Loop models

Bernard Nienhuis, les Houches 2008

Updated: July 16, 2008

1 Historical perspective

Loop models have come up in the theory of phase transitions and (other) critical phenomena. One may say that the best understanding of phase transitions has come from a succession of theories through the 20-th century. It started with Mean Field (MF) theory, which was the dominant theory until Renormalization Group (RG) took over around 1970. Much of the research began to be done in two dimensions, for two reasons. It is much easier to do numerical calculations in 2D than in 3D, and many of the critical exponents appeared to be rational (giving more hope to calculate them or even guess them). In 2D the Coulomb Gas (CG) method proved to be an effective way to calculate universal quantities. It is largely formulated in the language of loop models. Very many lattice models can be represented as loop models.

The notion that conformal invariance is both plausible and powerful led to the study of Conformal Field Theory (CFT) as the scaling limit of critical systems. It proved very effective in predicting many properties of the scaling limit. Finally around 2000 the Stochastic Löwner evolutions (SLE) allowed the mathematicians to prove what physicist had been so busy calculating all these years. While CFT, and (in its specialization of rigor) SLE are currently more powerful than CG, the latter approach is unique in that it makes direct contact with the (lattice) model and with the specific operator under investigation.

In parallel to all of these ran a series of exact solutions (e.g. partition sums) of specific models. The 2D Ising model had already shed doubt on the validity of MF theory in 1944. Much later followed the 6-vertex model, and the 8-vertex model. The hard square and hard hexagon model. In particular the formulation of the Yang Baxter equation (YBE) led to an explosion of solvable models.

2 Brief summary of renormalization theory

The ideas that led to the application of renormalization (usually designated Renormalization Group, or RG), to condensed matter systems can be formulated as follows. A many body system can have a number of relevant spatial scales. The microscopic scale is e.g. the size of the molecules or the lattice constant (in case the lattice is considered as given). Emerging length scales are correlation lengths. In generic systems these various length scales are of the same order of magnitude. However, when a continuous phase transition (or critical point) is approached the correlation lengths grow, and can become macroscopic. This implies that macroscopic properties involving space have a characteristic scale. However, when the critical point is reached, this scale diverges. Therefore the characteristic scale disappears, and the observables become scale invariant (that is for scales sufficiently

removed from the microscopic scales). If this is taken seriously then it is plausible that one can formulate a scale transformation for which the critical system is invariant and the generic system covariant. The possible existence of such transformation has already some important consequences that can be verified. If the transformation can be calculated or approximated more can be deduced.

To make this explicit consider a large but finite statistical system with N degrees of freedom s, associated with the sites of a lattice. It has a hamiltonian $H_N(g,s)$, where g stands for a number of coupling constants. The partition sum (or integral) is given by

$$Z_N(g) = \sum_s \mathrm{e}^{-H_N(g,s)} \tag{1}$$

(the factor kT is absorbed in the coupling constants). To formulate a renormalization transformation (RT) we assume that we can course grain the variables to a smaller number, say N'. This means we do the summation as far as the local small wavelength variables is concerned, so that the sum over the remaining variables would result in the original partition sum. The summand after the partial summation is written as an effective Boltzmann weight again, so that

$$Z_N(g) = \sum_s \mathrm{e}^{-H_N(g,s)} = \sum_{s'} \mathrm{e}^{-H_{N'}(g',s')} = Z_{N'}(g') \tag{2}$$

The notation reflects the hope that the effective Hamiltonian of the smaller system is of the same form as the original Hamiltonian but with altered values of the coupling constants. We further assume that the function $g'(g)$ is analytic or at least differentiable even at a phase transition. There is no need to assume that one knows how to do this, only that in principle it can be done.

Let $N = \ell^d N'$, with d the dimension of space, so that the linear change of scale is ℓ. In the limit that N is very large the free energy is proportional to N, so that the free energy per lattice site satisfies

$$f[g'(g)] = \ell^d f(g) \tag{3}$$

Going back to the correlation lengths, we introduce the symbol ξ for one of the correlation lengths, in units of the lattice constant. This must satisfy

$$\xi(g) = \ell \xi[g'(g)]\,. \tag{4}$$

This equation tells us that criticality at g implies criticality at $g'(g)$. Therefore critical manifolds are invariant under the map $g'(g)$. We may then assume that a critical manifold contains fixed points.

$$g'(g^*) = g^*\,, \tag{5}$$

which are going to play a leading role.

Exercise: *(1) Argue why limit cycles are not to be expected in these transformations. (2) What can one conclude concerning the value of ξ at a fixed point.*

If the microscopic scale is not given by a lattice, but by a (UV) integration cut-off, it is more convenient to take ℓ as a variable, and analyse

$$\beta(g) \equiv \lim_{\ell \to 1} \frac{\partial g'(g)}{\partial \ell} \tag{6}$$

called the β-function. In this case the equation for the fixed point is $\beta(g^*) = 0$.

What can we say about the behavior of the system in the vicinity of a fixed point? Sufficiently near g^* we can linearize the transformation, and diagonalize the derivative matrix $\partial g'(g)/\partial g$. The principle variations of g near g^* we call u_j, and we can write the free energy and correlation length in terms of these u_j. The significance of a specific u_j depends on the corresponding eigenvalue. If an eigenvalue (assumed to be real) is larger than unity, the corresponding u_j grows, and if it is less than 1, u_j decreases.

Since the eigenvalues are not independent on ℓ, it is useful to make that dependence explicit. Since successive RT's with rescaling factors ℓ and ℓ' correspond to a rescaling with factor $\ell\ \ell'$, and also the eigenvalues multiply, we write these eigenvalues in the form ℓ^y.

To illustrate the distinction of eigenvalues larger or less than one, or positive and negative exponent y, we first discuss a case with two coupling constants, and $y_2 < 0 < y_1$. Considering the 'flow lines' (i.e. the curves in the parameter space traced out by repeated renormalization), suggests/shows that the phase transition is at the line $u_1 = 0$, as that is the watershed between different limits of the flows. If e.g. the effective variables at a very coarse scale are strongly coupled, the original variables will have long ranged correlations, and if the effective variables are weakly coupled, the original variables have only short range correlations.

The functions $\xi(u_1, u_2)$ and $f(u_1, u_2)$ satisfy

$$\xi(u_1, u_2) = \ell \xi\left(u_1 \ell^{y_1}, u_2 \ell^{y_2}\right) \qquad \text{and} \qquad f(u_1, u_2) = \ell^{-d} f\left(u_1 \ell^{y_1}, u_2 \ell^{y_2}\right) \tag{7}$$

the solutions of these equation are given by

$$\xi(u_1, u_2) = |u_1|^{-1/y_1}\ X_\pm\left(u_2\,|u_1|^{-y_2/y_1}\right)\ , \qquad f(u_1, u_2) = |u_1|^{d/y_1}\ S_\pm\left(u_2\,|u_1|^{-y_2/y_1}\right)\ , \tag{8}$$

where X and S are unknown functions, and their suffix is the sign of u_1. Since the locus of the transition is $u_1 = 0$, nothing special happens at $u_2 = 0$. This implies that the thermodynamic functions are *analytic* at $u_2 = 0$. Therefore the functions X and S can be expanded in their argument.

$$\xi(u_1, u_2) = \sum_{k=0}^{\infty} X_{k,\pm}\ u_2^k\,|u_1|^{(-1-k\,y_2)/y_1} \quad \text{and} \quad f(u_1, u_2) = \sum_{k=0}^{\infty} S_{k,\pm}\ u_2^k\,|u_1|^{(d-k\,y_2)/y_1} \tag{9}$$

We see here that the singular behavior of the free energy and the correlation length are a sum of power-law terms, with increasing exponent. The non-integer powers concern the u_1 dependence. The leading term is determined by y_1. None of the exponents depends on the value of u_1 or u_2. Because the leading critical behavior is not affected by u_2, this parameter is called **irrelevant**. In contrast, the parameters with positive exponent y are called **relevant**. When $y = 0$ the corresponding parameter (or operator) is called marginal.

Exercise: *Argue that a marginal operator can be marginally relevant, marginally irrelevant, that this behavior will generally be different on either side of the fixed point, but need not be, and finally that one can also have strictly marginal operators.*

The amplitudes of the non-dominant terms depend explicitly on the value of the irrelevant parameter. When the nonlinear terms of $g'(g)$ are taken into account, also the coefficient of the leading term will depend on u_2. Finally, note that the exponents of the free energy and of the correlation length are not independent.

Now consider a case with two relevant variables, $y_1 > y_2 > 0$, omitting possible irrelevant variables. To make contact with some real life case, think of u_1 as the magnetic field, and of u_2 as the temperature variation with respect to the critical temperature. Equation (9) for the free energy can be used to calculate the various thermodynamic functions: magnetization M and susceptibility χ as first and second derivative of f w.r.t. the field h, the heat capacity C as second derivative of f w.r.t. the temperature T. The well-known critical exponents (at zero field: $C \propto |T - T_c|^{-\alpha}$, $M \propto (T_c - T)^\beta$, $\chi \propto |T - T_c|^{-\gamma}$, $\xi \propto |T - T_c|^{-\nu}$, and at $T = T_c$: $h \propto M^\delta$) can now readily be calculated readily. We get:

$$\alpha = 2 - \frac{d}{y_2}, \qquad \beta = \frac{d - y_1}{y_2}, \qquad \gamma = \frac{2y_1 - d}{y_2}, \qquad \delta = \frac{y_1}{d - y_1}, \qquad \nu = \frac{1}{y_2} \tag{10}$$

Exercise: *Verify these values of the critical exponents.*

Evidently these exponents are not independent but can expressed in ony two variables.

Also the overall scaling of critical correlation functions can be calculated within the RG framework. Let the (extensive) operator conjugate to one of the parameters u_j be Q_j. Now consider the fluctuation of Q_j in a finite system with linear size L, away from the fixed point only along the u_j-axis:

$$G(u_j, L) \equiv \langle Q_j Q_j \rangle - \langle Q_j \rangle^2 = L^d \frac{\partial^2 f_L(u_j)}{\partial u_j^2} \tag{11}$$

This scales as

$$G(u_j, L) = \ell^{2y_j}\ G(\ell^{y_j}\ u_j, L/\ell) \tag{12}$$

so that

$$G(0, L) \propto L^{2y_j} \tag{13}$$

Let Q_j be written as a sum (or integral) over local operators:

$$Q_j = \int_{L^d} \mathrm{d}^d r\ q_j(r) \tag{14}$$

then G can be written as an integral over the two-point function of $q_j(r)$:

$$Q_j(0, L) = \int_{L^d} \mathrm{d}^d r \int_{L^d} \mathrm{d}^d r'\ \langle\, q_j(r)\ q_j(r')\,\rangle\ -\ \langle\, q_j(r)\,\rangle\ \langle\, q_j(r')\,\rangle \tag{15}$$

This leads to a power law decay as

$$\langle\, q_j(0)\ q_j(r)\,\rangle\ -\ \langle\, q_j(0)\,\rangle\ \langle\, q_j(r)\,\rangle\ \propto\ |r|^{2(y_j - d)}\ . \tag{16}$$

It is convenient to introduce $x_j = d - y_j$. Note that also irrelevant operators have a power law decay for their two-point function, but these decay faster than $|r|^{-2d}$.

Exercise: *Consider a fixed point with only irrelevant parameters. All y_j are negative. Can you think of any further restriction on the possible values of these y_j?*

2.1 Summary

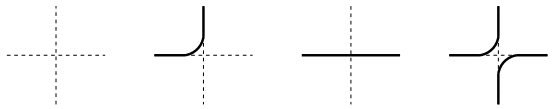
The existence of an RT is a strong and unproven assumption. It has the following consequences:

- **Relevant** and **irrelevant** parameters ($y \gtrless 0$) increase and decrease under (repeated) renormalization.
- Critical exponents and scaling functions do not depend on irrelevant parameters, but can only depend on invariants of the RT, e.g. symmetry and spatial dimension. In particular, they do not depend on the lattice. This claim is known as **universality**.
- Two-point functions decay with a power of the distance equal to $2x_j = 2(d - y_j)$, where y_j is the exponent of the coupling parameter conjugate to the operators correlated.
- All correlation lengths diverge with the same exponent, as a critical point is approached.
- Critical exponents are independent on the side from which the critical point is approached, but the corresponding amplitides are independent.
- Critical amplitudes and corrections to scaling do depend on the irrelevant parameters.
- The exponents for the various thermodynamic quantities are not independent, but satisfy a number of scaling relations.
- When integer linear combinations of the exponents y add up to d, corresponding integer power laws are modified with a logarithm.
- RG admits the existence of a line of fixed points, along which critical exponents may vary.
- At the point where an irrelevant parameter turns relevant, there are logarithmic corrections to the power law behavior.

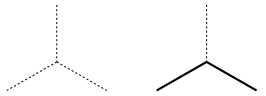
In all cases where exact solutions of model systems or accurate experimental or numerical data are available, (and these are many), these consequences (and more) of the RG are corroborated. For physics this is the ultimate test of a theory.

3 Loop models

We will discuss lattice gases of non-intersecting loops (polygons) on the lattice. The local configurations of these models can look like (up to rotations):



on the square lattice, or like



on the hexagonal lattice. The different local configurations on the square lattice, have weight 1, u, v and w, respectively and on the hexagonal lattice 1, and x. Each closed loop has weight n. The n=0 loop model denotes not just the $n \to 0$ limit, which is trivial, but the derivative of the partition sum w.r.t. n, which generates the configurations of a single loop. One may also take higher derivatives to describe the interaction between a fixed number of such loops.

Exercise: *Verify that the loop model for generic n is not local in the sense that a local change in the configuration changes the weight in a way that can not be determined by local inspection near the locus of the change. For what value(s) of n should one make an exception?*

Exercise: *Show that the hexagonal version can be recovered from the square by allowing the weights to be non-isotropic, and demanding that the weight on the square facorizes into two triangles.*

It is natural to make a distinction in the loop weights between contractible and non-contractible loops, and between true loops and paths that terminate on a boundary.

Similar models on the triangular lattice or more general semi-regular or even random lattices can be readily formulated. Also intersections, or loops in different colors can be introduced, or additional variables decorating the loops. These also have various 'applications' and have been studied. Here we consider only the simplest versions.

The configurations shown need not all have positive weight. If the first three (for the square lattice) have zero weight, every edge is covered. This model will be referred to as Completely Packed Loop (CPL) model. If only the first and the last are omitted, every vertex is visited once: the Fully Packed loop (FPL) model. This also has a hexagonal version by taking only the second of the hexagonal vertices.

When all vertices have non-zero weight (on the square or hexagonal lattice), one can still control the density of loops by the weight of each vertex configuration. The model has a non-critical phase for small step fugacity, then a continuous phase transition at into a critical phase for large step fugacity. At the phase transition the model is sometimes called the dilute loop model, and in the large step-fugacity phase the dense loop model.

These models have universal critical behavior (i.e. depending on n only) when the loop weight is $-2 \leq n \leq 2$. They also have other critical points but these are

induced by frustration due to the specific lattice structure, and are therefore not universal in the traditional sense of independence of the lattice.

3.1 ADE models

As explained in Cardy's lecture these loop models can be used to represent ADE models. These are models in which the faces of the lattice assume discrete states, represented by the nodes on a graph $\mathcal{G}$, with links between states that are allowed to be adjacent. The name ADE models comes from the classification of graphs with largest eigenvalue less than 2. These are the A_m series (an open linear graph with m nodes), the D_m series (one branch pont with three legs with respectively 1, 1, and m-3 edges), and the exceptional E_m diagrams with $m \in \{6, 7, 8\}$ (again a three-legged diagram with 1, 2 and m-4 edges respectively).

The adjacency matrix of the graph is called $A_{i,j}$. ($A_{i,j}$ is one if the nodes i and j are connected and zero otherwise) The weight of the ADE model is completely local, and written as a product over the vertices:

$$W_{\rm ADE} = \prod_{\text{vert } k} W(k) \prod_{\text{turns}} A_{ij} \left(\frac{S_j}{S_i}\right)^{\gamma_{\rm b}/2\pi} . \tag{17}$$

There is a factor $W(k)$ for each vertex depending only on the local configuration of domain walls, and a factor for each turn of the domain wall which also depends on the states of the faces on the inside (j) and outside (i) of the loop. The bending angles $\gamma_{\rm b}$ are counted positive where the loop bends inwards, so that the sum of the bending angles along a loop is $+2\pi$. Thus the weight of an entire closed domain wall is

$$W_{\rm loop} = A_{j,k} \frac{S_k}{S_j} , \tag{18}$$

and thus depends on the state k inside and j outside the domain wall. We now choose S_j to be an eigenvector of A: $\sum_j A_{i,j} S_j = S_i$. If we choose the Perron-Frobenius (PF) eigenvector with only positive elements, then the local weights are real and positive. When for a fixed configuration of domain walls the sum over compatible state configurations is performed, each closed domain will thus contributes a factor equal to the eigenvalue of $\mathbf{A}$, i.e. n, just as in the loop model. This works perfect on a singly connected bounded domain with all the boundary sites in the same state. With other boundary conditions the paths may terminate on the boundary, and these paths may have a weight different from n.

On a torus, however, there may be non-contractible loops, which do not have the weight $A_{j,k}\ S_k/S_j$, but simply $A_{j,k}$. The domains separated by these winding loops form a closed chain. Once all contractible domains are summed over, the only sum that remains is over the non-contractible domains. This sum is a trace over a power of the matrix A. The ADE partition sum on the torus therefore reads,

$$Z_{\rm ADE,T} = \sum_j \sum_{\rm cfg} \prod_{\text{vertices } k} W_k\ n^{N_{\rm c}}\ \Lambda_j^{N_{\rm w}} \tag{19}$$

where the first sum is over the eigenvalues of A, the second over all loop configurations, and $N_{\rm c}$ is the number of contractible loops, and $N_{\rm w}$ of winding loops. When

$N_{\rm w} = 0$, the sum on j is simply the sum over the states of the outermost domain. Each term in the sum on j corresponds a sector of the Transfer Matrix, and to a character in the conformal partition sum.

The other eigenvalues can also be used as a possible value for n. In this case the ADE model may have complex weights, but this is not necessarily the case. The model will have negative dimensions, which can be a blessing or a curse. It may be avoided by some selection rule that excludes the negative weight sectors (i.e. the sectors with eigenvalue $> n$).

3.1.1 ABF models

The models with adjacency diagram A_m have been studied by Andrews, Baxter and Forrester, and are called ABF models the restricted solid-on-solid (RSOS) model. It can be viewed as a spin-(m-1)/2 Ising model. Since the Boltzmann weights depend on the position on the diagram, the model has a parameter space that increases with m. The specific critical point that is obtained from the prescription described above, and based on the CPL model, is not the Ising critical point, except for m=3. For increasing m it gives a sequence of multicritical points, each one selecting the point in parameter space where the previous one turns first-order. The same is true for the general loop models in the dense phase. In the dilute loop models the same sequence of universality classes is found, with m shifted down with one unit.

Exercise: *Calculate the loop weights n for the ABF models. Note that the loop weight for the Ising critical point (or the points in universality class of the Ising critical point) can take two values. How are they related to SLE?*

3.1.2 The A_∞ model

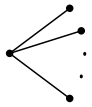
The A_m models in the limit $m \to \infty$ will play an important role in these lectures, since they are the basis for the Coulomb Gas (CG) theory. The position along the diagramm is sometimes referred to as height, and the model named height-model. They have a continuous spectrum of eigenvalues $-2 \le \Lambda \le 2$, which makes them very flexible. Also they have running wave modes as eigenstates, so that the elements of the eigenvector are all the same in magnitude, only different in phase. We will use them mainly as a calculational tool: they admit any possible loop weight in the interval [-2,2], and the operators in other models can usually be translated in the A_∞ language.

Only for $n = 2$ the A_∞ has (positive) real weights, and can be used to describe the behavior of a crystal surface. Each height representing a stack of atoms with respect to some reference position. The non-critical phase is called smooth, referring to the fact that all the heights globaly concentrate around one given value (symmetry breaking of the Z symmetry). The regions of other heights form small islands or (lakes) and occasionally islands within islands.

The dense phase is called rough, because here the heights fluctuate wildly, the local averages of heights take continous values, and the system (on a large scale) no longer has preference for integer values. The transition between the smooth and the rough phase is called the roughening transition. It is interesting to note that in real crystal the roughening temperature depends on the direction in which the height is

measures. The roughening temperature is lowest in the directions of high symmetry. As a result a crystal in equilibrium with a melt, a vapor or a solution, has smooth facets separated by rounded (rough) edges and corners. The experimental findings are in good agreement with the theory. A more open question is the shape of a crystal in various growth conditions.

3.1.3 The Potts model

Since symmetry is an important determinant of a universality class, it is useful to construct adjacency diagrams with special symmetries. An extensively investigated model is the Potts model with the permutation group of q elements as its symmetry (the symmetric group S_q). The appropriate diagram is  . with a general number of q legs. We can refer to the q nodes as $1,2,\cdots,q$, and to the neutral site as 0.

Exercise: *Verify that the largest eigenvalue is $\sqrt{q}$ and its eigenstate $(\sqrt{q},1,1,\ldots)$.*

It follows that the loop weight of the Potts model is $\sqrt{q}$. In the (homogeneous and isotropic) CPL model, since it is bipartite, one sublattice is completely frozen in the neutral state, while the equivalent q states live on the other sublattice. One may ignore the frozen sublattice and deal with the effective interactions on the dynamic sublattice only. The PF eigenvector has weight 1 for each of the q equivalent nodes, and $\sqrt{q}$ on the neutral node (up to normalization).

Exercise: *Verify that in each carre of four faces of the lattice, The weight of two unequal states across the diagonal of a square is 1, while the weight of two equal states is 1+$\sqrt{q}$.*

This formula for the Boltzmann weight suggests that there is also a critical point with the negative sign of the $\sqrt{q}$. This continuation sits in a critical thermal interval, as we shall see later. In any case this model, as it is built with the Pasquier construction, represents a simple Potts model on a sublattice of the original lattice.

In principle we can use any loop model in this construction, but then all q+1 states can sit anywhere in the lattice. Such models are known as *dilute* Potts models, referring to the neutral state as a vacancy or an unoccupied site.

Exercise: *What are the other eigenvalues of the adjacency diagram? The meaning of the weight changing operators associated with the other eigenvalues have a clear physica meaning. They will come back later, but in principle at this point you have the ingredients to figure out their meaning.*

3.1.4 Percolation, USP, SAW and θ SAW

Percolation is a description of transport behavior in a random mixture of permeable and non-permeable materials (or conducting and insulating). One prototypical model is site percolation on the triangular lattice. It is a perfect model for a mixture

of copper and glass spherical beads on a tray, (shaken in a triangular array). Only when more than half of the beads is copper, a large array will conduct. This model is well described by the hexagonal loop model with $n = 1$ and $x = 1$. Another prototypical model is bond percolation on the square lattice. In bond percolation neighboring sites are connected (by a conductor) with probability p, independently of each other. Again precisely at p=1/2 the model undergoes a transition between a conductor and an insulator. This model at the transition is described by the CPL loop model with n=1. This description makes it evident that percolation is the $q \to 1$ limit of the Potts model.

Uniform spanning trees (USP) are the set of all tree-like subgraphs of a lattice, with equal weight. The CPL loop model with n=0 is a description of this ensemble. The USP are the (or rather a) $q \to 0$ limit of the critical Potts model.

Exercise: *Convince yourself of all the above assertions in this subsection (except of course the definitions).*

Self avoiding walks (SAW) are used to describe linear polymers in solution. The $n = 0$ dense or dilute loop model give a partition sum of the closed SAW, grand canonical in the steps (or monomers). Small step fugacity describes small polymers, and critical (dilute) step fugacity the solution of a very long polymer.

In a poor solvent the polymer has a tendency to phase separate. The phase transition between a well-dissolved polymer and a co-agulated polymer is called the θ-point, and is described by the hexagonal loop model with n=1 and x=1. Of course this model itself has many loops. But by the special values of the weights one may, for a fixed configuration of one open or closed walk, sum over all compatible configurations of the other loops. This results in an attractive interaction between the segments of the original walk as they are spatially close, but distant along the chain.

Exercise: *What is the nature of the attractive interaction?*

3.1.5 Hard Hexagon and Hard Square model

With the A_m on the CPL model, not all m need to be distinguished. Since the lattice is bipartite, one sublattice can take only even nodes as their states, and the other only odd. This admits a simplification by identifying even and odd nodes as the same state. We identify the nodes as 1,0,0,1, succesively. The numbers are interpreted as the occupation number in a lattice gas. In this model every face can be occupied by at most one particle and two adjacent sites can not be simultaneously occupied. On the square lattice this model is called the Hard Square model, as the particles can be seen as tiles of size $\sqrt{2} \times \sqrt{2}$ placed in diagonal orientation with respect to the lattice. The neighbor exclusion is equivalent to the restriction that the tiles may not overlap. The critical point is in the universality class of the tricritical Ising model.

Exercise: *In the Hard Square model that follows from this construction, two particles at distance $\sqrt{2}$ have an attractive interaction, calculate the strength of this interaction.*

Exercise: *Verify that this Hard Hexagon model can be obtained from the CPL model in which the weights (up to a sign) have the same ratio as the elements of the eigenvector of the A_4 diagram (i.e. the golden mean $(\sqrt{5}+1)/2$). One need not take the PF eigenvector. To get full hexagonal symmetry, the two sites across one diagonal may not be occupied at the same time, and the two sites across the other diagonal must not interact.*

Exercise: *Motivate that the Hard Hexagon model must have the same critical exponents as the three-state Potts model.*

3.2 The O(n)model

Another representation of the loop model on the hexagonal model, is the O(n)models with

$$Z_{\mathrm{O}(n)} = \int \mathcal{D}s \prod_{\langle j,k\rangle} (1 + x s_j \cdot s_k) \tag{20}$$

The n-component spins sit on the vertices of the hexagonal lattice, and the interaction factors are between nearest neighbors. The name O(n)model, I derived from the global O(n)symmetry under which the measure is invariant. The single spin measure and the spin length are normalized such that $\int \mathcal{D}s = 1$, and $\int \mathcal{D}s\ s \cdot s = n$. The loop model is obtained as the expansion of $Z_{\mathrm{O}(n)}$ in powers of x: The product under the integral is expanded, and the second term in the binomial is indicated as a bond on the corresponding lattice link. The only terms in that expansion that survive the integration over the spins have an even number of bonds incident in each lattice sites. The spin integration involves a sum over spin components, which results in a factor n per loop.

Exercise: *Show that the square loop model can be obtained by O(n)spins on the vertices of the 4-8 lattice (a lattice with coordination number 3, where, at each vertex two, octagonal and one square face meet)*

Ordinarily one is interested in the critical behavior of the ordinary n-component Heisenberg model:

$$\int \mathcal{D}s\ \exp\left(\sum_{\langle j,k\rangle} K s_j \cdot s_k\right) \tag{21}$$

on any lattice. Universality would in principle assert that this model, having the same O(n)symmetry, has the same critical behavior as (20), but let us look critically at the difference between the two models.

By expanding the exponent in(3.2) one gets multiply occupied bonds (powers of $K s_j \cdot s_k$). One then has to sum over all ways to connect the elementary steps of the diagrams pairwise in each vertex. (see exercise below) Every closed polygon acquires a weight n by the summation over the indices. By the mutiple occupancy of the bonds, and also for singly occupied bonds on a four- or more-coordinated lattice, the polygons may cross each other as well as themselves. Thus the main differences are (i) multiply occupied bonds, and (ii) intersections.

Let us take the premise that the model (20) is exceptional in its critical behavior. This implies that it is a fixed point in a larger parameter space, and the deviation

from it is relevant. This means that such positive deviation will grow, but if it is zero it remains so.
When one looks at the graphs of both models (20) and (3.2) from a big distance, multiply occupied bonds can not be distinguished from neighboring parallel edges, each of which is singly occupied. This suggests that under RT multiply occupied bonds will be generated. This inconsistent with the premise that these multiply occupied bonds contribute to a relevant perturbation.

Thus loops seen on a great distance may begin to touch each other. However, they will not be seen to intersect, if they did not intersect to begin with. This suggests that intersections can not be generated by an RT, and therefore a model without intersections may well be a fixed point in the space of models where intersections are allowed. The non-intersecting loop model will be generic if the intersections are irrelevant, and exceptional if intersections are relevant. This will have to be verified.

Exercise: *Consider the O(n)spin model (20) on the square lattice, and find the set of possible diagrams in the formal expansion of Z in powers of x, and the prescription for the weight of these diagrams.*

3.2.1 Models with Cubic symmetry

The cubic symmetry in this context denotes the symmetry group of the n-dimensional hypercube. The spins can be discrete and point say to the vertices of the cube, or to the hyperfaces of the cube (i.e. the vertices of the hyper-octahedron), but they can also be continuous, and have a weight function with cubic rather than O(n)symmetry. As long as the partition sum is defined as in (20) (with the spin-measure adjusted), the small-x expansion is again the loop model.

Another possibility is to place spin on the faces of the lattice; let each of n spin components be an Ising spin, and let nearest neighbors be different in at most one of the components, at the weight cost x. Then the expansion of the partition sum in powers of x (in this case a Low Temperature expansion) is again a loop model, the loops signifying domain walls between domains of equal spin. Since two neighboring domains can differ in one of their n components the loop weight is again n. In fact this is a form of an ADE model, in which the graph $\mathcal{G}$ is the hypercube, with vertices (corners) as nodes, and edges as links.

3.3 Operators and Correlation functions

We will now calculate the one-point distribution (1PD) $P(k)$ of the ADE model, i.e., the probability that a face is in state k. Consider a loop well inside a large lattice. We assume that the 1PD is unaffected by the presence of the loop (or any other loop). In other words we assume that the 1PD conditional on the presence of a loop is the same as the unconditional 1PD. That this is plausible follows from the calculation of the partition sum above: the contribution to the partition sum of a particular domain is independent of the domains it is contained in, and it is independent of all the domains it contains, once the state of these domains has been summed over.

The conditional probability $P(k|j)$ that the inside domain of a loop is in state k, provided the outside domain is in a given state j, is determined by Eq. (18) as

$P(k|j) = A_{k,j}S_k/(n\,S_j)$. Thus we find the joint probability $P(k,j)$ that the outside of a loop is in state j *and* its inside in state k as

$$P(k,j) = P(j)P(k|j) = P(j)A_{k,j}\,\frac{S_k}{n\,S_j}\,. \tag{22}$$

Summation on j now yields the probability that the inside domain is in state k, which should be equal to $P(k)$:

$$\sum_j P(k,j) = P(k)\ . \tag{23}$$

Using the symmetry of A, one finds the unique solution to this consistency condition as

$$P(k) = (S_k)^2\ . \tag{24}$$

Here and below we assume that the eigenvectors are normalized, with $\sum_j (S_j)^2 = 1$, and in case the eigenvector is complex, the square should **not** be replaced by the absolute square.

An approach alternative to the condition that the loop considered is well inside a large lattice, is to consider a bounded lattice of arbitrary size, with the faces on the boundary all in the same state, *but with Eq. (24) as the probability distribution for that state*: the ideal fixed boundary condition. Then by induction the same distribution holds for the domains separated from the boundary by one domain wall, and so on recursively to the innermost domains. It is then assumed that in the thermodynamic limit the boundary condition should not matter, well away from the boundary.

Consider the function S^μ_k/S_k, i.e., the ratio of an arbitrary eigenvector S^μ and the Perron-Frobenius eigenvector S. If this function is part of a correlation function $\langle\cdots S^\mu_k/S_k\cdots\rangle$, where k is the state of a given face, it effectively changes the weight of the loops *surrounding the face*, as long as they do not surround other operator insertions. This is easily seen in the expression (18): the factor S^μ_k/S_k replaces the numerator by S^μ_k, so that the weight of the loop becomes that of the corresponding eigenvalue, say Λ_μ. We will call these functions weight-changing operators of weight Λ_μ.

A more interesting result comes from the two-point function

$$\left\langle \frac{S^\mu_j}{S_j}\,\frac{S^\nu_k}{S_k}\right\rangle\ ,$$

j and k being the state of two arbitrary faces. The weights of the loops surrounding either of these faces but not the other is changed into the respective eigenvalues Λ_μ and Λ_ν. corresponding to the eigenvectors S^μ and S^ν. Now consider the innermost domain wall that surrounds both faces. After the states of the domains nested inside it are summed over, the weight governing the state of the final domain is

$$\frac{S^\mu_j S^\nu_j}{(S_j)^2}A_{j,k}\ ,$$

where k is the state of the surrounding domain. This can be expanded as a linear combination of all eigenvectors:

$$S^\mu_j S^\nu_j = S_j \sum_\kappa C^\kappa_{\mu\,\nu} S^\kappa_j\ ,$$

where, (again provided the eigenvectors are normalized),

$$C^\kappa_{\mu\,\nu} = \sum_j \frac{S^\mu_j S^\nu_j S^\kappa_j}{S_j}.$$

Apparently the combination of two operators labeled μ and ν look from a distance like a linear combination of operators κ.

These structure constants of the operator product expansion, or fusion rules, may of course be readily calculated explicitly for any adjacency diagram, but here we only note that (i) they are symmetric in μ, ν and κ, that (ii) they take integer values for A_m, and (iii) that they vanish if one of the indices corresponds with the largest eigenvalue, and the others two differ.

Exercise: *Verify these properties for the A_m diagram.*

Property (iii) implies that the two-point correlation function of two different weight-changing operators vanishes in the thermodynamic limit. Obviously these structure constants may be used just as well in correlation functions of more than two operators.

3.4 Terminal operators

If the adjacency diagram $\mathcal{G}$ possesses symmetries, one may introduce cuts in the lattice accros which the nodes s of the diagram on one side of the cut are identified with Ps on the other, where P is in the symmetry group $\mathcal{S}_\mathcal{G}$ of $\mathcal{G}$. The end point of one or more of these cuts, acts as the terminal of a number of domain walls, (as many as there are steps to connect the equivalent nodes). The fusion rules of these operators are not so easy to formulate, but obviously depend on the properties of $\mathcal{S}_\mathcal{G}$.

4 The Coulomb Gas

The CG approach is based on the ADE model with state diagram A_∞. The states are called heights and traditionally take values in $\pi\mathrm{Z}$, rather than Z. We imagine an RG transformation in which the heights are not rescaled. Let us first consider the case n=2, so that the loops can be considered domain walls surrounding a hill or a valley both with weight 1.

On a large scale the heights will take continuous values with possibly a preference for integer multiples of π. Since the renormalized Hamiltonian can only depend on differences or gradients of h, it should be expandable in even powers of the gradient:

$$H = \int \mathrm{d}^2 r\ \frac{g}{4\pi}(\partial h)^2 + g_2(\partial h)^4 + \cdots - z_2\cos 2h - z_4\cos 4h - \cdots \tag{25}$$

The free field Hamiltonian, with only the first term, is unchanged under RG, which follows from simple power counting. By the same argument the higher order powers in ∂h will be irrelevant. To inspect the relevance of the terms z_j we can calculate the correlation function

$$\langle \cos(eh_o)\ \cos(eh_r)\ \rangle = \frac{1}{2}\langle \exp(\mathrm{i}eh_o - \mathrm{i}eh_r)\ \rangle \tag{26}$$

in which the prefactor e may take the values 2, 4 etc. To calculate the r dependence, we need to do the integral

$$\int \mathcal{D}h\ \exp\left(\int \mathrm{d}^2 r\ \Big[\ -\frac{g}{4\pi}\ (\partial h)^2 + \mathrm{i}eh_o - \mathrm{i}eh_r\Big]\right) \tag{27}$$

a Gaussian integral. Since $\partial \cdot \partial \log h = 2\pi\delta(h)$, the result is

$$\langle \cos(eh_o)\ \cos(eh_r)\ \rangle\ \propto\ \exp\left(\frac{-e^2}{g}\log|r|\right) = |r|^{-e^2/g} \tag{28}$$

The middle expression explains the notation e for the prefactor: the operators behave as particles with electric charge e (not to be confused with the base of the natural logaritm e. The $\cos(2h)$ term is relevant if the correlation funtion decays slower than $|r|^{-4}$, i.e. when $g > 1$. The higher harmonics will be less relevant, so we need only consider them if for some special reason z_2 vanishes.

To get a physical picture of what this all means, realize that positive values of z_2, imply a positive density of charges: hence the name Coulomb Gas. If g is large, the coupling of the charges is weak, and they may effectively screen all Coulombic interaction between test charges. However, when g is small, the charges are strongly coupled so that they may form charge neutral complexes. This is what causes the crossover between the smooth and the rough crystal phase. In the language of the charges they can be called a plasma and a molecular phase respectively. It may sound counterintuitive that the dynamic rough phase corresponds to the boring molecule phase.

The value of g, resulting from the original loop model, is not a priori known. It is evident that g will be a decreasing function of the original step fugacity. When the step fugacity is small almost all heights will be the same: the smooth phase. If the domain walls are sufficiently numerous, the height fluctuations bcome more probable and it is reasonable that eventually g will sink under the threshold value 1. We can, however, conclude that everywhere in the rough phase, the exponent x_e of the correlation function

$$\langle \cos(eh_o)\ \cos(eh_r)\ \rangle\ \propto\ |r|^{x_e} \tag{29}$$

is proportional to e^2, and that at the roughening transition the prefactor is 1.

It should be noted that the presence of charges, even when they are irrelevant, will still weaken the electric interaction, thus increase the value of g, by partial screening. We need not calculate the precise strength of this effect, but the sign is of importance.

4.1 Vortices

In some of our applications we will need the behavior of vortices or screw disclocations in the height model. Such an excitations abound in systems in which the height is defined on a periodic interval. In order to calculate the exponent of vortices, it is more convenient to consider only two vortex insertions, in the absence of other vortices or periodic fields. Consider thus the integral

$$\int \mathcal{D}h\ \exp\left(\int \mathrm{d}^2 r\ -\frac{g}{4\pi}\ (\partial h - 2\pi k)^2\right) \tag{30}$$

Here we take k a fixed vector field, zero almost everywhere, and the value m on a line that connects two vortex insertions, in the direction orthogonal to the line. We use the continuum space notation because it is more compact, but it is convenient to think of the ∂'s as lattice differences, and the spatial integral as a sum over the lattice. However, we may do 'partial sums' just as we may do partial integrals: $\int \mathrm{d}^2 rA\partial B = -\int \mathrm{d}^2 rB\partial A$ provided $\int \mathrm{d}^2 rAB$ vanishes.

We have a Gaussian integral with terms in the exponent, quadratic and linear in h. This can be done quickly after the introduction of the notation G for the inverse of the Laplace operator $\partial \cdot \partial$. It differs from the continuum version $\log(r - r')/2\pi$ only by a constant term and terms of order $\mathcal{O}(r^{-2})$. The action then is transformed as

$$\begin{gathered}
\Big[-\frac{g}{4\pi}\ (\partial h - 2\pi k)^2\Big]\ \ \rightarrow \\
\Big(\frac{g}{4\pi} h\partial\cdot\partial h - gh\partial\cdot k - \pi\ g\ k\cdot k\Big)\ \ \rightarrow \\
\text{(doing the Gaussian integration)}\ \ \Big(-\frac{g}{2}(\partial\cdot k)G(\partial\cdot k) - \pi\ g\ k\cdot k\Big)\ \ \rightarrow \\
\Big(-\frac{g}{2}(\partial\cdot k)G(\partial\cdot k) - \pi\ g\ k\cdot(\partial\cdot\partial)G\ k\Big)\ \ \rightarrow \\
\Big(-\frac{g}{2}(\partial\cdot k)G(\partial\cdot k) - \pi\ g\ k\cdot(\partial\cdot G\ \partial k)\Big)\ \ \rightarrow \\
\Big(-\frac{g}{2}(\partial\times k) \rightarrow G(\partial\times k)\Big)\ \ \rightarrow \\
|r|^{-gm^2}
\end{gathered} \tag{31}$$

This is very much like the correlation of two functions periodic in h, but here the exponent is proportional to g. They are viewed as magnetic rather than electric charges.

The interaction between electric and magnetic charges is fairly obvious: when an electric charge is taken around a vortex, it must gain a phase factor equal to $\exp(2\pi\mathrm{i}em)$, so that the interaction will look like

$$\exp[\mathrm{i}em\arg(r)] \tag{32}$$

4.2 Changing the loop weight

When we want to consider loop models with general n, we can still make use of the A_∞ model, but select a subdominant eigenvalue, say, $n = 2\cos\pi e_o \leq 2$. The corresponding eigenvector is $\exp(e_o \mathrm{i} h)$. With this information we can follow the protocol of Sec.3.1 and construct the weights of the height model. Each bend of the loop with angle γ_b towards the higher h, has a weight $\exp\left[\frac{e_o \mathrm{i}\gamma_b}{2\pi}\right]$. Note that it is the exponent that should be added to the Hamiltonian, and not the Boltzmann weight itself:

$$H \to H + \mathrm{i}e_o \sum_{\text{loops}} (h_{\text{in}} - h_{out}) \tag{33}$$

It is an interesting game, (or a difficult exercise) to note that this addition in the continuum has the form

$$\mathrm{i}e_o \frac{\partial h \cdot \partial\partial h \cdot \partial h}{\partial \cdot \partial h} \tag{34}$$

This object measures the curvature density of the contour lines (level sets) of h. Integrated over the entire configuration this takes the form of the sum of the values of h at the extrema, minus the sum of h at the saddle points minus twice h at infinity. This is fairly obvious only when it is taken into account where the expression comes from. On a curved surface it should be incremented with $\frac{\mathrm{i}e_o R h}{2\pi}$, where R is the local curvature.

Now consider the model on a compact domain, and take the heights at the boundary all equal to 0. This boundary condition precisely reproduces the loop model, where the loops are not permitted to penetrate the boundary. Now we introduce the weight-changing operator, based on the eigenvector $\exp(-e_o \mathrm{i} h)$, with eigenvalue n, and name it $\mathcal{B}$. When we insert $\mathcal{B}$ at some position in the lattice, the effect to the weights of the loops in the configuration, is nil, because the changed weight is the same as the original. The operator $\mathcal{B}$, however is not the identity operator. In particular, when it is inserted twice (possibly in different places), the configuration weights do change, because the weight of the loops that surround both insertions does change.

Expressed directly in the height variables the operator has the form

$$\exp\left(-2\mathrm{i}e_o h\right) \ , \tag{35}$$

an electric charge of strength $-2e_o$. It is sometimes called the background charge, or charge at infinity. This can be understood as follows: The loops surrounding a hill have weight $\exp\left(e_o \mathrm{i}\pi\right)$, while the loops that surround a valley have weight $\exp\left(-e_o \mathrm{i}\pi\right)$. For the loops that surround the operator insertion these two weights are interchanged. This can be taken to mean that for those loops inside and outside are interchanged. In other words, the operator insertion is effectively the point at infinity.

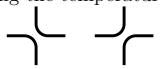
In summary we note that there are two alternative ways to calculate the partition sum: with and without the insertion of the background charge. This will play an important role in the sequel.

4.3 Application to the Potts model

In this section we will show that the CG predicts the critical exponents of the Potts model. First we remember that the loop weight for the q-state Potts model is $\sqrt{q}$, and that the background charge follows from $\sqrt{q} = 2\cos(e_o\pi)$ (up to a sign and additive multiples of 2). We will first concentrate on the thermal or energy exponent.

4.3.1 The energy operator

We have seen that the Potts model has weight $1 + \sqrt{q}$ for two neighboring equal spins and weight 1 for unequal spins. Changing the temperature of the Potts model would result in a change of the ratio of these weights. The operator conjugate to the (inverse) temperature is the energy operator $\mathcal{E}$.

What does $\mathcal{E}$ look like in the loop language. We should note that the Potts spins live on only one of the two sublattices of the faces of the CPL model. We place the Potts variables on the sublattice where the heights are an even multiple of π and the neutral state corresponds to the odd multiples of π. Changing the temperature results in a change in the relative weight of the two diagrams  .

Thus the energy gives an opposite sign to weight of these two diagrams. What does this do the the weight of the total configuration? Note that the Potts sites are always surrounded by an even number of loops. If we concentrate on the center of the two diagrams, this too must be surrounded by an even number of loops when the two Potts variables are 'connected', and an odd number otherwise. In the latter case it has a negative weight. As a result, we can identify $\mathcal{E}$ with a weight changing operator of weight $-n = -\sqrt{q}$.

We will now use this to see what $\mathcal{E}$ looks like in the heights. First we will inspect the weight of a sequence of plaquettes with increasing height (factors π in the height are omitted).

config.	$\begin{matrix}0 & 1\\ 1 & 0\end{matrix}$	$\begin{matrix}0 & 1\\ 1 & 2\end{matrix}$	$\begin{matrix}2 & 1\\ 1 & 2\end{matrix}$	$\begin{matrix}2 & 3\\ 1 & 2\end{matrix}$	$\begin{matrix}2 & 3\\ 3 & 2\end{matrix}$
height	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$
Boltzmann	$2\cos(e_o \mathrm{i}\pi/2)$	1	$2\cos(e_o \mathrm{i}\pi/2)$	1	$2\cos(e_o \mathrm{i}\pi/2)$
$\mathcal{E}$	$2\mathrm{i}\sin(e_o \mathrm{i}\pi/2)$	-1	$-2\mathrm{i}\sin(e_o \mathrm{i}\pi/2)$	1	$2\mathrm{i}\sin(e_o \mathrm{i}\pi/2)$

(36)

First we note that the unperturbed Boltzmann weight alternates between integer and half-integer values of h. This corresponds with the operator $\cos(2h)$, or positive and negative charges of magnitude 2. We must conclude that these charges are ubiquitous, and must be irrelevant for the model to be critical.

The energy operator alternates with period 2π in h. This corresponds to charges of magnitude 1. However, it is not symmetric in positive and negative charges. It is dominated by a positive charge. Since the integer-height states and the half-integer

height states are not equivalent, one can not say how a single positive charge would look like precisely, as a function of h. For the moment we take the point of view that $\mathcal{E}$ is just a positive charge.

We can de the reverse and ask which charges correspond to the appropriate change of loop weigt that corresponds with the operator $\mathcal{E}$. Triple charges or in general all odd magnitude charges have the same effect as single charges on the total loop weight. It should be possible to combine the operators $\mathcal{E}$ and $\mathcal{B}$. The operator $\mathcal{B}$ inverts the factors $\exp(e_o \mathrm{i}\pi)$ for every loop surrounding the position. This means that the the effect of $\mathcal{E}$ is the complex conjugate of what it was without the insertion of $\mathcal{B}$. Conclusion, the operator $\mathcal{E}$ can be represented by a unit positive charge, but equally well by a charge -1-2e_o. It is important to note that the charge 1 and the charge -1-2e_o are not just good alternatives for $\mathcal{E}$. The statement is stronger: *these charge operator represent have exactly the same effect on the weight of the loop configuration.*

4.3.2 The energy two-point function

Having specified the nature of $\mathcal{E}$, we are in the position to calculate its two-point function. In the loop language the $\mathcal{E}$ two-point function picks up a minus-sign for each loop that surrounds precisely one of the $\mathcal{E}$ insertions, and leaves unchanged the loops that surround both. Since there are two versions of the single $\mathcal{E}$ operator, there are three candidates for the two-point function, to wit the following charge pairs: $(-1-2e_o, -1-2e_o)$, $(1, -1-2e_o)$, and (1,1). The weight of a loop is given by $2\cos(e\pi + e_o\pi)$, where e is the total charge enclosed. The first candidate gives a minus-sign to the loops that surround a single charge, but the loops that surround both will have weight $2\cos(3\pi e_o)$. The second works fine: both $2\cos(\pi + e_o\pi)$ and $2\cos(-\pi + e_o\pi)$ are equal to $-\sqrt{q}$, $2\cos(-e_o\pi) = \sqrt{q}$. The corresponding two-point function behaves like

$$\langle\, \mathcal{E}(o)\, \mathcal{E}(r) \rangle \;\propto\; |r|^{\frac{-1-2e_o}{g}} \tag{37}$$

The third candidate works equally perfect for the loop weights. However, in the CG language it has the difficulty that it is not charge neutral. We have learned to accept configurations of a total charge $-2e_o$, but not 2. This problem, however can be corrected by noting that charges of magnitude 2 float freely in the system. Therefore the interaction between the two positive charges can be screened by a free floating negative charge. Thus we get:

$$\langle\, \mathcal{E}(o)\, \mathcal{E}(r) \rangle \;\propto\; \int \mathrm{d}^2 r' |r|^{\frac{1}{g}}\, |r - r'|^{\frac{-2}{g}}\, |r'|^{\frac{-2}{g}} \;\propto\; |r|^{2-\frac{3}{g}}\,, \tag{38}$$

from simple power counting. Note that, depending on the value of g, as yet unspecified, one may have to regularize the UV singularities of the integral. If in the scaling limit $\mathcal{E}$ consists of positive charges only, they must be prevented to form a unit neative charge by picking up a -2.

Since these two calculations of the two-point function represent exactly the same loop weights in the corresponding loop correlation function, they must be equal.

$$\frac{-1-2e_o}{g} = 2 - \frac{3}{g} \quad \text{so that} \quad e_o = 1 - g \tag{39}$$

This argument resolves the uncertainty of the value of g, and is more satisfying than the argument, originally used by the author, that relies on an exact result from another source. It is (to some) more convincing than the requirement that the screening charges are marginal. This latter argument is sometimes ascribed to Kondev, but was first published 8 years earlier by Foda and the author, and is really due to Pasquier.

Exercise: *Acting on the premise that $\mathcal{E}$ also contains an admixture of negative charge, one may still use the argument that the exponent with and without the operator $\mathcal{B}$ should be the same. Show that this leads to a contradiction. (it is a bit messy from the many cases that should be checked)*

Exercise: *A third way to calculate the two point function is to take the integral (38), and insert $\mathcal{B}$ at an arbitrary fixed position. Try to show that this integral leads indeed to the same result. Hint: note that the integrand is explicitly Moebius invariant, i.e. invariant for $z \to (a+bz)/(c+dz)$ with $ad-bc=1$, in complex coordinates. Then judiciously choose a Moebius transformation that simplifies the integral.*

Thus we have determined the thermal exponent of the q-state Potts model as

$$x_\varepsilon = 1 - \frac{3}{2g}, \quad \text{where} \quad \sqrt{q} = -2\cos\pi g, \quad \text{and} \quad g \le 1 \tag{40}$$

The last condition goes back to the requirement that the charges ± 2 should be irrelevant, and the notation with $\sqrt{q}$ emphasises that we need the positive root.

Exercise: *Calculate the α and the ν exponent of the three-state Potts model*

4.3.3 Critical and tricritical and antiferromagnetic points

The Potts model is defined for any positive integer q and its low or high temperature expansion for all q. It has been known for a long time that the transition is continous for $q \le 4$ and first-order for $q > 4$. The transition point is one analytic curve for all q, but the analytic continuation of the thermal exponent beyond $q = 4$, folds back to smaller q. This suggests that it describes critical behavior of another kind in a model with the same S_q symmetry. Continuity of the RG picture indicates that the other kind of critical behavior is a tricritical point, i.e. a point where the transition changes from continuous to first-order.

When we constructed the Potts model with the ADE construction, we noted that in the case of the dense or dilute loop models, the central node of the diagram plays the role of a vacancy or unoccupied site. Now, if a vacancy is introduced with large weight, and the Potts variables on the other hand have a strong interaction, there must be a first-order transition between the S_q symmetric dilute phase, and a S_{q-1} symmetric ordered phase. Thus, in a dilute Potts model one may expect a critical and a tricritical RG fixed point which are the analytic continuation of each other, meeting at $q = 4$. This gives us an interpretation for another branch of Eq.(40) for $g \ge 1$.

At this branch, the condition that the double charges are irrelevant is no longer necessary, because in order to reach the tricritical point one has to tune another

variable, which precisely suppresses the double charges. Without further motivation, we state that not all the double charges are suppressed, only the positive ones. This leads to a crossover exponent, which generates the RG flow from the tricritical to the critical point. The operator associated with this flow must be the presence of positive double charges. Like the leading thermal exponent, it can be calculated from two constructions of the two-point function. One is by letting two double positive charges be screened by two negative ones, and the other is by the simple correlation of a $+2$ with $-2-2e_o$. As in the case of the leading thermal exponent, the result is the same.

$$x_{\varepsilon,2} = 4 - 2g \quad \text{with} \quad g \leq \frac{3}{2} \tag{41}$$

This is the exponent that governs the RG flow on the transition manifold from the tricitical into to the critical regime, and further on into the critical fixed point (where it has a smaller value g and is irrelevant).

Exercise: *Show that both construction result in this same result*

The analytic continuation on the other side, i.e. beyond $q = 0$, of the critical Boltzmann weight, has already been mentioned. It is in the antiferromagnetic domain, where unequal spins have higher weight than equal spins. Having calculated the leading thermal exponent we see that in this AF regime, the temperature is an irrelevant parameter, and there should be a whole critical interval (in T), attracted to what is the analytic continuation of the FM critical point. This phase is known as the Kadanoff-Berker phase.

4.3.4 The magnetic exponent, spin and disorder operators

In the disordered phase of the Potts model, the probability that two distant spins are in the same state approaches exponentially the chance value $1/q$. At the critical point it still approaches the same value, but as a power law. The exponent is called the magnetic exponent.

Exercise: *Show that in the Potts model on the square lattice, the quantity* $\langle q\delta(s_o, s_r) - 1\rangle / (q-1)$ *is equal to the probability that the sites o and r are not separated by a bond, in the $n = \sqrt{q}$ CPL model.*

Taking the result of the exercise for granted, the magnetic or spin operator $\mathcal{S}$ is a weight changing operator of weight zero: any loop that surrounds precisely one out of the two operators of a spin two-point function, separates, and does therefore not contribute. A charge $e_s = \pm\frac{1}{2} - e_o$ does the job precisely, as it changes the weight $2\cos(e_o\pi)$ into $2\cos(\pi/2)$, moreover, the sum of the two different candidates is precisely $-2e_o$, the background charge. As a result the magnetic exponent of the Potts model is given by

$$x_s = \frac{\frac{1}{4} - (1-g)^2}{2g} \tag{42}$$

Exercise: *Verify this result and try to argue that a next to leading magnetic exponent is obtained by replacing the $\frac{1}{4}$ by $\frac{9}{4}$.*

4.3.5 The energy four-point function

The critical two-point functions are simple power laws, and thus contain no information beyond the value of the exponent. Also three-point functions, given that they are conformally invariant, are completely determined by their exponents. Four-point functions, however, are non-trivial. Here we sketch the way how to calculate them, mainly to demonstrate the power of the CG approach.

One approach is to take four positive unit charges to represent the operators $\mathcal{E}$. They should be screened by two double negative charges. This fourfold integral

$$\langle \mathcal{E}(r_1)\mathcal{E}(r_2)\mathcal{E}(r_3)\mathcal{E}(r_4)\rangle = \prod_{1\leq j<k\leq 4} |r_j - r_k|^{1/g} \int \mathrm{d}^2w_1\, \mathrm{d}^2w_2\ |w_1 - w_2|^{4/g} \prod_{j=1}^{2}\prod_{k=1}^{4} |w_j - r_k|^{-2/g} \tag{43}$$

is not very attractive to even attempt. It becomes simpler if one of the $\mathcal{E}$ is combined with a $\mathcal{B}$, so that quasi-neutrality is achieved with one less screening charge. The result is

$$\langle \mathcal{E}(r_1)\mathcal{E}(r_2)\mathcal{E}(r_3)\mathcal{E}(r_4)\rangle = \prod_{1\leq j<k\leq 3} |r_j - r_k|^{1/g} \prod_{k=1}^{3} |r_k - r_4|^{2-3/g} \times \int \mathrm{d}^2w\ |w - r_4|^{6/g-4} \prod_{k=1}^{3} |w - r_k|^{-2/g} \tag{44}$$

One may verify that the integrand is invariant for Moebius (see exercise above eq.(40)) transformations, which can be used to simplify the expression. The area integral can reduced to a contour integral, making use of the following theorem (it may even have a name), using complex notation for the plane. Let $p(u,v) = \partial/\partial v P(u,v)$ be functions analytic in a domain D, respectively $\bar{D}$

$$\int_D \mathrm{d}^2z\ p(z,\bar{z}) = \frac{1}{2\mathrm{i}} \oint_{\partial D} \mathrm{d}z\ P(z,\bar{z}). \tag{45}$$

The domain to be taken is the complex plane with appropriate cuts to exclude non-analyticities due to the fractional powers. When the dust settles the result is, expressed in the complex variables the variables $u_i = (r_i - r_4)(r_j - r_k)$ in which the indices i, j, k are a cyclic permutation of $1, 2, 3$.

$$\langle \mathcal{E}(r_1)\mathcal{E}(r_2)\mathcal{E}(r_3)\mathcal{E}(r_4)\rangle = -\frac{1}{2}\tan\frac{\pi}{g}\left\{\left(1 + 2\cos\frac{2\pi}{g}\right)|G|^2 + |H|^2\right\} \tag{46}$$

in which the contributions G and H are expressed in the hypergeometric function F and the Beta function B.

$$\begin{aligned} G &= \mathrm{B}\Big(\tfrac{3}{g} - 1,\ 1 - \tfrac{1}{g}\Big)\left(\frac{u_1}{u_2}\right)^{1/2g} u_3^{1-3/2g}\, \mathrm{F}\Big(\tfrac{1}{g},\ 1 - \tfrac{1}{g};\ \tfrac{2}{g};\ -\frac{u_1}{u_2}\Big) \\ \text{and} \qquad & \qquad (47) \\ H &= \mathrm{B}\Big(1 - \tfrac{1}{g},\ 1 - \tfrac{1}{g}\Big)\left(\frac{u_1 u_3}{u_2}\right)^{1-3/2g} \mathrm{F}\Big(2 - \tfrac{3}{g},\ 2 - \tfrac{1}{g};\ 2 - \tfrac{2}{g};\ -\frac{u_1}{u_2}\Big) \end{aligned}$$

It goes without saying that the 4-point function found here agrees with the special case of the Ising model, that was known at the time, and that it can be confirmed also for other q by conformal field theory. In particular it can be verified that this four-point function is conformally invariant. We stress that this is a result, and not a property that has been inserted in the procedure.

An interesting result of this calculation, even irrespective of the explicit outcome, is the prediction that the integrals (43) and (44) are the same up to an overal factor (which may actually be calculable).

4.3.6 Retrospective and Perspective

Before we continue, we wish to point out that the CG theory and more specifically the calculations done on the Potts model, is based on a number of hypotheses.

1. The RG is taken as valid.
2. While one can justify that the infinity of operators that separate the Potts models from the Gaussian free field with screening charges and background charge are irrelevant, *at the Gaussian model itself*, it may still be counted a miracle that no catastrofes happen along the way.
3. That the temperature of the Potts model traces out the special line where only positive unit charges are present is another miracle. However, this need not be taken as a hypothesis, as the contrary can be shown incorrect.

As in the discussion of RG, the proof of the pudding is in the eating, or in the biblical fashion, a rotten tree does not produce healthy fruit. In my opinion the many healthy results of this approach prove it to be correct (though not necessarily fully understood) As RG has already been confirmed so extensively, and the third premise seems to be internally compelling, the second hypothesis may be the weakest point. However, it looks very much like the central limit theorem, (in a field rather than a number). It may well be open to mathematical proof.

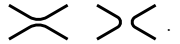
4.4 The O(n) spin model

In the case of the Potts model the critical point was known beforehand, (by duality) which was a significant advantage. In the O(n) spin model this is not the case, and in this section we will deal with precisely that problem. By using (20) as a starting point, we have already resolved a few difficulties. Every link can be occupied by at most one bond, and because the lattice is three-coordinated, intersections are not possible. The intention of this section is to make use of the knowledge we have acquired about the Potts model, to apply it to the O(n)model. The difficulty we will have to overcome, is that in the Potts model, every link in the model sits on a domain wall of the height model, while in the O(n)model it is essential that some edges are inside a domain. The way we will try to resolve this is by the observation that two adjacent domain walls may well be opposite in sign, and therefore neutralize each other on a slightly coarser lattice.

As a starting point we make the following observations. Above we argued that the Potts model at non-critical temperature admits positive unit charges, but not

negative. When negative unit charges are introduced, this will affect the renormalization of g. If the product of fugacities $z_1 z_{-1} \in \mathrm{R}_+$, the coupling constant g will grow, thus weakening the electric interaction by screening. However, if $z_1 z_{-1} \in \mathrm{R}_-$, the effect is opposite and g will decrease. At first $z_{\pm 1}$ will grow because they are relevant. This will strengthen the effect on the value of g, which will decrease faster. However, this makes it unavoidable that g passes the value where the $z_{\pm 1}$ are marginal, and turn irrelevant. From thereon, $z_{\pm 1}$ will diminish, and the model returns to the Gaussian model, but now a much smaller value of g. All this sums up to the conclusion that the Potts model (parametrized by its temperature) is a critical locus in a larger parameter space. If the height version of the Potts model is embedded in a slightly more general class of height models, it is highly unlikely that this more general class of height model does not contain $z{-}1$. Thus we propose: if a model $M(T)$, parametrized by its temperature, in a more general parameter space, intersects with the Potts model, it is to be expected that that intersection is the critical point of M. The pleasant side of these observations, is that (in order to use our knowledge about the Potts model) the O(n)model need not coincide with a Potts model entirely, but only at one temperature.

The speculative element of all this, is that the long flow away from the Gaussian line in the direction of a relevant parameter, all the way to another segment of the Gaussian model, where this same parameter is irrelevant, may well be interupted by other fixed points far away from the Gaussian line. However, in the case of the flow of the Potts tricritical point towards the critical point, precisely this mechanism is at work. Thus the speculation is not without hope.

Let's see if we can make it work. Consider the Potts adjacency diagram, placed on the faces of a kagome lattice, using the ADE construction with the CPL model on the Kagome lattice. The Potts variables live on the hexagons, and the neutral state in the trangles. This is equivalent with a Potts model on the triangular lattice, of which the critical point is known, but we do not need that here. The Potts model traces out a 2-dimensional parameter space of temperature and q. The temperature is parametrized by the weight ratio of the diagrams  .

Keeping the same height configurations, we can distinguish four local configurations that have different weights:

	0	2	2	0
config.	1 1	1 3	1 1	1 1
	2	2	2	0
Potts	1	B	$p + B/p^2$	$1/p + Bp^2$
O(n)	1	CD	C^2	D^2

(48)

up to even shifts in height.

If the heights in the triangles decouple from each other, they can be summed out, with the heights on the hexagons remaining. Neighbors can be the same or different

by 2 units. This has no other option than to be the A_∞ model on the hexagonal loop model.

The Potts model can be parametrized by the two weights of the local loop configurations, say 1 and B, and the factor p, for a 120 degree bend around a hill, as shown in the table. (note: $\sqrt{q} = p^3 + p^{-3}$)

The decoupling condition for the triangles requires that the leftmost and rightmost height in the configurations are independent. This is formulated in the weights on the line labelled O(n).

From here it is formula manipulation to see if the models intersect. But, since both models trace a two-dimensional manifold, and there are just three weights to match, we can expect a one-dimensional intersection. At that intersection, we then expect the O(n)model to have its phase transition, because the phases 'on either side' of the Potts model are very different.

Exercise: *Calculate the intersection of the model and show that, at the intersection point (i) the O(n)parameters x and n are related by $x^2(2 \pm \sqrt{n-2}) = 1$, and that (ii) the Potts model is in the antiferromagnetic regime and at zero temperature, and (iii) that the relation between n and the number q of states in the Potts model is $n = 2 - (2-q)^2$.*

The fact that the intersection of the O(n)model and the Potts model takes place in the antiferromagnetic regime of the Potts model, means that the thermal parameter of the Potts model is irrelevant, and that therefore the sensitive condition, that the RG flow has to take one far away from the Gaussian line, and back again to it on a different place (without meeting any RG) is no longer necessary. The critical point of the O(n)model will simply flow towards the nearest Potts fixed point in the direction of its irrelevant thermal operator. Now that we have done the calculation the outcome is such that we can have confidence that it is meaningful.

Assertion (iii) of the exercise implies (see below) that $n = -2\cos(4\pi g)$. This is different from the accepted notation in the literature i.e. $n = -2\cos(\pi g)$. This is related to the fact that in the model just constructed, the steps across domain walls have amplitude 2π, since they correspond to two steps of the Potts model. To keep contact with most of the literature, we will rescale the heights $h_{\rm new} = h_{\rm old}/2$, and rescale the charges $e_{\rm new} = 2e_{\rm old}$, and the coupling constant $g_{\rm old} = g_{\rm new}/4$ accordingly.

4.4.1 The O(n)thermal exponent

Now remember that the energy operator of the Potts model was a positive unit charge, which now becomes a positive double charge. The energy operator of the O(n)model is the negative charge of the same magnitude, that the Potts model had managed to avoid so carefully. This leads to the following (charge neutral) construction of the two-point energy function of the O(n)model

$$|r_1 - r_2|^{4/g} \int \mathrm{d}^2 r_3 \, \mathrm{d}^2 r_4 \, |r_3 - r_4|^{4/g} \prod_{i=1,2;j=3,4} |r_i - r_j|^{-4/g} \;=\; |r_1 - r_2|^{4-8/g} \,, \quad (49)$$

the result from simple power counting. This corresponds to the exponent, $x_\varepsilon = \frac{4}{g} - 2$ .. One may recognize the expression as the second thermal exponent of

the Potts model (41). The verification that the same result is obtained from the construction with e_o must wait until we have considered the precise value of e_o.

Again, like in the case of the Potts model, the exponent can be analytically continued beyond $n = 2$. In this case, the same is true for the critical point (see exercise above). The case $g \geq 1$ belongs to $x^{-2} = 2 + (2-n)^{1/2}$ and $g \leq 1$ to $x^2 = 2 - (2-n)^{1/2}$. But since the thermal exponent for large x is irrelevant, we conclude that the whole interval $x^{-2} < 2 + (2-n)^{1/2}$ is attracted to the same fixed point. Apparently the phase with relatively large x, the phase dense with loops, is also critical. The behavior is a perfect image of that at the transition manifold of the Potts model, including the value of the exponents. It turns out, that we have now established two relations between Potts and O(n)models, one in which $n = \sqrt{q}$ (transition manifold of dilute Potts model, on whole (n,T) sheet of O(n)model), and one with $n = 2 - (2-q)^2$ (O(n)critical $\leftrightarrow$ Potts AF).

The value of the background charge has changed in a non-trivial way in last the equivalence. This is because an "O(n)loop" is not simply two Potts loops with the same sign. It may be a local collaboration of Potts loops that go side by side for a while, and separate later. (a picture here may explain things). The background charge is determined completely by the magnitude of the step represented by a domain wall (again π after the rescaling above), and the weight that the closed loops are supposed to have, i.e.

$$n = 2 - (2-q)^2 \quad \text{with} \quad q = (2\cos\pi g_{\rm old})^2 \;\rightarrow\; n = -2\cos(\pi g_{\rm new}) \qquad (50)$$

As a result,

$$e_o = g - 1 \qquad (51)$$

somewhat surprisingly, the same expression as (39), but with opposite sign. The sign actually does not follow from the reasoning above, but from the careful considerations of the coincidence equations. Now one can verify that the thermal exponent can be calculated with 'the other construction' as well, but it is just a repetition of the second thermal exponent of the Potts model.

4.4.2 The O(n)magnetic exponent

There are many other operators that one can construct in the O(n)model, the most notable probably the spin operator itself, conjugate to the magnetic field. In this paragraph we will calculate its exponent. As before a way to approach this is to inspect the two-point function, which is $\langle s(o) \cdot s(r) \rangle$. The diagrams that contribute to that, in the expansion in powers of x, there is another set of diagrams that survives, than in the case of the partition sum: now the number of bonds incident on o and r must be odd, while in all other vertices it is even. In the height language this has an obvious interpretation: in site o and r there is a screw dislocation of one step π, i.e. a magnetic charge of strength $m = \pm\frac{1}{2}$.

At first sight one may expect an exponent equal to $gm^2/2 = g/8$, but this is not to be. The the domain walls that terminate on o or r gain a phase factor when they curve. In closed domain walls the integrated curvature is always the same, but with these open domain walls, this is not the case. The domain walls can spiral around their terminals indefinitely, and accordingly pick up arbitrary phase factors. We will have to correct this.

Note, that as the domain walls spiral about their terminal, do so, they either drill down the height of their terminal, or they screw it up: along a fixed path from o to infinity the observer will have to take as many extra steps up as the domain walls spiral more around o, and will measure the height $h(o)$ accordingly. Thus we can counteract the phase factor of the spiralling by the extra insertion of an electric charge, neutralizing the curvature effect. This is done precisely by a charge $-e_o$. It takes a little thought to conclude that the same charge is needed at the negative and the positive vortex.

As a result, the spin-spin correlation function is that of the combination of a magnetic charge $m = \frac{1}{2}$ and an electric charge $e = -e_o$, on one side and $m = \frac{-1}{2}$ with $e = -e_o$ on the other. The resulting exponent is

$$x_s = \frac{g}{8} - \frac{(g-1)}{2g} \tag{52}$$

Exercise: *It may not be obvious that this electro-magnetic charge has no spin, i.e. that there is no phase factor associated with turning the position of $s(r)$ around that of $s(o)$. Verify that this is the case.*

4.4.3 Intersections

In this whole calculation we have considered only O(n)models of which the diagrams of the high-temperature have no intersections. This is OK if intersections are irrelevant. The two-point funtion of intersections is the same as the two-point function of joint terminals of 4 legs.

Exercise: *Verify this*

This can be calculated completely analogously with the calculation of the magnetic exponent, and thus results in

$$x_\times = 2g - \frac{(g-1)^2}{2g} \tag{53}$$

Indeed this is irrelevant at the O(n)critical point.

Unfortunately, it is relevant in the dense phase, and for this reason this phase can not serve as the low temperature phase of the generic O(n)model.

5 Perspective

- The applications of loop models are surprisingly numerous. That is fortunate, because both CG and SLE are based upon it. The language allows for methods and investigation of operators that are not accessible in completely local theories, and these operators do make physical sense (consider e.g. the probability that two spins in the Ising model sit in the same up cluster, or that two unequal spins sit on the same domain boundary as an unequal pair elsewhere.

- In these lectures we have only considered models that admit a single component height representation. Several models with a more dimensional height lattice, have been studied by these methods (Jacobsen, Kondev, Henley), with remarkable results. It appears that we have only scratched the surface of what is possible. In this area the CG still has an edge against SLE, CFT or exact solutions.

- The little excursion in the true continuum theory should be developed further.

- The assertion that the alternative calculations of multipoint correlation functions are the same, leads to interesting conjectures for integral expressions. It would be interesting to find a more general form in which these conjecture can be cast, or to find a proof for them.

- The way that the critical point of the dilute loop model was found is general enough that it may be used to find critical points of other loop models. In particular a conjecture concerning the critical point of the SAW model on the square lattice may thus be confirmed.

- Perhaps the most challenging: In principle the CG approach should be able to give off-critical scaling functions. (equations of state) So far there are no exact results on these objects.