields V and B receive the additional terms:

$$\begin{aligned}
\delta V_\mu &= \cdots + 2i\phi X \varepsilon^{ij}\bar\epsilon_i\psi_{\mu\, j} + \text{h.c.}\,,\\
\delta B_{\mu\nu} &= \cdots - 2\,\bar\epsilon^i\gamma_{[\mu}\psi_{\nu]i}\bar X\Big(2\eta_{11}\phi^2 X + \phi\bar X\partial_{\bar I}\bar g X^I - 4i\text{Re}\,[\partial_I(Xb)]X^I\Big)\\
&\quad +2i\,\varepsilon^{ij}\bar\epsilon_i\psi_{j[\mu}V_{\nu]}\Big(X(\eta_{11}\phi - g)\Big)\\
&\quad +2\,\varepsilon^{ij}\bar\epsilon_i\psi_{j[\mu}W^0_{\nu]}X\Big(\eta_{11}\phi^2 - \phi g - 4ib\Big)\\
&\quad +2\varepsilon^{ij}\bar\epsilon_i\psi_{j[\mu}W^A_{\nu]}\eta_{AB}X^B + \text{h.c.}\,,
\end{aligned} \tag{6.29}$$

and the first line of the fermionic variation is cast in a superconformally covariant form as follows:

$$\delta\lambda_i = \Big(\not{D}\phi - i\hat{\not{V}}^{(z)}\Big)\epsilon_i - \frac{i}{2X}\varepsilon_{ij}\sigma\cdot\Big(\mathscr{F}^-(V) - i\phi\mathscr{F}^{-0}\Big)\epsilon^j + 2\varepsilon_{ij}\bar X\phi^{(z)}\epsilon^j + \cdots\,, \tag{6.30}$$

(the dots represent the remaining terms in (6.24)). Interestingly enough, all extra covariantizations are implicitly contained in covariant derivatives and field strengths, except for the explicit gravitino fields in the variations of V_μ and $B_{\mu\nu}$. The latter terms could have been expected, as can be seen from the following argument: in the calculation of the supersymmetry commutator on the gauge fields V_μ and $B_{\mu\nu}$ based on the rigid transformation rules (6.24) we freely took derivatives over the supersymmetry parameters to extract gauge transformations like in (6.18) and (6.21). To arrive at the same result in the case of local supersymmetry, the above ${\psi_\mu}^i$-proportional terms are required and sufficient. The above supersymmetry variations lead to the following gauge parameters in the commutator of two supersymmetry transformations, *c.f.* (6.28):

$$\begin{aligned}
\theta^1(\epsilon_1,\epsilon_2) &= 4i\phi X\,\varepsilon^{ij}\bar\epsilon_{i\,2}\epsilon_{j\,1} + \text{h.c.}\,,\\
\Lambda_\mu(\epsilon_1,\epsilon_2) &= 2\bar\epsilon_2^i\gamma_\mu\epsilon_{i\,1}\,\bar X\Big(2\eta_{11}\phi^2 X + \phi\bar X X^I\partial_{\bar I}\bar g - 4iX^I\text{Re}\,[\partial_I(Xb)]\Big)\\
&\quad +2i\varepsilon^{ij}\bar\epsilon_{i\,2}\epsilon_{j\,1}X\Big(V_\mu(\eta_{11}\phi - g) - iW^0_\mu(\eta_{11}\phi^2 - \phi g - 4ib)\Big)\\
&\quad +2\varepsilon^{ij}\bar\epsilon_{i\,2}\epsilon_{j\,1}W^A_\mu\eta_{AB}X^B + \text{h.c.}\,.
\end{aligned} \tag{6.31}$$

We proceed with the definitions of a number of quantities that appear in (6.30) or are related to them. The supercovariant field strengths for the vector-tensor multiplet gauge fields are equal to:

$$\begin{aligned}
\mathscr{F}_{\mu\nu}(V) &= 2\partial_{[\mu}V_{\nu]} - 2W^0_{[\mu}V^{(z)}_{\nu]} + \tfrac{1}{4}i\phi\Big[\bar{X}T^{ij}_{\mu\nu}\varepsilon_{ij} - \text{h.c.}\Big] \\
&\quad -i\Big[\varepsilon^{ij}\bar{\psi}_{i\,[\mu}\gamma_{\nu]}\Big(2X\lambda_j + \phi\Omega^0_j\Big) + \phi X\varepsilon^{ij}\bar{\psi}_{\mu\,i}\psi_{\nu\,j} - \text{h.c.}\Big]\,, \\
H^\mu &= \tfrac{1}{2}ie^{-1}\varepsilon^{\mu\nu\lambda\sigma}\Big[\partial_\nu B_{\lambda\sigma} - \eta_{11}V_\nu\,\partial_\lambda V_\sigma - \eta_{1A}V_\nu\,\partial_\lambda W^A_\sigma \\
&\qquad -\eta_{AB}W^A_\nu\partial_\lambda W^B_\sigma - W^0_\nu\Big(B^{(z)}_{\lambda\sigma} + \eta_{11}V_\lambda V^{(z)}_\sigma\Big)\Big] \\
&\quad -\Big[i\bar{\psi}_\nu{}^i\sigma^{\mu\nu}\Big(2|X|^2\Big(2\eta_{11}\phi - \text{Re}\,[g]\Big)\lambda_i \\
&\qquad +\bar{X}\Big(2\eta_{11}\phi^2\Omega^0_i + \phi\bar{X}\partial_{\bar{I}}\bar{g}\,\Omega^I_i - 4i\text{Re}\,[\partial_I(Xb)]\Omega^I_i\Big)\Big) + \text{h.c.}\Big] \\
&\quad +\tfrac{1}{4}ie^{-1}\varepsilon^{\mu\nu\lambda\sigma}\bar{\psi}_\nu{}^i\gamma_\lambda\psi_{\sigma\,i}\Big[\bar{X}\Big(2\eta_{11}\phi^2X + \phi\bar{X}X^I\,\partial_{\bar{I}}\bar{g} - 4iX^I\,\text{Re}\,[\partial_I(Xb)]\Big) + \text{h.c.}\Big]\,.
\end{aligned} \tag{6.32}$$

The Bianchi identities corresponding to the field strengths (6.32) are straightforward to determine and read:

$$\begin{aligned}
&D_\mu\Big(\tilde{\mathscr{F}}^{\mu\nu}(V) + \tfrac{1}{4}i\phi(\bar{X}T^{\mu\nu\,ij}\varepsilon_{ij} + XT^{\mu\nu}_{ij}\varepsilon^{ij})\Big) \\
&\quad = -V^{(z)}_\mu\Big[\tilde{\mathscr{F}}^{0\,\mu\nu} - \tfrac{1}{4}(\bar{X}T^{ij\,\mu\nu}\varepsilon_{ij} - XT^{\mu\nu}_{ij}\varepsilon^{ij})\Big] - \tfrac{3}{4}i\Big[\varepsilon_{ij}\bar{\chi}^i\gamma^\nu(2\bar{X}\lambda^j + \phi\Omega^{j\,0}) + \text{h.c.}\Big]\,, \\
&D_\mu H^\mu \\
&\quad = -\tfrac{i}{4}\Big[\eta_{11}\,\tilde{\mathscr{F}}_{\mu\nu}(V)\,\mathscr{F}^{\mu\nu}(V) + \eta_{1A}\,\tilde{\mathscr{F}}_{\mu\nu}(V)\,\mathscr{F}^{\mu\nu\,A} + \eta_{AB}\,\tilde{\mathscr{F}}^A_{\mu\nu}\,\mathscr{F}^{\mu\nu\,B} + 2\tilde{\mathscr{F}}^0_{\mu\nu}\hat{B}^{\mu\nu\,(z)}\Big] \\
&\qquad -\tfrac{i}{16}\Big[T^{\mu\nu}_{ij}\Big(2\eta_{11}\,\phi X\,\mathscr{F}_{\mu\nu}(V) + \eta_{1A}(X^A\,\mathscr{F}_{\mu\nu}(V) + i\phi X\mathscr{F}^A_{\mu\nu}) + 2\eta_{AB}\,X^A\,\mathscr{F}^B_{\mu\nu} \\
&\qquad\quad +2X\hat{B}^{(z)}_{\mu\nu} + X\,\mathscr{F}^0_{\mu\nu}(\eta_{11}\phi^2 - \phi\,g - 4ib)\Big) - \text{h.c.}\Big] \\
&\qquad +3i(\bar{\lambda}_i\chi^i - \bar{\lambda}^i\chi_i)\,|X|^2(2\eta_{11}\phi - \text{Re}\,[g]) \\
&\qquad -\tfrac{3}{2}i\Big[X\,\bar{\chi}_i(2\eta_{11}\phi^2\Omega^{i\,0} + \phi X\partial_I g\Omega^{I\,i} + 4i\text{Re}\,[\partial_I(Xb)]\Omega^{I\,i}) - \text{h.c.}\Big]\,.
\end{aligned} \tag{6.33}$$

Furthermore, the following quantities appear in the above formulae, which are the supercovariant part of the z-transformed vector and tensor fields:

$$\begin{aligned}
\hat{V}^{(z)}_a &= -\frac{1}{2|X|^2(2\eta_{11}\phi - \text{Re}\,[g])} \\
&\qquad \times\Big(H_a - \Big[iXD_a\bar{X}^I\Big(2\eta_{11}\phi^2\delta_I{}^0 + \phi\bar{X}\partial_{\bar{I}}\bar{g} - 4i\text{Re}\,[\partial_I(Xb)]\Big) + \text{h.c.}\Big]\Big) \\
&\quad +\cdots\,, \\
\hat{B}^{(z)}_{ab} &= -\tfrac{1}{2}\text{Im}\,[g]\,\mathscr{F}_{ab}(V) + \tfrac{1}{2}i(2\eta_{11}\phi - \text{Re}\,[g])\tilde{\mathscr{F}}_{ab}(V) - \tfrac{1}{2}\phi(\eta_{11}\phi - \text{Re}\,[g])\mathscr{F}^0_{ab} \\
&\qquad +\tfrac{1}{2}i\phi\text{Im}\,[X\partial_I g]\tilde{\mathscr{F}}^I_{ab} + 4\text{Im}\Big[\partial_I(Xb)\mathscr{F}^{-\,I}_{ab}\Big] + \cdots\,,
\end{aligned} \tag{6.34}$$

where the dots indicate fermionic contributions. The hatted fields are fully covariant with respect to all local symmetries; they do not coincide with the image of V_μ and $B_{\mu\nu}$ under central-charge transformations, $V^{(z)}_\mu$ and $B^{(z)}_{\mu\nu}$. The latter are given by:

$$\begin{aligned}
V^{(z)}_\mu &= e_\mu{}^a\hat{V}^{(z)}_a + \tfrac{1}{2}\Big(i\bar{\psi}_\mu{}^i\lambda_i + \text{h.c.}\Big)\,, \\
B^{(z)}_{\mu\nu} &= e_\mu{}^{[a}e_\nu{}^{b]}\hat{B}^{(z)}_{ab} - \eta_{11}V_{[\mu}V^{(z)}_{\nu]}
\end{aligned}$$

$$+\tfrac{1}{2}\Big[X\varepsilon^{ij}(\bar\psi_{\mu\,i}\psi_{\nu\,j}+\tfrac{1}{4}T_{\mu\nu\,ij})(\eta_{11}\phi^2-\phi g-4ib)+2X\varepsilon^{ij}\bar\psi_{i\,[\mu}\gamma_{\nu]}\lambda_j\,(2\eta_{11}\phi-g)$$
$$+\varepsilon^{ij}\bar\psi_{i\,[\mu}\gamma_{\nu]}\Big(\eta_{11}\phi^2\Omega_j^0-i\eta_{1A}\phi\Omega_j^A-4i\partial_I(Xb)\Omega_j^I\Big)+\text{h.c.}\Big]\,. \tag{6.35}$$

There are of course similar expressions for $\lambda_i^{(z)}$ and $\phi^{(zz)}$, which are of less direct relevance. Because the fields ϕ and λ_i are themselves covariant, the action of the central charge will yield covariant expressions.

The results for the central-charge transformations are determined from the commutator:

$$[\,\delta_Q(\epsilon),\delta_z(z)\,]=\delta_{\text{vector}}\Big(iz\bar\epsilon^i\lambda_i+\text{h.c.}\Big)+\delta_{\text{tensor}}\Big(\Lambda_\mu(\epsilon,z)\Big)\,, \tag{6.36}$$

where:

$$\begin{aligned}\Lambda_\mu(\epsilon,z)\;&=\;\tfrac{1}{2}z\varepsilon^{ij}\bar\epsilon_i\gamma_\mu\Big(2X(2\eta_{11}\phi-g)\lambda_j+\eta_{11}\phi^2\Omega_j^0-i\eta_{1A}\phi\Omega_j^A-4i\partial_I(Xb)\Omega_j^I\Big)\\&\quad+z\varepsilon^{ij}\bar\epsilon_i\psi_{\mu\,j}X\Big(\eta_{11}\phi^2-\phi g-4ib\Big)+\tfrac{1}{2}iz\eta_{11}V_\mu\bar\epsilon^i\lambda_i+\text{h.c.}\,,\end{aligned} \tag{6.37}$$

which implies that the supersymmetry transformations of $\phi^{(z)}$, $\lambda_i^{(z)}$ are just the z-transformed versions of $\delta_Q\phi$, $\delta_Q\lambda_i$ as given in (6.24) and (6.30). Hence, with the exception of $\phi^{(z)}$ all the z-transformed fields are subject to constraints. By acting on these constraints with central-charge transformations, one recovers an infinite hierarchy of constraints. These relate the components of the higher multiplets ($\phi^{(z)}, V_\mu^{(z)}, B_{\mu\nu}^{(z)}, \lambda_i^{(z)}, \phi^{(zz)}$), etcetera to the lower ones, in such a way as to retain precisely $8+8$ independent degrees of freedom.

We close this section with a number of superconformal variations of various quantities defined above. The supercovariant field strengths transform as follows:

$$\begin{aligned}\delta\mathscr{F}_{ab}(V)\;&=\;-2i\varepsilon^{ij}\bar\epsilon_i\gamma_{[a}D_{b]}\Big(2X\lambda_j+\phi\Omega_j^0\Big)-2\varepsilon^{ij}\bar\epsilon_i\gamma_{[a}\Omega_j^0\,\hat V_{b]}^{(z)}-i\bar\epsilon^i\lambda_i\mathscr{F}_{ab}^0\\&\quad-2i\varepsilon^{ij}\bar\eta_i\sigma_{ab}\Big(2X\lambda_j+\phi\Omega_j^0\Big)+\text{h.c.}\,,\\
\delta H^a\;&=\;4i\bar\epsilon^i\sigma^{ab}D_b\Big[|X|^2\Big(2\eta_{11}\phi-\text{Re}\,[g]\Big)\lambda_i\Big]\\&\quad+2i\bar\epsilon^i\sigma^{ab}D_b\Big[\bar X\Big(2\eta_{11}\phi^2\Omega_i^0+\phi\bar X\partial_{\bar I}\bar g\,\Omega_i^I-4i\text{Re}\,[\partial_I(Xb)]\Omega_i^I\Big)\Big]\\&\quad+\tfrac{3}{2}i\bar\epsilon^i\gamma^a\chi_i\,\bar X\Big(2\eta_{11}\phi^2X+\phi\bar X\partial_{\bar I}\bar gX^I-4i\text{Re}\,[\partial_I(Xb)]X^I\Big)\\&\quad-\tfrac{1}{2}\varepsilon^{ij}\bar\epsilon_i\gamma_b\,\tilde{\mathscr{F}}^{ba}(V)\Big(2\eta_{11}(2X\lambda_j+\phi\Omega_j^0)-i\eta_{1A}\Omega_j^A\Big)\\&\quad+\tfrac{1}{2}i\varepsilon^{ij}\bar\epsilon_i\gamma_b\,\tilde{\mathscr{F}}^{ba\,0}\Big(2X(2\eta_{11}\phi-g)\lambda_j+\eta_{11}\phi^2\Omega_j^0-i\eta_{1A}\phi\Omega_j^A-4i\partial_I(Xb)\Omega_j^I\Big)\\&\quad+\tfrac{1}{2}i\varepsilon^{ij}\bar\epsilon_i\gamma_b\,\tilde{\mathscr{F}}^{ba\,A}\Big(i\eta_{1A}(2X\lambda_j+\phi\Omega_j^0)+2\eta_{AB}\Omega_j^B\Big)\\&\quad+i\varepsilon^{ij}\bar\epsilon_i\gamma_b\Omega_j^0\,\tilde{\hat B}{}^{(z)\,ba}\\&\quad-\tfrac{1}{4}i\bar\epsilon_i\gamma_b\,T^{ba\,ij}\Big[2|X|^2\Big(2\eta_{11}\phi-\text{Re}\,[g]\Big)\lambda_j\\&\qquad\qquad+\bar X\Big(2\eta_{11}\phi^2\Omega_j^0+\phi\bar X\partial_{\bar I}\bar g\,\Omega_j^I-4i\text{Re}\,[\partial_I(Xb)]\Omega_j^I\Big)\Big]\\&\quad+\tfrac{3}{2}i\bar\eta^i\gamma^a\Big[2|X|^2\Big(2\eta_{11}\phi-\text{Re}\,[g]\Big)\lambda_i\\&\qquad+\bar X\Big(2\eta_{11}\phi^2\Omega_i^0+\phi\bar X\partial_{\bar I}\bar g\,\Omega_i^I-4i\text{Re}\,[\partial_I(Xb)]\Omega_i^I\Big)\Big]+\text{h.c.}\,.\end{aligned} \tag{6.38}$$

Note that the variation of covariant higher z-transformed components can be obtained from the original variations by simply attaching superscript (z)'s. However, the variations of the non-covariant gauge fields V and B cannot be extended to higher-z level in a simple way. Therefore we give the variation of the covariant fields $\hat{V}_a^{(\mathrm{z})}$ and $\hat{B}_{ab}^{(\mathrm{z})}$:

$$\begin{aligned}
\delta\hat{V}_a^{(\mathrm{z})} &= i\varepsilon^{ij}\bar{\epsilon}_i\gamma_a\Big(2X\lambda_j+\phi\Omega_j^0\Big)^{(\mathrm{z})}+i\bar{\epsilon}^iD_a\lambda_i-\tfrac{1}{8}i\bar{\epsilon}_i\gamma_a\sigma\cdot T^{ij}\lambda_j-\tfrac{1}{2}i\bar{\eta}^i\gamma_a\lambda_i+\text{h.c.}\,,\\
\delta\hat{B}_{ab}^{(\mathrm{z})} &= -4\bar{\epsilon}^i\sigma_{ab}|X|^2\Big((2\eta_{11}\phi-\mathrm{Re}\,[g])\lambda_i\Big)^{(\mathrm{z})}\\
&\quad -2\bar{\epsilon}^i\sigma_{ab}\,\phi^{(\mathrm{z})}\bar{X}\Big(4\eta_{11}\phi\Omega_i^0+\bar{X}\partial_{\bar{I}}\bar{g}\,\Omega_i^I\Big)\\
&\quad -\varepsilon^{ij}\bar{\epsilon}_i\gamma_{[a}D_{b]}\Big(2X(2\eta_{11}\phi-g)\lambda_j+\eta_{11}\phi^2\Omega_j^0-i\eta_{1A}\phi\Omega_j^A-4i\partial_I(Xb)\Omega_j^I\Big)\\
&\quad +i\varepsilon^{ij}\bar{\epsilon}_i\gamma_{[a}\hat{V}_{b]}^{(\mathrm{z})}\Big(2\eta_{11}(2X\lambda_j+\phi\Omega_j^0)-i\eta_{1A}\Omega_j^A\Big)\\
&\quad +i\Big(\eta_{11}\mathscr{F}_{ab}(V)+\tfrac{1}{2}\eta_{1A}\mathscr{F}_{ab}^A\Big)\bar{\epsilon}^i\lambda_i\\
&\quad -\varepsilon^{ij}\bar{\eta}_i\sigma_{ab}\Big(2X(2\eta_{11}\phi-g)\lambda_j+\eta_{11}\phi^2\Omega_j^0-i\eta_{1A}\phi\Omega_j^A-4i\partial_I(Xb)\Omega_j^I\Big)+\text{h.c.}\,.
\end{aligned}\tag{6.39}$$

The same structure is repeated as one goes higher up in the central-charge hierarchy. It was already observed in [110] that the transformations of the higher-z fields involve objects both at the next and at the preceding level. The transformations of the basic vector-tensor fields as given in (6.24), (6.29) and (6.30) are special in this respect. They involve only the next level as there is no lower level. The consistency of this is ensured by the gauge transformations of the fields V_μ and $B_{\mu\nu}$, which allows for a truncation of the central-charge hierarchy from below.

6.5 Actions for vector-tensor multiplets

In this section we present the construction of invariant actions for the vector-tensor multiplet, using the multiplet calculus described in section 2.4. We start by constructing a general linear multiplet depending on the vector-tensor fields and the background vector-multiplet components. From this linear multiplet we construct the associated supergravity actions, using the density formula for the linear multiplet discussed in section 2.4. Their dual description in terms of vector multiplets alone is the issue of the following section.

In order to construct a linear multiplet from the components of the vector-tensor multiplet and the background vector multiplets, we first of all need an expression for the lowest component L_{ij}, which is an $\mathrm{SU}(2)_R$ triplet of weights $w=2$, $c=0$. The only vector-tensor component that transforms under $\mathrm{SU}(2)_R$ is the fermion λ_i. For the vector multiplets, only the fermions Ω_i^I and the auxiliary fields Y_{ij}^I transform non-trivially under $\mathrm{SU}(2)_R$. Therefore, the most general linear multiplet must be based on an L_{ij} of the following form:

$$\begin{aligned}
L_{ij} &= X\mathcal{A}\,\bar{\lambda}_i\lambda_j+\bar{X}\bar{\mathcal{A}}\,\varepsilon_{ik}\varepsilon_{jl}\bar{\lambda}^k\lambda^l+X\mathcal{B}_I\,\bar{\lambda}_{(i}\Omega_{j)}^I+\bar{X}\bar{\mathcal{B}}_{\bar{I}}\,\varepsilon_{ik}\varepsilon_{jl}\bar{\lambda}^{(k}\Omega^{l)I}\\
&\quad +\mathcal{C}_{IJ}\,\bar{\Omega}_i^I\Omega_j^J+\bar{\mathcal{C}}_{\bar{I}\bar{J}}\,\varepsilon_{ik}\varepsilon_{jl}\bar{\Omega}^{Ik}\Omega^{Jl}+\mathcal{G}_IY_{ij}^I\,,
\end{aligned}\tag{6.40}$$

where $\mathcal{A}$, $\mathcal{B}_I$, $\mathcal{C}_{IJ}$ and $\mathcal{G}_I$ are functions of ϕ, X^I and $\bar{X}^I$. In this section the index I does not take the value $I = 1$. In order that L_{ij} has weights $w = 2$ and $c = 0$, the functions $\mathcal{A}$ and $\mathcal{G}_I$ must have weights $w = c = 0$, while $\mathcal{B}_I$ and $\mathcal{C}_{IJ}$ have weights $w = -c = -1$. Obviously, the reality condition on L_{ij} requires that $\mathcal{G}_I$ be real. We suppress the superscript zeroes of the central-charge vector multiplet for the sake of clarity. We also expect the linear multiplet to transform only under the central charge and not under the gauge transformations associated with the other vector multiplets, but this is not important for most of the construction.

Requiring that L_{ij} transforms into a spinor doublet as indicated in (2.51), puts stringent requirements on each of the functions $\mathcal{A}(\phi, X^I, \bar{X}^I)$, $\mathcal{B}_I(\phi, X^I, \bar{X}^I)$, $\mathcal{C}_{IJ}(\phi, X^I, \bar{X}^I)$ and $\mathcal{G}_I(\phi, X^I, \bar{X}^I)$. These are encapsulated by a system of coupled first-order, linear differential equations, which are determined as follows. Upon varying (6.40) with respect to supersymmetry, one finds that the resulting three-fermion terms and terms involving Y_{ij}^I take the required form if and only if the following conditions are satisfied:

$$\begin{aligned}
\mathcal{E}\partial_\phi\mathcal{A} &= -4\eta_{11}\bar{\mathcal{A}}\,, & \mathcal{E}\partial_{\bar{I}}\mathcal{B}_J &= \bar{\mathcal{B}}_{\bar{I}}\partial_J g\,,\\
\mathcal{E}\partial_I\mathcal{A} &= (\mathcal{A}+\bar{\mathcal{A}})\partial_I g + 2\eta_{11}\mathcal{B}_I\,, & \mathcal{E}\partial_\phi\mathcal{C}_{IJ} &= 2i\bar{\mathcal{A}}\partial_I\partial_J(Xb)\,,\\
\mathcal{E}\partial_{\bar{I}}\mathcal{A} &= -2\eta_{11}\bar{\mathcal{B}}_{\bar{I}}, & \mathcal{E}\partial_{\bar{K}}\mathcal{C}_{IJ} &= i\bar{\mathcal{B}}_{\bar{K}}\partial_I\partial_J(Xb)\,,\\
\mathcal{E}\partial_\phi\mathcal{B}_I &= 2\bar{\mathcal{A}}\partial_I g\,, & \partial_\phi\mathcal{G}_I &= -X\mathcal{B}_I - 2\mathcal{A}P_I\,,\\
\partial_{(I}(X^2\mathcal{E}\mathcal{B}_{J)}) &= 4i(\mathcal{A}+\bar{\mathcal{A}})X\partial_I\partial_J(Xb)\,, & \partial_I\mathcal{G}_J &= -2\mathcal{C}_{IJ} - \mathcal{B}_I P_J\,,
\end{aligned} \tag{6.41}$$

where g and b were defined in (6.24) and:

$$\begin{aligned}
\mathcal{E} &= -4\eta_{11}\phi + g + \bar{g}\,,\\
P_I &= -\tfrac{1}{2}\phi\delta_I{}^0 - i\mathcal{E}^{-1}\mathrm{Im}\Big(\phi X\partial_I g + 4i\partial_I(Xb)\Big)\,.
\end{aligned} \tag{6.42}$$

Furthermore, the reality condition on L_{ij} requires that $\mathcal{G}_I$ be real. It is satisfying that the system of equations (6.41) turns out to be integrable, despite its complexity. After some work, one can prove that the general solution decomposes as a linear combination of three distinct solutions, each with an independent physical interpretation. The most interesting of these is given by:

$$\begin{aligned}
[\mathcal{A}]_1 &= \eta_{11}(\phi+i\zeta) - \tfrac{1}{2}g\,,\\
[\mathcal{B}_I]_1 &= -\tfrac{1}{2}(\phi+i\zeta)\partial_I g - 2i\partial_I b\,,\\
[\mathcal{C}_{IJ}]_1 &= -\tfrac{1}{2}i(\phi+i\zeta)\partial_I\partial_J(Xb)\,,\\
[\mathcal{G}_I]_1 &= \mathrm{Re}\Big[(\tfrac{1}{3}\eta_{11}(\phi+i\zeta)^3 - \tfrac{1}{2}i\zeta(\phi+i\zeta)g)\delta_I{}^0 + \tfrac{1}{2}(\phi+i\zeta)X\partial_I(g\phi+4ib)\Big]\,,
\end{aligned} \tag{6.43}$$

where:

$$\zeta(\phi, X^I, \bar{X}^I) = \frac{\mathrm{Im}\,[\phi g + 4ib]}{2\eta_{11}\phi - \mathrm{Re}\,[g]}\,. \tag{6.44}$$

In terms of the action, which will be discussed shortly, this solution provides the couplings which involve the vector-tensor fields. The remaining two solutions, which we discuss next, give rise either to a total divergence or to interactions which involve only the background fields. The second solution takes the form:

$$\begin{aligned} [\mathcal{A}]_2 &= i\eta_{11}\zeta' - i\alpha\,, \\ [\mathcal{B}_I]_2 &= -\tfrac{1}{2}i\zeta'\partial_I g - 2i\partial_I\gamma\,, \\ [\mathcal{C}_{IJ}]_2 &= \tfrac{1}{2}\zeta'\partial_I\partial_J(Xb)\,, \\ [\mathcal{G}_I]_2 &= \mathrm{Re}\Big[2iX\phi\partial_I\gamma + \tfrac{i}{2}\zeta' X\phi\partial_I g - 2\zeta'\partial_I(Xb)\Big]\,, \end{aligned} \tag{6.45}$$

where $\gamma = \frac{1}{4}i\alpha_A X^A/X$ is a holomorphic homogeneous function of the background scalars X^A and X^0; α and α_A are arbitrary real parameters. Furthermore:

$$\zeta'(\phi, X^I, \bar{X}^I) = \frac{2\alpha\phi + 4\mathrm{Re}\,[\gamma]}{2\eta_{11}\phi - \mathrm{Re}\,[g]}\,. \tag{6.46}$$

Note that this solution could be concisely included into the first solution by redefining $g \to g + 2i\alpha$ and $b \to b + \gamma$. In fact, this second solution indicates that the functions g and b are actually defined modulo these shifts. In terms of the action, this ambiguity is analogous to the shift of the theta-angle in an ordinary Yang-Mills theory.

The third and final solution is given by:

$$\begin{aligned} [\mathcal{A}]_3 &= 0\,, \\ [\mathcal{B}_I]_3 &= 0\,, \\ [\mathcal{C}_{IJ}]_3 &= -\tfrac{1}{8}i\partial_I\partial_J(f(X)/X)\,, \\ [\mathcal{G}_I]_3 &= -\tfrac{1}{2}\mathrm{Im}\,[\partial_I(f(X)/X)]\,. \end{aligned} \tag{6.47}$$

Where $f(X)$ is a holomorphic function of X^0 and X^A, homogeneous of degree 2. In terms of the action, this solution corresponds to interactions amongst the background vector multiplets alone. Note that it corresponds to the alternative derivation of the vector-multiplet action, alluded to with formula (3.7). The function $f(X)$ provides the holomorphic prepotential.

All solutions have in common that they are homogeneous functions of X^I and $\bar{X}^I$: $\mathcal{A}$ and $\mathcal{G}_I$ are of degree 0 and $\mathcal{B}_I$ and $\mathcal{C}_{IJ}$ are of degree -1. This is a result of the fact that the field ϕ has $w = 0$. Furthermore we note the identities:

$$X^I\,\mathcal{B}_I = X^I\,\mathcal{C}_{IJ} = 0\,, \tag{6.48}$$

which ensure that L_{ij} is invariant under S-supersymmetry, in accordance with (2.51).

Now that we have determined the scalar triplet L_{ij}, in terms of the specific functions $\mathcal{A}(\phi, X^I, \bar{X}^I)$, $\mathcal{B}_I(\phi, X^I, \bar{X}^I)$, $\mathcal{C}_{IJ}(\phi, X^I, \bar{X}^I)$, and $\mathcal{G}_I(\phi, X^I, \bar{X}^I)$ given above, we can generate the remaining components of the linear multiplet, φ_i, G, and E_μ by varying (6.40) with respect to supersymmetry. Given the complexity of the transformation rule for λ_i found in (6.30) and (6.24), it is clear that a fair amount of work is involved in carrying out this process. However, since our prime interest goes to the bosonic part of the action, we can limit ourselves to the bosonic part of E_a and G, viz. (2.56).

The higher components of the linear multiplet are then given by:

$$
\begin{aligned}
\varphi^i &= -\bar X(\not{D}\phi + i\hat{\not{V}}^{(z)})(\bar{\mathcal{A}}\lambda^i + \tfrac12\bar{\mathcal{B}}_{\bar I}\Omega^{I\,i}) + \mathcal{G}_I\not{D}\Omega^{I\,i} \\
&\quad -\tfrac{i}{2}\varepsilon^{ij}\sigma\cdot(\mathscr{F}(V) - i\phi\mathscr{F}^0)(\mathcal{A}\lambda_j + \tfrac12\mathcal{B}_I\Omega^I_j) \\
&\quad +\tfrac12\varepsilon^{ij}\sigma\cdot\mathscr{F}^I(X\mathcal{B}_I\lambda_j + 2\mathcal{C}_{IJ}\Omega^J_j) \\
&\quad -\not{D}\bar X^I(\bar X\bar{\mathcal{B}}_{\bar I}\lambda^i + 2\bar{\mathcal{C}}_{\bar I\bar J}\Omega^{J\,i}) \\
&\quad -|X|^2\phi^{(z)}\varepsilon^{ij}(2\mathcal{A}\lambda_j + \mathcal{B}_I\Omega^I_j) \\
&\quad +\tfrac12 Y^{I\,ij}\Big((\partial_\phi\mathcal{G}_I)\lambda_j + (\partial_J\mathcal{G}_I)\Omega^J_j\Big) + 3\,\text{fermion terms}\,, \\
G &= \bar X\bar{\mathcal{A}}\,(D_a\phi + i\hat V_a^{(z)})(D^a\phi + i\hat V^{a\,(z)}) \\
&\quad +2\bar X\bar{\mathcal{B}}_{\bar I}\,D_a\bar X^I(D^a\phi + i\hat V^{a\,(z)}) \\
&\quad +4\bar{\mathcal{C}}_{\bar I\bar J}\,D_a\bar X^I\,D^a\bar X^J - 2\mathcal{G}_I\,D_aD^a\bar X^I \\
&\quad +\frac{1}{4X}(\mathscr{F}(V)^- - i\phi\mathscr{F}^{0\,-})_{ab}(\mathcal{A}(\mathscr{F}(V)^- - i\phi\mathscr{F}^{0\,-}) + 2iX\mathcal{B}_I\mathscr{F}^{I\,-})^{ab} \\
&\quad -\mathcal{C}_{IJ}\mathscr{F}^{I\,-}_{ab}\mathscr{F}^{J\,-\,ab} - 4\bar X|X|^2\mathcal{A}(\phi^{(z)})^2 \\
&\quad -\tfrac14(\partial_{(I}\mathcal{G}_{J)} + X^{-1}P_{(I}\,\partial_\phi\mathcal{G}_{J)})\,Y^I_{ij}Y^{J\,ij} \\
&\quad -\tfrac12\mathcal{G}_I\,\mathscr{F}^{I\,+}_{ab}\,T^{ab}_{ij}\varepsilon^{ij} + \text{fermion terms}\,, \\
E_a &= \mathrm{Re}\Big[-4|X|^2\phi^{(z)}(\bar{\mathcal{A}}\,(D_a\phi + i\hat V_a^{(z)}) + \mathcal{B}_I\,D_aX^I) \\
&\qquad -2i(D^b\phi + i\hat V^{(z)\,b})(\mathcal{A}\,(\mathscr{F}(V)^-_{ab} - i\phi\mathscr{F}^{0\,-}_{ab}) + iX\mathcal{B}_I\mathscr{F}^{I\,-}_{ab}) \\
&\qquad -2D^bX^I(i\mathcal{B}_I\,(\mathscr{F}(V)^-_{ab} - i\phi\mathscr{F}^{0\,-}_{ab}) - 4\mathcal{C}_{IJ}\,\mathscr{F}^{J\,-}_{ab}) \\
&\qquad -2\mathcal{G}_I\,D^b(\mathscr{F}^{I\,-}_{ab} - \tfrac14\bar X^IT^{ij}_{ab}\varepsilon_{ij})\Big] + \text{fermion terms}\,.
\end{aligned}
\tag{6.49}
$$

The appearance of terms containing T^{ij}_{ab} may seem strange because this field does not appear in the transformation rules for λ_i and Ω_i. However, this field appears in the variation of $\not{D}\Omega_i$ and in the Bianchi identities for $\mathscr{F}^I_{ab}$, which have to be used to obtain G and E_a. Having derived the complete linear multiplet we turn to the construction of the action.

Since this linear multiplet (6.49) transforms under the central charge, we have to use the central-charge vector multiplet in the density formula, as explained in section 2.4. This yields an action that is both invariant under local superconformal symmetries and local gauge transformations. Carrying out this calculation we note the following terms in the Lagrangian density:

$$
\mathscr{L} = 4e\,\bar X\mathcal{C}_{IJ}\,D^aX^ID_aX^J - 2e\,\mathcal{G}_IX\,\Box_C\bar X^I\cdots\,,
\tag{6.50}
$$

which we rewrite by first splitting off the $f_\mu{}^\mu$-proportional part, analogous to the manipulations leading to (2.59), and then performing a partial integration. The latter step involves derivatives of the function $\mathcal{G}_I$, for which we substitute the differential equations (6.41). Afterwards, the bosonic terms of the full action read:

$$
\begin{aligned}
e^{-1}\mathscr{L} &= -2\mathcal{G}_I\,X\bar X^I(\tfrac16R - D) \\
&\quad +|X|^2\mathcal{A}\,(\partial_\mu\phi - i\hat V_\mu^{(z)})^2 + 2|X|^2\mathcal{B}_I\,D^\mu X^I(\partial_\mu\phi - i\hat V_\mu^{(z)})
\end{aligned}
$$

$$
\begin{aligned}
&-4X\mathcal{C}_{IJ}\,D^\mu X^I D_\mu \bar{X}^J - 2\bar{X}(X\mathcal{B}_I + \mathcal{A}\,P_I)\partial_\mu\phi\,D^\mu X^I \\
&-2X(\mathcal{B}_I\,P_J D_\mu X^I + \bar{\mathcal{B}}_{\bar{I}}\,\bar{P}_{\bar{J}}\,D_\mu \bar{X}^I)\,D^\mu \bar{X}^J + 2\mathcal{G}_I\,D_\mu X\,D^\mu \bar{X}^I \\
&+\mathcal{A}\,(\mathscr{F}(V)^{-\,\mu\nu} - i\phi\mathscr{F}^{-\,0\,\mu\nu})\Big(\tfrac{1}{4}(\mathscr{F}(V)^-_{\mu\nu} - i\phi\mathscr{F}^{-\,0}_{\mu\nu}) + iW^0_\mu(\partial_\nu\phi - i\hat{V}^{(\mathrm{z})}_\nu)\Big) \\
&+iX\mathcal{B}_I\,\mathscr{F}^{-\,I\,\mu\nu}\Big(\tfrac{1}{2}(\mathscr{F}(V)^-_{\mu\nu} - i\phi\mathscr{F}^{-\,0}_{\mu\nu}) + iW^0_\mu(\partial_\nu\phi - i\hat{V}^{(\mathrm{z})}_\nu)\Big) \\
&+i\mathcal{B}_I\,(\mathscr{F}(V)^{-\,\mu\nu} - i\phi\mathscr{F}^{-\,\mu\nu})W^0_\mu D_\nu X^I \\
&-\mathcal{C}_{IJ}\,\mathscr{F}^{I\,-\,\mu\nu}\Big(X\mathscr{F}^{J\,-}_{\mu\nu} + 4W^0_\mu D_\nu X^J\Big) \\
&-|X|^2\mathcal{A}\,(W^0_\mu\,W^{\mu\,0} + 4|X|^2)(\phi^{(\mathrm{z})})^2 \\
&-\tfrac{1}{4}(X\,\partial_{(I}\mathcal{G}_{J)} + P_{(I}\,\partial_\phi\mathcal{G}_{J)})Y^I_{ij}Y^{J\,ij} - \tfrac{1}{4}\mathcal{G}_I\,Y^0_{ij}Y^{I\,ij} \\
&-\tfrac{1}{2}\mathcal{G}_I X\mathscr{F}^{I\,+}_{ab}\,T^{ab}_{ij}\varepsilon^{ij} + \mathcal{G}_I\,W^0_a\,D_b(\mathscr{F}^{-\,I\,ab} - \tfrac{1}{4}\bar{X}^I T^{ab\,ij}\varepsilon_{ij}) + \text{h.c.}\,,
\end{aligned}
\tag{6.51}
$$

where we have made the terms proportional to W^0_μ in the covariant derivatives explicit. The above result represents general couplings of a vector-tensor multiplet to a background of n vector multiplets. Note that each term involves a factor of the functions $\mathcal{A}(\phi,\, X^I,\, \bar{X}^I)$, $\mathcal{B}_I(\phi,\, X^I,\, \bar{X}^I)$, $\mathcal{C}_{IJ}(\phi,\, X^I,\, \bar{X}^I)$, $\mathcal{G}_I(\phi,\, X^I,\, \bar{X}^I)$ or $P_I(\phi,\, X^I,\, \bar{X}^I)$, for which explicit solutions have been given in (6.43), (6.45) and (6.47). These solutions describe the local couplings of the vector-tensor multiplet components, a total derivative and the self-interactions of the background, respectively. As a result of this, the Lagrangian (6.51) can be written as a sum of three analogous pieces: a vector-tensor piece, a total-derivative piece and a background piece.

This form of the action would be a suitable starting point to consider the breaking of superconformal gravity into Poincaré gravity. As was indicated in section 2.4, an additional compensator *e.g.* a hypermultiplet would be needed to be able to define a gauge for the chiral $\mathrm{SU}(2)_R$ and to obtain a consistent field equation for the field D. Moreover, the fermionic terms in the action would be needed to fix the gauge of S-supersymmetry and to obtain a decomposition rule like (2.64). The derivation then proceeds completely analogous to the case described in [32]. Some additional remarks concerning the gauge-equivalent Poincaré action can be found in [120]. However, since we are more interested in the duality with a model describing only vector multiplets, we leave the matter as a possible future endeavour.

So far we have considered the action based on the transformation rules with general Chern-Simons coefficients η_{11}, η_{1A} and η_{AB}. However, as we have seen at the end of section 6.3, one can always distinguish between the non-linear and the linear vector-tensor multiplet. It is therefore instructive to specify the functions involved in the definition of the Lagrangian density (6.51) in these two cases.

The non-linear vector-tensor multiplet:
As described above, when the parameter η_{11} does not vanish, we can take $\eta_{11} = 1$ and $\eta_{1A} = 0$ without loss of generality. In this case the functions $\mathcal{A}(\phi, X^I, \bar{X}^I)$, $\mathcal{B}_I(\phi, X^I, \bar{X}^I)$, $\mathcal{C}_{IJ}(\phi, X^I, \bar{X}^I)$, and $\mathcal{G}_I(\phi, X^I, \bar{X}^I)$ which define the linear multiplet and, more importantly,

the vector-tensor Lagrangian (6.51) are given by the following expressions:

$$\begin{aligned} \mathcal{A} &= \phi + i\phi^{-1}(b+\bar{b})\,, \\ \mathcal{B}_I &= -2i\partial_I b\,, \\ \mathcal{C}_{IJ} &= -\tfrac{1}{2}i(\phi + i\phi^{-1}(b+\bar{b})\partial_I\partial_J(Xb) - \tfrac{1}{8}i\partial_I\partial_J(X^{-1}f)\,, \\ \mathcal{G}_I &= \mathrm{Re}\Big[\tfrac{1}{3}\phi^3\,\delta_I{}^0 + 2i\phi X\partial_I b - 2\phi^{-1}(b+\bar{b})\,\partial_I(Xb)\Big] - \tfrac{1}{2}\mathrm{Im}\,[\partial_I(X^{-1}f)]\,. \end{aligned} \tag{6.52}$$

For the sake of clarity, we have absorbed the parameters α and α_A into the functions b and g in the manner described immediately after equation (6.46). Substituting these functions in the Lagrangian (6.51), it is easy to see that the action contains, besides the total derivative and terms that depend only on the background vector multiplet fields, a cubic part and a linear part in vector-tensor fields.

The linear vector-tensor multiplet:
As described previously, if $\eta_{11} = 0$, implying the absence of the $V\wedge \mathrm{d}V$ Chern-Simons coupling, we obtain the linear vector-tensor multiplet. In this case, it is not possible to perform a field redefinition to remove all of the η_{1A} parameters. The functions $\mathcal{A}(\phi, X^I, \bar{X}^I)$, $\mathcal{B}_I(\phi, X^I, \bar{X}^I)$, $\mathcal{C}_{IJ}(\phi, X^I, \bar{X}^I)$, and $\mathcal{G}_I(\phi, X^I, \bar{X}^I)$ which define the linear multiplet and the vector-tensor Lagrangian (6.51) are now given by the following expressions:

$$\begin{aligned} \mathcal{A} &= -\tfrac{1}{2}g\,, \\ \mathcal{B}_I &= -\frac{1}{g+\bar{g}}\Big(\phi\bar{g}\partial_I g + 2i(g+\bar{g})\overleftrightarrow{\partial}_I(b+\bar{b})\Big), \\ \mathcal{C}_{IJ} &= -\frac{1}{g+\bar{g}}\Big(i\phi\bar{g} + 2(b+\bar{b})\Big)\partial_I\partial_J(Xb) - \tfrac{1}{8}i\partial_I\partial_J(X^{-1}f)\,, \\ \mathcal{G}_I &= \frac{1}{g+\bar{g}}\mathrm{Re}\Big[\phi\bar{g}X\partial_I(\phi g + 4ib) - 2i(b+\bar{b})\partial_I(X(\phi g + 4ib))\Big]. \end{aligned} \tag{6.53}$$

As above, for the sake of clarity we have absorbed the parameters α and α_A into the functions b and g in the manner described immediately after equation (6.46). Substituting these functions into the Lagrangian (6.51), one obtains a Lagrangian that contains, besides the total derivative terms and a part that depends exclusively on the background mentioned above, a quadratic part and a linear part in vector-tensor fields.

6.6 Dual vector-multiplet actions

As we already mentioned in the introduction, a vector-tensor multiplet is classically equivalent to a vector multiplet. The theory which we have presented, involving one vector-tensor multiplet and n vector multiplets is classically equivalent to a theory involving $n+1$ vector multiplets. Since these latter theories are well understood, it is of interest to determine which subset of vector-multiplet models is classically equivalent to vector-tensor models. Furthermore, low-energy effective string Lagrangians with $N = 2$ supersymmetry are usually described in terms of vector multiplets, such that by going to the vector multiplet language one can more easily verify which (if any) string theories are described by the vector-tensor multiplets we constructed above. As we have seen in chapter 3, the scalar fields of $N=2$ vector multiplets take their values in special Kähler spaces. For the case of effective Lagrangians corresponding to heterotic $N=2$ supersymmetric string compactifications, this space must contain, at least at weak string coupling, an $\mathrm{SU}(1,1)/\mathrm{U}(1)$ coset factor parameterized in terms of the complex scalar corresponding to the axion/dilaton complex. According to a well-known theorem [121] this uniquely specifies the special Kähler space.

We will find that the special Kähler space parameterized by the dual vector multiplets fails to exhibit the $\mathrm{SU}(1,1)/\mathrm{U}(1)$ factor, at least if one insists that it is the vector-tensor scalar and the scalar that arises after dualization of the tensor field that parameterize this subspace [110, 111]. Therefore it is impossible to associate this scalar and the tensor field with the (perturbative) heterotic dilaton-axion complex. However, they do play a natural role in the description of the non-perturbative heterotic string effects we alluded to in the introduction of this chapter.

One goes about constructing the dual vector-multiplet formulation in the usual manner, by introducing a Lagrange multiplier field a, which, upon integration, enforces the Bianchi identity on the field strength H_μ. The relevant term to add to the Lagrangian is therefore:

$$
\begin{aligned}
e^{-1}\mathscr{L}(a) &= a\,D_\mu H^\mu \\
&\quad +\tfrac{1}{4}ia\Big(\eta_{11}\,\tilde{\mathscr{F}}_{\mu\nu}(V)\,\mathscr{F}^{\mu\nu}(V)+\eta_{1A}\,\tilde{\mathscr{F}}_{\mu\nu}(V)\,\mathscr{F}^{\mu\nu\,A}+\eta_{AB}\,\tilde{\mathscr{F}}^{A}_{\mu\nu}\,\mathscr{F}^{\mu\nu\,B}+2\tilde{\mathscr{F}}^{0}_{\mu\nu}\hat{B}^{\mu\nu\,(\mathrm{z})}\Big) \\
&\quad +\tfrac{1}{16}ia\Big(T^{\mu\nu}_{ij}\Big(2\eta_{11}\,\phi X\,\mathscr{F}_{\mu\nu}(V)+\eta_{1A}(X^A\,\mathscr{F}_{\mu\nu}(V)+i\phi X\mathscr{F}^{A}_{\mu\nu})+2\eta_{AB}\,X^A\,\mathscr{F}^{B}_{\mu\nu} \\
&\qquad\qquad +2X\hat{B}^{(\mathrm{z})}_{\mu\nu}+X\,\mathscr{F}^{0}_{\mu\nu}(\eta_{11}\phi^2-\phi\,g-4ib)\Big)-\mathrm{h.c.}\Big)\,.
\end{aligned}
\tag{6.54}
$$

Note that we dropped the explicit fermionic terms, as we will do in the remainder of this section. Including the Lagrange multiplier term, we treat H_μ as unconstrained and integrate it out in the action, thereby trading the single on-shell degree of freedom represented by $B_{\mu\nu}$ for the real scalar a. Doing this, we obtain a dual theory involving only vector multiplets. To perform these operations, it is instructive to note that all occurrences of H_μ in (6.51) and (6.54) are most conveniently written in terms of $\hat{V}^{(\mathrm{z})}_\mu$, which can be done using (6.34). Because we are suppressing the fermions in what follows, we will henceforth drop the hat on $V^{(\mathrm{z})}_\mu$. All such terms can then be collected, and written as follows:

$$
\mathscr{L}(V^{(\mathrm{z})}_\mu)=\tfrac{1}{4}e(2\eta_{11}\phi-\mathrm{Re}\,[g])\Big(W^{0\,\mu}W^{0\,\nu}-(W^0_\lambda W^{0\,\lambda}+4|X|^2)g^{\mu\nu}\Big)\Big(V^{(\mathrm{z})}_\mu V^{(\mathrm{z})}_\nu-2V^{(\mathrm{z})}_\mu\partial_\nu(a-\zeta)\Big)\,,
\tag{6.55}
$$

where ζ was defined in (6.44). It is interesting how the terms involving $V^{(\mathrm{z})}_\mu$ factorize into the form given in (6.55). The equation of motion for H_μ is conveniently written in terms of $V^{(\mathrm{z})}_\mu$, which follows immediately from (6.55). It is given by the following simple expression:

$$
V^{(\mathrm{z})}_\mu=\partial_\mu(a-\zeta)\,.
\tag{6.56}
$$

We also impose the equations of motion for the auxiliary fields, $\phi^{(z)} = Y_{ij}^I = 0$ (up to fermionic terms). After substituting these solutions, we manipulate the result into the form (3.13) for the bosonic part of the vector-multiplet Lagrangian density, conveniently rescaled by a factor $-1/2$:

$$\begin{aligned} e^{-1}\mathscr{L} &= \tfrac{1}{2}i(F_I\bar{X}^I - X^I\bar{F}_I)\Big(-\tfrac{1}{6}R + D\Big) + \tfrac{1}{2}i\Big(\mathcal{D}_\mu F_I\,\mathcal{D}^\mu\bar{X}^I - \mathcal{D}_\mu X^I\,\mathcal{D}^\mu\bar{F}_I\Big) \\ &\quad -\tfrac{1}{8}i\bar{F}_{IJ}F_{\mu\nu}^{+I}F^{+\mu\nu J} - \tfrac{1}{16}i(F_I - X^J\bar{F}_{JI})F_{\mu\nu}^{+I}T_{ij}^{\mu\nu}\varepsilon^{ij} \\ &\quad +\tfrac{1}{128}i(F_I - X^J\bar{F}_{JI})X^I\Big(T_{\mu\nu\,ij}\varepsilon^{ij}\Big)^2 + \text{h.c.}\,, \end{aligned} \tag{6.57}$$

characterized by a prepotential that depends on *all* the scalar fields, $F = F(X^0, X^1, X^A)$. The natural bosonic components in the dual theory are found to be:

$$\begin{aligned} X^1 &= X^0\Big((a-\zeta) + i\phi\Big)\,, \\ W_\mu^1 &= V_\mu + (a-\zeta)W_\mu^0\,, \end{aligned} \tag{6.58}$$

and one can check that these transform as components of a common vector multiplet. For the general case, the dual theory obtained in this manner is described by the following prepotential:

$$\begin{aligned} F(X^0, X^1, X^A) &= -\frac{1}{X^0}\Big(\tfrac{1}{3}\eta_{11}X^1X^1X^1 + \tfrac{1}{2}\eta_{1A}X^1X^1X^A + \eta_{AB}X^1X^AX^B\Big) \\ &\quad -\alpha X^1X^1 + \alpha_A X^1X^A + f(X^0, X^A)\,. \end{aligned} \tag{6.59}$$

The quadratic terms proportional to α and α_A (defined in section 6.5) give rise to total derivatives since their coefficients are real. The term involving the function $f(X^0, X^A)$ represents the self-interactions of the background vector multiplets. The first three terms in (6.59) encode the couplings of the erstwhile vector-tensor fields, ϕ and a, and it is these that we are most interested in. As mentioned above, it is relevant to investigate whether the Kähler space described by this prepotential function can contain an $\mathrm{SU}(1,1)/\mathrm{U}(1)$ factor parameterized by the field X^1/X^0. According to the theorem of [121], this requires that X^1/X^0 appears linearly in the prepotential. This is obviously not the case for (6.59), as we have quadratic and cubic terms which cannot be removed by absorbing some of the other fields into the would-be dilaton field X^1/X^0. As discussed earlier, the best one can do is to remove *either* η_{11} or η_{1A}. We recall that these parameters are related to the Chern-Simons couplings of the tensor field in the dual formulation. The obstruction to removing the unwanted terms in the prepotential derives from the inability to formulate an interacting off-shell vector-tensor theory without any such Chern-Simons couplings.

In the present supergravity context it is important to note that the duality transformation we just described, does not interfere with the fields of the Weyl multiplet. This can be seen by noting that (6.55), (6.56) and (6.58) are completely identical to the relations found in [111] in the rigid supersymmetric case. This implies that the Weyl multiplet is not involved in the duality transformation and can be kept off-shell. The vector multiplets are not realized off-shell after the duality transformation, but the auxiliary fields Y_{ij}^I can be reinstated afterwards. In this respect it is instructive to compare our results to the analysis performed in [105]. Here the most general vector-multiplet theories admitting a (reverse) dualization into an antisymmetric tensor theory, were considered. They were found to precisely comprise the cases described here, plus the $\eta_{11} = 0$, $\eta_{1A} = 0$ case which is relevant for weakly coupled heterotic strings. However, in this last case the dualization into an antisymmetric tensor theory can no longer be carried out with the Weyl multiplet as a spectator. In particular, one is forced to first eliminate the U(1) chiral gauge field A_μ, which in the Poincaré theory plays the role of an auxiliary field.

Irrespective of these considerations, we note that the results we obtained in this article are a concise description of two very different situations. As described in detail in section 6.3, depending on whether the parameter η_{11} is zero or non-zero, the theory takes on very distinct characters. It is instructive then, to summarize our results independently for each of these two cases.

For the non-linear vector-tensor multiplet, we obtain a dual description involving only vector multiplets, characterized by the following holomorphic prepotential:

$$F = -\frac{X^1}{X^0}\Big(\tfrac{1}{3}\eta_{11}X^1X^1 + \eta_{AB}X^AX^B\Big) - \alpha X^1X^1 + \alpha_A X^1X^A + f(X^0, X^A)\,. \tag{6.60}$$

As already mentioned above, the quadratic terms proportional to α and α_A represent total derivatives, and the last term involves the background self-interactions. Notice that in this case the prepotential is cubic in X^1. No higher-dimensional tensor theory is known that gives rise to this coupling.

For the linear vector-tensor multiplet the dual description in terms of only vector multiplets is characterized by the prepotential:

$$F = -\frac{X^1}{X^0}\Big(\tfrac{1}{2}\eta_{1A}X^1X^A + \eta_{AB}X^AX^B\Big) - \alpha X^1X^1 + \alpha_A X^1X^A + f(X^0, X^A)\,. \tag{6.61}$$

Again, as discussed above, the quadratic terms involving α and α_A represent total derivatives, while the last term involves the background self-interactions. Notice that in this case the prepotential has a term quadratic in X^1, which cannot be suppressed. Such a term also arises from the reduction of six-dimensional tensor multiplets to four dimensions. In that case, the presence of the quadratic term is inevitable, because it originates from the kinetic term of the tensor field [108]. Observe that we have at least three abelian vector fields coupling to the vector-tensor multiplet, namely W_μ^0, W_μ^1 and $\eta_{1A}W_\mu^A$.

The work presented in this chapter represents an exhaustive analysis of the $N = 2$ vector-tensor multiplet coupled to (conformal) supergravity and a number of background vector multiplets. One of these vector multiplets provides the gauge field that couples to the central charge. Although we considered only a single vector-tensor multiplet, our methods can be applied straightforwardly to theories where several of these multiplets are present. We have presented the complete and general superconformal transformation rules in this context, and have shown that these actually include two distinct cases, one of which is non-linear in the vector-tensor components, and the other of which is linear. The difference between these two cases is encoded in the coefficients of the Chern-Simons couplings, denoted by η_{IJ}. Furthermore we have constructed a supersymmetric action for this system and exhibited its bosonic part. The dual descriptions in terms of vector multiplets have been obtained and the respective prepotentials determined.

Appendix A

Notation and conventions

Throughout this thesis we use $\mu, \nu, \cdots = 0,1,2,3$ to denote curved indices, and $a, b, \cdots = 0,1,2,3$ for local Lorentz indices. Our (anti)symmetrizations are always with weight one, so *e.g.*:

$$[ab] = \tfrac{1}{2}(ab - ba)\,, \qquad (ab) = \tfrac{1}{2}(ab + ba)\,. \tag{A.1}$$

We take gamma-matrices, such that:

$$\gamma_a \gamma_b = \eta_{ab} + 2\sigma_{ab}\,, \qquad \gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3\,, \tag{A.2}$$

where η_{ab} is of signature $(-+++)$. Furthermore, a charge conjugation matrix C is defined, such that:

$$-\gamma_\mu^T = C\gamma_\mu C^{-1}, \qquad \gamma_5^T = C\gamma_5 C^{-1}, \qquad C^T = -C. \tag{A.3}$$

The completely antisymmetric tensor satisfies:

$$\varepsilon^{abcd} = e^{-1}\varepsilon^{\mu\nu\lambda\sigma} e_\mu{}^a e_\nu{}^b e_\lambda{}^c e_\sigma{}^d\,, \qquad \varepsilon^{0123} = i\,, \tag{A.4}$$

which implies:

$$\sigma_{ab} = -\tfrac{1}{2}\varepsilon_{abcd}\sigma^{cd}\gamma_5\,. \tag{A.5}$$

The dual of an antisymmetric tensor field F_{ab} is given by:

$$\tilde{F}_{ab} = \tfrac{1}{2}\varepsilon_{abcd}F^{cd}, \tag{A.6}$$

and the (anti)selfdual part of F_{ab} reads:

$$F^\pm_{ab} = \tfrac{1}{2}(F_{ab} \pm \tilde{F}_{ab}). \tag{A.7}$$

Note that under hermitian conjugation (h.c.), selfdual becomes antiselfdual and *vice versa*. Under complex conjugation $\mathrm{SU}(2)_R$ indices $i, j, \ldots$ change place, for instance:

$$(T_{ab\,ij})^* = T_{ab}^{ij}, \tag{A.8}$$

and likewise $\mathrm{SU}(2)_R$ indices on spinors are uppered or lowered. Furthermore, the derivative $\overleftrightarrow{\partial}_\mu$, placed between two fields A and B denotes:

$$A \overleftrightarrow{\partial}_\mu B = A(\partial_\mu B) - (\partial_\mu A)B. \tag{A.9}$$

As far as spinors are concerned, we define the Dirac conjugate $\bar{\psi}$ of a spinor ψ by:

$$\bar{\psi} = \psi^\dagger \gamma_0. \tag{A.10}$$

A Majorana spinor by definition satisfies the (pseudo) reality condition:

$$\bar{\psi} = \psi^T C. \tag{A.11}$$

For two spinors ψ and ϕ, complex conjugation gives:

$$(\bar{\psi}\gamma_a\phi)^* = -\bar{\phi}\gamma_a\psi, \qquad (\bar{\psi}\phi)^* = \bar{\phi}\psi, \tag{A.12}$$

from which identities for complex conjugates of various other bilinears can be derived. Furthermore, bilinears of *Majorana* spinors ψ and ϕ can be transposed as follows:

$$\bar{\psi}\gamma_a\phi = -\bar{\phi}\gamma_a\psi, \qquad \bar{\psi}\phi = \bar{\phi}\psi. \tag{A.13}$$

If two spinors ψ and ϕ do not form a bilinear, their product can be decomposed on a basis of four-by-four matrices by means of a Fierz rearrangement:

$$\phi\,\bar{\psi} = -\tfrac{1}{4}(\bar{\psi}\phi)\mathbb{I} - \tfrac{1}{4}(\bar{\psi}\gamma^a\phi)\gamma_a - \tfrac{1}{4}(\bar{\psi}\gamma_5\phi)\gamma_5 + \tfrac{1}{4}(\bar{\psi}\gamma^a\gamma_5\phi)\gamma_a\gamma_5 + \tfrac{1}{2}(\bar{\psi}\sigma^{ab}\phi)\sigma_{ab}. \tag{A.14}$$

which is useful, for example, when computing QQ-commutators for fermion fields.

Finally, we note the identities:

$$\begin{aligned}
\sigma_{ab} &= -\tfrac{1}{2}\varepsilon_{abcd}\sigma^{cd}\gamma_5\,, & \gamma^b\gamma_a\gamma_b &= -2\gamma_a\,,\\
\sigma^{ab}\sigma_{ab} &= -3\,, & \sigma^{cd}\sigma_{ab}\sigma_{cd} &= \sigma_{ab}\,,\\
\gamma^c\sigma_{ab}\gamma_c &= 0\,, & \sigma^{bc}\gamma_a\sigma_{bc} &= 0\,,\\
[\gamma^c,\sigma_{ab}] &= 2\delta_{[a}{}^c\gamma_{b]}\,, & \{\gamma^c,\sigma_{ab}\} &= \varepsilon_{ab}{}^{cd}\gamma_5\gamma_d\,,\\
[\sigma_{ab},\sigma^{cd}] &= -4\delta_{[a}{}^{[c}\sigma_{b]}{}^{d]}\,, & \{\sigma_{ab},\sigma^{cd}\} &= -\delta_{[a}{}^c\delta_{b]}{}^d + \tfrac{1}{2}\varepsilon_{ab}{}^{cd}\gamma_5\,.
\end{aligned} \tag{A.15}$$

Appendix B

Useful expressions in $N = 2$ conformal supergravity

We use the following definition of *covariant* quantities and derivatives [36]: a quantity is covariant if transformations on that quantity do not result in expressions that depend on the derivatives of the transformation parameters. In particular, given a covariant quantity, we define the covariant derivative of that quantity by the operator:

$$D_\mu = \partial_\mu - \sum_A \delta_A(h_\mu(A)) \,, \tag{B.1}$$

which leads to another covariant quantity. Here $h_\mu(A)$ is the gauge field associated with δ_A. For the superconformal transformations, the gauge fields are normalized like in [32]:

$$\begin{array}{rclrcl} h_\mu^{ab}(M) &=& \omega_\mu{}^{ab} \,, & h_\mu(D) &=& b_\mu \,, \\ h_\mu(U(1)) &=& A_\mu \,, & h_\mu{}^i{}_j(SU(2)) &=& -\frac{1}{2}\mathcal{V}_\mu{}^i{}_j \,, \\ h_\mu^i(Q) &=& \frac{1}{2}\psi_\mu^i \,, & h_\mu^i(S) &=& \frac{1}{2}\phi_\mu^i \,, \\ h_\mu^a(K) &=& f_\mu^a \,. & & & \end{array} \tag{B.2}$$

When vector and/or vector-tensor multiplets are present, additional covariantizations must be included, which depend on the relevant gauge fields. The same holds for the central charge. A covariant box $\Box_C$ is defined as: $\Box_C = \eta^{ab} D_a D_b$. We use $\mathscr{D}_\mu$ to denote a derivative that is covariant with respect to M, D, U(1), SU(2) and gauge transformations[1]. For example:

$$\begin{aligned} \mathscr{D}_\mu X^I &= (\partial_\mu - b_\mu + iA_\mu)X^I - g f^I_{JK} W^J_\mu X^K, \\ \mathscr{D}_\mu \Omega^I_i &= (\partial_\mu - \tfrac{1}{2}\omega_\mu{}^{ab}\sigma_{ab} - \tfrac{3}{2}b_\mu + \tfrac{1}{2}iA_\mu)\Omega^I_i - \tfrac{1}{2}\mathcal{V}_\mu{}^j{}_i\Omega^I_j - g f^I_{JK} W^J_\mu \Omega^K_i \,, \\ \mathscr{D}_\mu \psi_{\nu\, i} &= (\partial_\mu - \tfrac{1}{2}\omega_\mu{}^{ab}\sigma_{ab} + \tfrac{1}{2}b_\mu - \tfrac{1}{2}iA_\mu)\psi_{\nu\, i} - \tfrac{1}{2}\mathcal{V}_\mu{}^j{}_i \psi_{\nu\, j} \,, \end{aligned} \tag{B.3}$$

[1] However, in case of the hypermultiplet and the vector-tensor multiplet we do not include the central charge transformation $\delta_z(W^0_\mu)$ in $\mathscr{D}_\mu$.

Examples of fully covariant quantities are the (covariant) field strengths defined throughout the text. Likewise, the curvature tensors $\hat{R}_{\mu\nu}(Q)^i$, $\hat{R}_{\mu\nu}(\mathrm{U}(1))$, etcetera, and the auxiliary fields T_{ab}^{ij}, χ^i and D in the Weyl multiplet are covariant. Moreover, covariant quantities transform only into other covariant quantities, for instance, the Q-supersymmetry variation of T_{ab}^{ij} is proportional to $\hat{R}_{\mu\nu}(Q)^i$. In actual calculations, one may benefit from the following observation: if one is calculating a variation of a covariant derivative, only the covariant terms in the variations of the gauge fields are explicitly written down in the end-result, because all term proportional to (non-covariant) gauge-fields contribute only as covariantization terms. Keep in mind, however, that derivatives of gauge fields can lead to (covariant) curvature terms.

The covariant general coordinate transformation is defined as follows:

$$\delta^{\mathrm{cov}}(\xi) = \delta_{\mathrm{g.c.t.}}(\xi) + \sum_A \delta_A(-\xi^\mu h_\mu(A)), \tag{B.4}$$

where the sum is over all superconformal (except the g.c.t) and additional gauge transformations, each with parameter $-\xi^\mu h_\mu(A)$. Note that for covariant quantities, δ^{cov} is generated by the covariant derivative and for gauge fields involves the corresponding curvature tensor, for example: $\delta^{\mathrm{cov}}(\xi)\, W_\mu = -\xi^\nu F_{\mu\nu}(W)$.

Furthermore, we give some useful expressions and the definitions of a number of quantities that have been used in the text. First of all, the composite gauge fields $\omega_\mu{}^{ab}, \phi_\mu{}^i$ and $f_\mu{}^a$ contained in the Weyl multiplet, are given by:

$$\begin{aligned}
\omega_\mu{}^{ab} &= -2e^{\nu[a}\partial_{[\mu}e_{\nu]}{}^{b]} - e^{\nu[a}e^{b]\sigma}e_{\mu c}\partial_\sigma e_\nu{}^c - 2e_\mu{}^{[a}e^{b]\nu}b_\nu \\
&\quad -\tfrac{1}{4}(2\bar\psi_\mu{}^i\gamma^{[a}\psi^{b]}{}_i + \bar\psi^{ai}\gamma_\mu\psi^b{}_i + \text{h.c.})\,, \\
\phi_\mu{}^i &= (\sigma^{\rho\sigma}\gamma_\mu - \tfrac{1}{3}\gamma_\mu\sigma^{\rho\sigma})(\mathscr{D}_\rho\psi_{\sigma\, i} - \tfrac{1}{8}\sigma\cdot T^{ij}\gamma_\rho\psi_{\sigma\, j} + \tfrac{1}{2}\sigma_{\rho\sigma}\chi^i), \\
f_\mu{}^\mu &= \tfrac{1}{6}R - D - \Big(\tfrac{1}{12}e^{-1}\varepsilon^{\mu\nu\rho\sigma}\bar\psi_\mu{}^i\gamma_\nu\mathscr{D}_\rho\psi_{\sigma\, i} - \tfrac{1}{12}\bar\psi_\mu{}^i\psi_\nu{}^j\, T_{ij}^{\mu\nu} - \tfrac{1}{4}\bar\psi_\mu{}^i\gamma^\mu\chi_i + \text{h.c.}\Big).
\end{aligned} \tag{B.5}$$

The following supercovariant curvatures appear in the main text:

$$\begin{aligned}
\hat{R}_{\mu\nu}(Q)^i &= 2\mathscr{D}_{[\mu}\psi_{\nu]}^i - \gamma_{[\mu}\phi_{\nu]}^i - \tfrac{1}{4}\sigma\cdot T^{ij}\gamma_{[\mu}\psi_{\nu]j}\,, \\
\hat{R}_{\mu\nu}(\mathrm{U}(1)) &= 2\partial_{[\mu}A_{\nu]} - i\Big(\tfrac{1}{2}\bar\psi_{[\mu}^i\phi_{\nu]i} + \tfrac{3}{4}\bar\psi_{[\mu}^i\gamma_{\nu]}\chi_i - \text{h.c.}\Big), \\
\hat{R}_{\mu\nu}(\mathrm{SU}(2))^i{}_j &= 2\partial_{[\mu}\mathcal{V}_{\nu]}^i{}_j + \mathcal{V}_{[\mu}^i{}_k\mathcal{V}_{\nu]}^k{}_j + \Big(2\bar\psi_{[\mu}^i\phi_{\nu]\,j} - 3\bar\psi_{[\mu}^i\gamma_{\nu]}\chi_j - (\text{h.c.}\,;\,\text{traceless})\Big)\,, \\
\hat{R}_{\mu\nu}(M)^{ab} &= 2\partial_{[\mu}\omega_{\nu]}^{ab} - 2\omega_{[\mu}^{ac}\omega_{\nu]}^{cb} - 4f_{[\mu}^{[a}e_{\nu]}^{b]} + (\bar\psi_{[\mu}^i\sigma^{ab}\phi_{\nu]\,i} + \text{h.c.}) \\
&\quad +\tfrac{1}{2}\bar\psi_{[\mu}^iT_{ij}^{ab}\psi_{\nu]}^j - \tfrac{3}{2}\bar\psi_{[\mu}^i\gamma_{\nu]}\sigma^{ab}\chi^i - \bar\psi_{[\mu}^i\gamma_{\nu]}\hat{R}^{ab}(Q)_i + \text{h.c.}\,.
\end{aligned} \tag{B.6}$$

Other covariant curvatures can be found in [29]. In computations, one may benefit from using the following relationships:

$$\begin{aligned}
\gamma^\mu(\hat{R}_{\mu\nu}(Q)^i + \sigma_{\mu\nu}\chi^i) &= 0\,, \\
2\mathscr{D}_{[\mu}e_{\nu]}^a - \bar\psi_{[\mu}^i\gamma^a\psi_{\nu]i} &= 0.
\end{aligned} \tag{B.7}$$

Appendix C

Superconformal multiplets

The components of the *Weyl multiplet* transform as:

$$
\begin{aligned}
\delta e_\mu{}^a &= \bar\epsilon^i \gamma^a \psi_{\mu\, i} + \text{h.c.}\,, \\
\delta \psi_\mu{}^i &= 2\mathscr{D}_\mu \epsilon^i - \tfrac{1}{4}\sigma\cdot T^{ij}\gamma_\mu \epsilon_j - \gamma_\mu \eta^i\,, \\
\delta b_\mu &= \tfrac{1}{2}\bar\epsilon^i \phi_{\mu\, i} - \tfrac{3}{4}\bar\epsilon^i\gamma_\mu \chi_i - \tfrac{1}{2}\bar\eta^i \psi_{\mu\, i} + \text{h.c.} + \Lambda_K^a\, e_{\mu\, a}\,, \\
\delta A_\mu &= \tfrac{1}{2} i\bar\epsilon^i \phi_{\mu\, i} + \tfrac{3}{4} i\bar\epsilon^i \gamma_\mu \chi_i + \tfrac{1}{2} i\bar\eta^i \psi_{\mu\, i} + \text{h.c.}\,, \\
\delta \mathcal{V}_\mu{}^i{}_j &= 2\,\bar\epsilon_j \phi_\mu{}^i - 3\,\bar\epsilon_j \gamma_\mu \chi^i + 2\,\bar\eta_j \psi_\mu{}^i - (\text{h.c.}\,;\ \text{traceless})\,, \\
\delta T_{ab}^{ij} &= 8\,\bar\epsilon^{[i}\hat R_{ab}(Q)^{j]}\,, \\
\delta \chi^i &= -\tfrac{1}{6}\sigma^{ab} \not{D} T_{ab}^{ij}\epsilon_j + \tfrac{1}{3}\hat R(\mathrm{SU}(2))^i{}_j \cdot \sigma \epsilon^j - \tfrac{2}{3} i\hat R(\mathrm{U}(1))\cdot\sigma\epsilon^i \\
&\quad + D\,\epsilon^i + \tfrac{1}{6}\sigma\cdot T^{ij}\eta_j\,, \\
\delta D &= \bar\epsilon^i \not{D}\chi_i + \text{h.c.}\,,
\end{aligned} \tag{C.1}
$$

and the dependent fields transform as:

$$
\begin{aligned}
\delta\omega_\mu{}^{ab} &= -\bar\epsilon^i\sigma^{ab}\phi_{\mu\, i} - \tfrac{1}{2}\bar\epsilon^i T_{ij}^{ab}\psi_\mu{}^j + \tfrac{3}{2}\bar\epsilon^i\gamma_\mu\sigma^{ab}\chi_i \\
&\quad + \bar\epsilon^i\gamma_\mu \hat R^{ab}(Q)_i - \bar\eta^i\sigma^{ab}\psi_{\mu\, i} + \text{h.c.} + 2\Lambda_K^{[a}\, e_\mu{}^{b]}\,, \\
\delta\phi_\mu{}^i &= -2 f_\mu{}^a\gamma_a\epsilon^i - \tfrac{1}{4}\not{D} T^{ij}\cdot\sigma\gamma_\mu\epsilon_j + \tfrac{3}{2}[(\bar\chi_j\gamma^a\epsilon^j)\gamma_a\psi_\mu{}^i - (\bar\chi_j\gamma^a\psi_\mu{}^j)\gamma_a\epsilon^i] \\
&\quad + \tfrac{1}{2}\hat R(\mathrm{SU}(2))^i{}_j\cdot\sigma\gamma_\mu\epsilon^j + i\hat R(\mathrm{U}(1))\cdot\sigma\gamma_\mu\epsilon^i + 2\mathscr{D}_\mu\eta^i + \Lambda_K^a\gamma_a\psi_\mu{}^i\,, \\
\delta f_\mu{}^a &= -\tfrac{1}{2}\bar\epsilon^i\psi_\mu{}^j\, D_b T_{ij}^{ba} - \tfrac{3}{4} e_\mu{}^a\bar\epsilon^i\not{D}\chi_i - \tfrac{3}{4}\bar\epsilon^i\gamma^a\psi_{\mu\, i}\, D \\
&\quad + \bar\epsilon^i\gamma_\mu D_b\hat R^{ba}(Q)_i + \tfrac{1}{2}\bar\eta^i\gamma^a\phi_{\mu\, i} + \text{h.c.} + \mathscr{D}_\mu\Lambda_K^a\,.
\end{aligned} \tag{C.2}
$$

The superconformal transformation rules of the *chiral multiplet* with arbitrary Weyl weight w:

$$\begin{aligned}
\delta_Q(\epsilon)\, A &= \bar{\epsilon}^i \Psi_i \,, \\
\delta_Q(\epsilon)\, \Psi_i &= 2\not{D} A\, \epsilon_i + B_{ij}\epsilon^j + \sigma \cdot F^- \varepsilon_{ij}\epsilon^j + 2wA\, \eta_i \,, \\
\delta_Q(\epsilon)\, B_{ij} &= 2\bar{\epsilon}_{(i}\not{D}\Psi_{j)} + 2\varepsilon_{k(i}\bar{\epsilon}^k\Lambda_{j)} + (1-w)\bar{\eta}_{(i}\Psi_{j)} \,, \\
\delta_Q(\epsilon)\, F^-_{ab} &= \varepsilon^{ij}\bar{\epsilon}_i\not{D}\sigma_{ab}\Psi_j + \bar{\epsilon}^i\sigma_{ab}\Lambda_i - (1+w)\varepsilon^{ij}\bar{\eta}_i\sigma_{ab}\Psi_j \,, \\
\delta_Q(\epsilon)\, \Lambda_i &= -\sigma^{ab}\not{D}F^-_{ab}\epsilon_i + \varepsilon^{kj}\not{D}B_{ij}\epsilon_k + \varepsilon_{ij}C\epsilon^j - \tfrac{3}{2}\varepsilon^{jk}(\bar{\chi}_{[i}\gamma^a\Psi_{j]})\gamma_a\epsilon_k \\
&\quad +\tfrac{1}{2}\varepsilon^{jk}\Big((\not{D}A)T_{ij}\cdot\sigma + wA\not{D}T_{ij}\cdot\sigma\Big)\epsilon_k - (1+w)B_{ij}\varepsilon^{jk}\eta_k + (1-w)\sigma\cdot F^-\eta_i \,, \\
\delta_Q(\epsilon)\, C &= -2\varepsilon^{ij}\bar{\epsilon}_i\not{D}\Lambda_j - 6\bar{\epsilon}_i\chi_j B_{kl}\varepsilon^{ik}\varepsilon^{jl} \\
&\quad -\tfrac{1}{2}\bar{\epsilon}_i\Big((w-1)\sigma_{ab}(\not{D}T^{ab}_{jk})\Psi_l + \sigma\cdot T_{jk}\not{D}\Psi_l\Big)\varepsilon^{ij}\varepsilon^{kl}.
\end{aligned} \tag{C.3}$$

The superconformal transformation rules of *non-abelian vector multiplets*:

$$\begin{aligned}
\delta\, X^I &= \bar{\epsilon}^i\Omega_i^I \,, \\
\delta\, \Omega_i^I &= 2\not{D}X^I\epsilon_i + \varepsilon_{ij}\sigma\cdot\mathscr{F}^{I-}\epsilon^j + Y_{ij}^I\epsilon^j - 2gf_{JK}{}^I X^J\bar{X}^K\varepsilon_{ij}\epsilon^j + 2X^I\eta_i \,, \\
\delta\, W_\mu^I &= \varepsilon^{ij}\bar{\epsilon}_i\gamma_\mu\Omega_j^I + 2\varepsilon_{ij}\bar{\epsilon}^i\bar{X}^I\psi_\mu{}^j + \text{h.c.} \,, \\
\delta\, Y_{ij}^I &= 2\bar{\epsilon}_{(i}\not{D}\Omega_{j)}^I + 2\varepsilon_{ik}\varepsilon_{jl}\bar{\epsilon}^{(k}\not{D}\Omega^{l)\,I} - 4gf_{JK}{}^I\,\varepsilon_{k(i}\Big(\bar{\epsilon}_{j)}X^J\Omega^{k\,K} - \bar{\epsilon}^k\bar{X}^J\Omega_{j)}^K\Big) \,,
\end{aligned} \tag{C.4}$$

The superconformal transformation rules of a *linear multiplet* coupled to (non-abelian) vector multiplets:

$$\begin{aligned}
\delta\, L_{ij} &= 2\,\bar{\epsilon}_{(i}\varphi_{j)} + 2\,\varepsilon_{ik}\varepsilon_{jl}\,\bar{\epsilon}^{(k}\varphi^{l)} \,, \\
\delta\, \varphi^i &= \not{D}L^{ij}\epsilon_j + \not{E}\varepsilon^{ij}\epsilon_j - G\epsilon^i + 2g\,\bar{X}L^{ij}\varepsilon_{jk}\epsilon^k + 2\,L^{ij}\eta_j \,, \\
\delta\, G &= -2\,\bar{\epsilon}_i\not{D}\varphi^i - \bar{\epsilon}_i\Big(6\chi_j L^{ij} + \tfrac{1}{2}\varepsilon^{ij}\varepsilon^{kl}\sigma\cdot T_{jk}\varphi_l\Big) \\
&\quad +2g\,\bar{X}\Big(\varepsilon^{ij}\bar{\epsilon}_i\varphi_j - \varepsilon_{ij}\bar{\epsilon}^i\varphi^j\Big) - 2g\,\bar{\epsilon}_i\Omega^j L^{ik}\varepsilon_{jk} + 2\bar{\eta}_i\varphi^i \,, \\
\delta\, E_a &= 2\,\varepsilon_{ij}\bar{\epsilon}^i\sigma_{ab}D^b\varphi^j + \tfrac{1}{4}\bar{\epsilon}^i\gamma_a\Big(6\varepsilon_{ij}\chi_k L^{jk} - \tfrac{1}{2}\sigma\cdot T_{ij}\varepsilon^{jk}\varphi_k\Big) \\
&\quad +2g\,\bar{X}\bar{\epsilon}^i\gamma_a\varphi_i + g\,\bar{\epsilon}^i\gamma_a\Omega^j L_{ij} + \tfrac{3}{2}\bar{\eta}^i\gamma_a\varphi^j\varepsilon_{ij} + \text{h.c.}\,.
\end{aligned} \tag{C.5}$$

The representation of the gauge group on the linear multipelts has been included in the vector-multiplet fields: where we write X, we mean X^I contracted with the representation matrix of the Ith generator of the representation for the linear multiplets.

The superconformal transformation rules of the *vector-tensor multiplet*:

$$
\begin{aligned}
\delta\,\phi &= \bar\epsilon^i\lambda_i + \bar\epsilon_i\lambda^i\,,\\
\delta\,V_\mu &= i\varepsilon^{ij}\bar\epsilon_i\gamma_\mu(2X\lambda_j+\phi\Omega^0_j) - iW^0_\mu\,\bar\epsilon^i\lambda_i + 2i\phi X\varepsilon^{ij}\bar\epsilon_i\psi_{\mu\,j} + \text{h.c.}\,,\\
\delta\,B_{\mu\nu} &= -2\,\bar\epsilon^i\sigma_{\mu\nu}|X|^2\Big(4\eta_{11}\phi - 2\,\text{Re}\,[g]\Big)\lambda_i\\
&\quad -2\,\bar\epsilon^i\sigma_{\mu\nu}\bar X\Big(2\eta_{11}\phi^2\Omega^0_i + \phi\bar X\partial_{\bar I}\bar g\,\Omega^I_i - 4i\text{Re}\,[\partial_I(Xb)]\Omega^I_i\Big)\\
&\quad -2\,\bar\epsilon^i\gamma_{[\mu}\psi_{\nu]i}\bar X\Big(2\eta_{11}\phi^2 X + \phi\bar X\partial_{\bar I}\bar g X^I - 4i\text{Re}\,[\partial_I(Xb)]X^I\Big)\\
&\quad +i\varepsilon^{ij}\bar\epsilon_i\gamma_{[\mu}V_{\nu]}\Big(\eta_{11}(2X\lambda_j+\phi\Omega^0_j) - i\eta_{1A}\Omega^A_j\Big)\\
&\quad +2i\,\varepsilon^{ij}\bar\epsilon_i\psi_{j[\mu}V_{\nu]}\Big(X(\eta_{11}\phi - g)\Big)\\
&\quad +\varepsilon^{ij}\bar\epsilon_i\gamma_{[\mu}W^0_{\nu]}m\Big(2X(2\eta_{11}\phi - g)\lambda_j + \eta_{11}\phi^2\Omega^0_j - i\eta_{1A}\phi\Omega^A_j - 4i\partial_I(Xb)\Omega^I_j\Big)\\
&\quad +2\,\varepsilon^{ij}\bar\epsilon_i\psi_{j[\mu}W^0_{\nu]}X\Big(\eta_{11}\phi^2 - \phi g - 4ib\Big)\\
&\quad +\varepsilon^{ij}\bar\epsilon_i\gamma_{[\mu}W^A_{\nu]}\eta_{AB}\Omega^B_j + 2\varepsilon^{ij}\bar\epsilon_i\psi_{j[\mu}W^A_{\nu]}\eta_{AB}X^B\\
&\quad -i\eta_{11}W^0_{[\mu}V_{\nu]}\bar\epsilon^i\lambda_i + \text{h.c.}\,,\\
\delta\,\lambda_i &= \Big(\not{D}\phi - i\hat{\not{V}}^{(\text{z})}\Big)\epsilon_i - \frac{i}{2X}\varepsilon_{ij}\sigma\cdot\Big(\mathscr{F}^-(V) - i\phi\mathscr{F}^{-0}\Big)\epsilon^j + 2\varepsilon_{ij}\bar X\phi^{(\text{z})}\epsilon^j\\
&\quad -\frac{1}{X}(\bar\epsilon^j\lambda_j)\Omega^0_i - \frac{1}{X}(\bar\epsilon^j\Omega^0_j)\lambda_i\\
&\quad -\frac{1}{2X(2\eta_{11}\phi - \text{Re}\;g)}\epsilon^j\Big[\,2\eta_{11}\phi^2Y^0_{ij} + \phi\bar X\partial_{\bar I}\bar g\,Y^I_{ij} - 4i\text{Re}\,[\partial_I(Xb)]Y^I_{ij}\\
&\qquad\qquad -2\eta_{11}\Big(X\bar\lambda_i\lambda_j - \bar X\varepsilon_{ik}\varepsilon_{jl}\bar\lambda^k\lambda^l\Big)\\
&\qquad\qquad +X\Big(X\partial_I g\,\bar\Omega^I_{(i}\lambda_{j)} - \bar X\varepsilon_{ik}\varepsilon_{jl}\partial_{\bar I}\bar g\,\bar\Omega^{I(k}\lambda^{l)}\Big)\\
&\qquad\qquad +i\Big(\partial_I\partial_J(Xb)\,\bar\Omega^I_i\Omega^J_j + \varepsilon_{ik}\varepsilon_{jl}\,\partial_{\bar I}\partial_{\bar J}(\bar X\bar b)\bar\Omega^{Ik}\Omega^{Jl}\Big)\Big]\,.
\end{aligned}
\tag{C.6}
$$

The superconformal transformation rules for *hypermultiplets* coupled to (non-abelian) vector multiplets:

$$
\begin{aligned}
\delta A_i{}^\alpha &= 2\,\bar\epsilon_i\zeta^\alpha + 2\,\rho^{\alpha\beta}\varepsilon_{ij}\,\bar\epsilon^j\zeta_\beta\,,\\
\delta\zeta^\alpha &= \not{D}A_i{}^\alpha\,\epsilon^i + 2g\,X^I(t_I)^\alpha{}_\beta A_i{}^\beta\,\varepsilon^{ij}\epsilon_j + 2\,X^0 A^{(\text{z})}{}_i{}^\alpha\,\varepsilon^{ij}\epsilon_j + A_i{}^\alpha\eta^i.
\end{aligned}
\tag{C.7}
$$

The transformation rules for higher z-level components are given by adding superscript (z)'s.

	Weyl multiplet											parameters	
field	$e_\mu{}^a$	$\psi_\mu{}^i$	b_μ	A_μ	$\mathcal{V}_\mu{}^i{}_j$	T_{ab}^{ij}	χ^i	D	$\omega_\mu{}^{ab}$	$f_\mu{}^a$	$\phi_\mu{}^i$	ϵ^i	η^i
w	-1	$-\frac{1}{2}$	0	0	0	1	$\frac{3}{2}$	2	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
c	0	$-\frac{1}{2}$	0	0	0	-1	$-\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
γ_5		$+$					$+$				$-$	$+$	$-$

Table C.I: Weyl and chiral weights (w and c, respectively) and fermion chirality (γ_5) of the Weyl multiplet component fields and of the supersymmetry transformation parameters.

	vector multiplet				hypermultiplet			linear multiplet			
field	X^I	Ω_i^I	W_μ^I	Y_{ij}^I	A_i^α	ζ^α	$A_i^{\alpha(\mathrm{z})}$	L_{ij}	φ^i	G	E_a
w	1	$\frac{3}{2}$	0	2	1	$\frac{3}{2}$	1	2	$\frac{5}{2}$	3	3
c	-1	$-\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	1	0
γ_5		$+$				$-$			$+$		

Table C.II: Weyl and chiral weights (w and c, respectively) and fermion chirality (γ_5) of the vector, hyper and linear multiplet component fields.

	vector-tensor multiplet				
field	ϕ	V_μ	$B_{\mu\nu}$	λ_i	$\phi^{(\mathrm{z})}$
w	0	0	0	$\frac{1}{2}$	0
c	0	0	0	$\frac{1}{2}$	0
γ_5				$+$	

Table C.III: Scaling and chiral weights (w and c, respectively) and fermion chirality (γ_5) of the vector-tensor component fields.

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Samenvatting

De vereniging van het standaard model met Einsteins algemene relativiteitstheorie vormt één van de belangrijkste vraagstukken binnen de hedendaagse theoretisch natuurkunde. Het is duidelijk geworden dat een antwoord op dit vraagstuk een mogelijkerwijs volkomen nieuw maar in ieder geval diepgaand inzicht vereist in het gedrag van de natuur op de allerkleinste lengteschalen en bij de allerhoogste energieën. Onder die extreme omstandigheden ontstaan er problemen met de quantummechanische beschrijving die veldentheoretische modellen geven van het gravitatieveld. Een alternatief wordt geboden door string modellen: een beschrijving van elementaire deeltjes in termen van een string die het puntdeeltje vervangt, gedraagt zich quantummechanisch onder deze extreme omstandigheden aanzienlijk beter. Consistentie van string theorieën vereist dat ze worden geformuleerd in een ruimte-tijd met dimensie tien en bovendien supersymmetrisch zijn. Superstring modellen worden al sinds het midden van de jaren tachtig uitgebreid en kritisch bestudeerd en stellen nog steeds een belangrijke kandidatuur voor de formulering van quantumgravitatie.

De vraag die zich natuurlijk onmiddellijk aandient, is hoe de tien-dimensionale ruimte-tijd van superstring modellen in overeenstemming moet worden gebracht met het klaarblijkelijk vier-dimensionale karakter van 'onze' ruimte-tijd. Een iets bredere vraagstelling zou zijn: "hoe kunnen superstring modellen gerelateerd worden aan vier-dimensionale gravitatie en ijktheorieën zoals het standaard model en welke vier-dimensionale modellen kunnen worden opgevat als een effectieve beschrijving van een superstring model?".

Bij de formulering van het antwoord op die vragen is het begrip compactificatie onontbeerlijk: merk namelijk op dat superstring modellen een eis stellen aan de dimensie van de ruimte-tijd, maar niet aan zijn vorm. Zoals een (twee-dimensionale) cylinder kan worden opgevat als het produkt van een (één-dimensionale) rechte lijn met een (één-dimensionale) cirkel, zo kunnen sommige tien-dimensionale ruimtes worden opgedeeld als het produkt van vier-dimensionale en een zes-dimensionale ruimte. De twee factoren zijn in dat geval 'onze' vier-dimensionale ruimte-tijd en een zeer kleine factor die de overige zes dimensies omvat. Dat wil zeggen dat zich op ieder punt van de vier-dimensionale ruimte-tijd een zes-dimensionale ruimte bevindt, die zo klein is dat de natuurkunde van experimenteel bereikbare energieschalen een te 'grof' instrument vormt om iets van die kleine ruimte te kunnen meten of merken. Pas zodra de energieschaal omhoog gaat en in de buurt komt van de schaal die relevant is voor superstring modellen wordt het instrument 'fijner' en begint de compacte zes-dimensionale ruimte merkbaar te worden.

Om de relatie tussen superstring modellen en de vier-dimensionale veldentheorieën zoals algemene relativiteit en ijktheorieën te bestuderen zijn in principe twee benaderingswijze voorhanden. Ten eerste bestaat de mogelijkheid om de compactificatie te behandelen binnen het kader van superstring theorie en de vier-dimensionale aspecten te scheiden van de overige, hoog-energetische aspecten. Zodoende vindt men vier-dimensionale veldentheorieën die kunnen worden opgevat als een effectieve beschrijving van superstrings bij lage energieën. Daartoe dient echter eerst een keuze te worden gemaakt voor de zes-dimensionale compactificatie factor. De kenmerken van de effectieve beschrijving blijken sterk afhankelijk te zijn van die keuze. In het bijzonder bepaalt de compactificatie factor welk deel van de supersymmetrie van de tien-dimensionale beschrijving overblijft in de vierdimensionale.

De tweede benaderingswijze begint bij de vier-dimensionale veldentheorieën. Stel dat we ons beperken tot een bepaalde klasse van superstring compactificaties, bijvoorbeeld die compactificaties met $N = 2$ supersymmetrie, is het dan mogelijk om een classificatie te geven van de mogelijkheden die bestaan voor de effectieve superstring modellen, uitsluitend op basis van vier-dimensionale overwegingen? Dat is het oogpunt dat gehanteerd wordt in dit proefschrift: we beschouwen vier-dimensionale, $N = 2$ supersymmetrische veldentheorieën en proberen daarin een overzicht te verkrijgen van de mogelijkheden voor de representaties van supersymmetrie en de mogelijke veldentheoretische koppelingen die kunnen worden geformuleerd tussen die representaties. Ieder $N = 2$ supersymmetrisch effectief superstring model moet vervolgens te formuleren zijn als één van die veldentheorieën.

Helaas is het niet mogelijk om in het bestek van dit proefschrift een volledig overzicht te geven, noch is het mogelijk om een volledig beeld te geven van superstring modellen en compactificaties. In dit proefschrift komen de volgende onderwerpen aan bod:

1. *Introduction*
   Naast een inleiding waarin de context geschetst wordt, heeft dit hoofdstuk tot doel enige belangrijke concepten te introduceren op een zo eenvoudig mogelijk niveau, zodat in latere hoofdstukken waar dezelfde concepten in ingewikkeldere vorm terugkomen, verwezen kan worden. Behandeld worden achtereenvolgens supersymmetrie, niet-lineaire sigma-modellen, dualiteitstransformaties en ijk-equivalentie.

2. *Supersymmetry and Supergravity*
   De basis voor alle modellen die in dit proefschrift aan de orde komen, wordt gevormd door $N = 2$ supersymmetrie en supergravitatie. In dit hoofdstuk worden deze twee onderwerpen ingeleid, met de nadruk op superconforme methodes en de constructie van supersymmetrische acties.

3. *Vector Multiplets and Special Geometry*
   De in hoofdstuk twee geïntroduceerde concepten worden vervolgens toegepast in de

discussie van vector multipletten en hun koppelingen. Belangrijke begrippen in dit verband zijn de prepotential, symplectische transformaties en special geometry.

4. *Hypermultiplet Couplings*
   In het vierde hoofdstuk komt het hypermultiplet aan bod. Naast de superconforme behandeling van transformatieregels, centrale lading en supersymmetrische acties, wordt gekeken naar de koppelingen van on-shell rigide hypermultipletten. Hierbij speelt hyperkähler meetkunde een belangrijke rol. Daarnaast bevat dit hoofdstuk secties over de koppeling met een vector-multiplet achtergrond en het Weyl multiplet.

5. *The Mirror Map*
   Vier-dimensionale compactificaties van type-II superstring modellen vertonen een dualiteit die bekend staat onder de naam mirror symmetry. In hoofdstuk vijf wordt de formulering van mirror symmetry onderzocht in de context van drie- en vier-dimensionale supersymmetrische veldentheorieën.

6. *Vector-Tensor Multiplets*
   In hoofdstuk zes, tenslotte, worden de mogelijke koppelingen van vector-tensor multipletten beschouwd in een achtergrond van vector multipletten en het Weyl multiplet.

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Curriculum Vitae

Ik ben geboren op 15 oktober 1970, te Nijmegen. Van 1983 tot 1989 bezocht ik het Rythovius College in Eersel, alwaar ik in juni 1989 mijn eindexamen VWO behaalde. Vervolgens ben ik begonnen met mijn studie Natuurkunde aan de Universiteit Utrecht. In juni 1990 ontving ik de propaedeuse Natuurkunde en de propaedeuse Wiskunde (cum laude) en in augustus 1994 volgde het doctoraal examen Natuurkunde (cum laude). Tijdens het derde jaar van mijn studie heb ik een student-assistentschap vervuld, een onderwijstaak voor de begeleiding van werkcolleges. Mijn doctoraalscriptie theoretische natuurkunde heb ik geschreven onder begeleiding van Prof. dr. B. Q. P. J. de Wit, die ook de supervisie van mijn promotieonderzoek op zich heeft genomen. Het resultaat van dat onderzoek, gestart in september 1994, verricht aan het instituut voor theoretisch fysica te Utrecht in dienst van de stichting FOM en beëindigd in mei 1998, ligt voor u.