Bethe’s Ansatz for random tiling models

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1. Quasi-crystals
2. Random tilings
3. The square triangle tiling
4. Bethe Ansatz equations
5. Solution to the BAE
6. Field theoretic aspects
7. Rectangle triangle tilings with other symmetries
8. Sketch of nested B.A. for octagonal case.
Quasi-Crystallography

The crystalline phase is invariant for discrete translations and rotations.

Rotational symmetries are exclusively two, three, four or six-fold.

Why?

Let $f$ be some density function.

Translations: $f(\bar{x}) = f(\bar{x} + \bar{a})$

Rotations: $f(\bar{x}) = f(R\bar{x})$

Then also: $f(\bar{x}) = f(\bar{x} + R\bar{a} + R^4\bar{a})$

But $\bar{a}$ and $R\bar{a} + R^4\bar{a}$ have same direction, but irrational length ratio.

$\Rightarrow$ If $f$ is continuous it is constant.
3D is only slightly more complicated: 
\( \vec{a} \) need not be orthogonal to rotation axis. 
\[ \vec{b} = \vec{a} + R\vec{a} + R^2\vec{a} + R^3\vec{a} + R^4\vec{a} \] points along rotation axis. 
Compare now: \( 5(R\vec{a} + R^4\vec{a}) - 2\vec{b} \) and \( 5\vec{a} - \vec{b} \).

Yet many materials exist with icosahedral symmetry. (which has 5-fold axes). 
Mostly binary and ternary (Al) alloys.

There are also solids which are periodic in one direction and have five-, eight-, ten- or twelve-fold symmetry in the orthogonal plane.

Rather than periodic these materials could be quasi-periodic.
Periodic functions have discrete Fourier spectrum.

But the reverse is not true: take

$$\sin(x) + \sin(\pi x)$$

This function is almost periodic, but not periodic.

The function $\sin(x) + \sin(\pi x)$ is obviously the restriction of $\sin(x) + \sin(y)$ with the line $y = \pi x$.

In general, quasi-periodic functions are the restriction of a higher dimensional periodic function.
Let
\[ W(\vec{r}) = \sum_{m_1,m_2,\ldots,m_D} A_{\vec{m}} \exp \left( i \sum_{j=1}^{D} m_j \vec{a}_j \cdot \vec{r} \right) \]
with \( D > d \) (dimension of space).

Necessarily \( \{\vec{a}_j|j = 1, \cdots, D\} \) are linearly dependent.
But we choose them independent under the integers.
Note however, that \( \{\sum_{j=1}^{D} m_j \vec{a}_j|\vec{m} \in \mathbb{Z}^D\} \) lie dense in \( \mathbb{R}^d \).

View the vectors \( \vec{a}_j \) as \( d \)-dimensional restrictions of \( D \)-dimensional vectors, say:
\[ \vec{p}_j = \vec{a}_j + \vec{b}_j, \quad \text{with} \quad \vec{r} \perp \vec{b}_k \]
Then, \( W(\vec{r}) \) is simply a restriction of a \( D \)-dimensional periodic function.

The basis vectors \( \vec{a}_j \) can have any symmetry, crystallographic and otherwise.
If the $D$-dimensional function is zero except on periodically repeated objects of $(D - d)$ dimensions, the restriction to real space consists of $\delta$-peaks. (Then both the function and its Fourier Transform consist of $\delta$-functions.)

In this $(d = 1, \ D = 2)$ example the real-space function has $\delta$-peaks between (irregularly) alternating long and short intervals.

If $D > 3$ it is possible to find 'irrational' subspaces, which have special symmetries.
In this example objects are placed on the sites of a 4-dimensional hypercubic lattice, and intersected with a plane.
But what holds these materials in check? Is the quasi-periodic arrangement the minimum of some interparticle potential?

In periodic crystals each atom has the same environment. In a quasi-periodic arrangements, there are very many different local environments. Are they all potential minima?
It is at least as plausible, that these many local environments are almost equally likely, and that the *rotational symmetry* is the result of *thermal averaging*.

Effectively, in this scenario, the minimization of *potential energy* is replaced by minimization of *free energy*.

Asymmetric local configuration occur in all symmetric equivalent orientations.
Take this to the extreme, and make the potentials strictly short-range, and many local configurations strictly of equal energy. In this case minimization of free energy is effectively replaced by maximization of entropy.

This is for instance achieved by binary hard disk packings:

This naturally leads to …
Random tilings

are complete covering of space by copies of one or more rigid geometric objects.

A well-known example is the dimer model: space is covered by dominos (1×2 rectangles, or 1×1×2 bricks).

Is is much more interesting if the building blocks permit (or even force) non-crystallo-graphic structures.
For general binary ball packings: if the potential is strictly short range, e.g. hard disks, individual configurations need not be symmetric or quasiperiodic.

But the ensemble of all tight packings shows a discrete rotational symmetry, and may come out to be quasiperiodic.

The tight binary disk packing is equivalent to a so-called Random Tiling with (a subset of) three kinds of triangles:

![Diagram of three kinds of triangles](image)

The angles of the isosceles triangles may be such that they support a non-crystallographic rotational symmetry.

Both the rotational symmetry and the quasiperiodicity may be subject to symmetry breaking

Other well-studied tilings in 2D are rhombus tilings and in 3D rhombohedron tilings.
The straight intersection required by strict quasi-periodicity:

is in a random tiling replaced by an intersection with a fluctuating line:

A natural description would be a free field theory, compactified to the $D$-dimensional lattice.
a $D = 2 + 2$ example:

**Square-Triangle tiling**

The specific problem: tiling of the plane by squares and equilateral triangles.

A configuration of the square triangle random tiling
strong dependence on relative densities:

predominantly squares
even mixture
predominantly triangles

Relation between total numbers of tiles in the different orientations:

\[ S_1S_2 + S_2S_3 + S_3S_1 = \frac{1}{4}T_1T_2 \]

expressed in areal densities:

\[ \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 = \frac{4}{3}\tau_1\tau_2 \]
Towards a solution:

Discrete statistical problem:
Yet no discrete underlying lattice

tile edges can be in one of 6 orientations:
(say angle with horizontal is $\pi/12$, $3\pi/12$, $5\pi/12$ etc.)

Deform the tiles by rotating each edge to nearest multiple of $\pi/3$ (dotted direction).
The above configuration is then deformed into the following:

edges need to be marked to distinguish original orientations
Now we have a lattice model:
We mark edges with colored lines.

Five configurations of each triangle:

horizontal edges:

\[\] \[\] \[\] \[\] \[\]

diagonal edges:

\[\] \[\] \[\] \[\] \[\]

Continuity of the colored lines guarantees the tiles to fit without holes or overlaps.

The number of red and green lines cutting each horizontal is conserved.
transfer matrix:

$Z_N(\alpha, \beta)$:
Partition sum of bounded cylinder,

$T_{\alpha,\beta} = Z_1(\alpha, \beta)$:
weight of single layer of tiles

$\alpha, \beta, \gamma$:
state of row of horizontal edges

\[
Z_N(\alpha, \beta) = \sum_\gamma Z_{N-1}(\alpha, \gamma) T_{\gamma,\beta}
\]

toroidal partition sum:

$Z_N = \sum_\alpha Z_N(\alpha, \alpha) = \text{Trace } T^N$
Geometrical considerations:

In the transfer matrix the numbers of red and green marked horizontal edges, say $n_r$, and $n_g$ are conserved. How are these numbers related to the tile densities?

We had for the tile numbers:

$$S_1 S_2 + S_2 S_3 + S_3 S_1 = \frac{1}{4} T_1 T_2$$

and their areal densities:

$$\sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_3 \sigma_1 = \frac{4}{3} \tau_1 \tau_2$$

In a symmetric phase,

$$\sigma_1 = \sigma_2 = \sigma_3 = \frac{\sigma}{3}$$

$$\tau_1 = \tau_2 = \frac{\tau}{2}$$

Therefore $\sigma = \tau$.

In the fully symmetric phase half the area is covered by triangles and half by squares.
Most symmetric phase dominated by squares is e.g. $\sigma_1 > \sigma_2 = \sigma_3$ and $\tau_1 = \tau_2$.

On average rectangular areas $\ell \times (\ell + 1)$:

\[
S_1 = \ell(\ell + 1) \quad S_2 = S_3 = \frac{1}{2}
\]

\[
T_1 = T_2 = 2(2\ell + 1)
\]

\[
\tau = \frac{\sqrt{3}(2\ell + 1)}{2\ell^2 + 2\ell + 2 + (2\ell + 1)\sqrt{3}}
\]

\[
\frac{n_g}{L} = \frac{n_r}{L} = \frac{1}{\ell + 2}
\]

This density regime has square symmetry.
Most symmetric phase with predominantly triangles: $\sigma_1 = \sigma_2 = \sigma_3$ and $\tau_1 > \tau_2$.

On average hexagonal areas with sides $\ell$:

$$T_1 = 6\ell^2 \quad T_2 = 2$$

$$S_1 = S_2 = S_3 = \ell$$

$$\tau = \frac{3\ell^2 + 1}{(1 + \ell\sqrt{3})^2}$$

$$\frac{n_r}{L} = \frac{\ell + 1}{3\ell^2 + 3\ell + 1} \quad \frac{n_g}{L} = \frac{3\ell^2 + \ell}{3\ell^2 + 3\ell + 1}$$

This density regime has hexagonal symmetry.
Diagonalization of transfer matrix by Bethe’s Ansatz (M. Widom)

reference state (pseudo vacuum):

\[ \psi(x) = e^{ipx} \equiv u^x \]

\[ \wedge \psi(x) = (T\psi)(x) = t \psi(x + 1) \]

\[ \wedge = tu \]

\( t \): weight of two triangles
single green:

\[ \psi(y) = v^y \]

\[ \land \psi(y) = (T\psi)(y) = t \psi(y - 1) \]

\[ \land = t/v \]

one red and one green (isolated):

\[ \psi(x, y) = u^x v^y \]

\[ \land \psi(x, y) = t^2 \psi(x + 1, y - 1) \]

\[ \land = t^2 u/v \]
red followed by green:

Involves both \( x < y \) and \( y < x \).
Amplitude may depend on the order:

\[
\psi(x, y) = A_{r,g} u^x v^y \quad \text{with} \quad x < y
\]

\[
\psi(y, x) = A_{g,r} u^x v^y \quad \text{with} \quad x > y
\]

(arguments ordered)

The equation from above diagram:

\[
\wedge \psi(x - 1, x + 1) = t \psi(x - 2, x) + t \psi(x, x + 2)
\]

with \( \wedge = t^2 u/v \)

Substitution of the Ansatz results in:

\[
t A_{r,g} = A_{g,r} \left( u^2 + v^{-2} \right)
\]
Now consider three particles (say two red and one green), and try a product of three plane waves (wave numbers $u_1, u_2$ and $v$).

Equations when particles are isolated yield:

$$\Lambda = t^3 u_1 u_2 v^{-1}$$

Collisions (interchanges) between red and green particles force the amplitude to depend on the red-green sequence.

We allow the red particles to interchange momentum:

$$\psi(x_1, x_2, y) = A_{1,2,v} u_1^{x_1} u_2^{x_2} v y + A_{2,1,v} u_2^{x_1} u_1^{x_2} v y$$

$$\psi(x_1, y, x_2) = A_{1,v,2} u_1^{x_1} v y u_2^{x_2} + A_{2,v,1} u_2^{x_1} v y u_1^{x_2}$$

$$\psi(y, x_1, x_2) = A_{v,1,2} v y u_1^{x_1} u_2^{x_2} + A_{v,2,1} v y u_2^{x_1} u_1^{x_2}$$

(arguments ordered)
When the green particle collides with one of the red, we find: together with 

\[ A_{1,2,v} = t^{-1} A_{1,v,2} \left( u_2^2 + v^{-2} \right) \]

etc.

But when three particles collide at once:

\[
\wedge \psi(x - 2, x, x + 2) = \\
t \psi(x - 3, x - 1, x + 1) + \\
t^2 \psi(x - 1, x + 1, x + 3)
\]

This results in

\[ A_{1,2,v} = -A_{2,1,v} \]

and

\[ A_{v,1,2} = -A_{v,2,1} \]
The natural generalization: a linear combination of products of plane waves

\[ \psi(\vec{c}, \vec{x}) = \sum_{\vec{\sigma}} \tilde{A}_{\vec{\sigma}} \prod_j (w_{\sigma_j}^{c_j} x_j) \]

\( \vec{c} \): the color sequence,
\( \vec{x} \): position sequence \((x_j < x_{j+1})\), (both of the non-blank edges).
\( w_j^r = u_j \) and \( w_k^g = v_k \) are the wave numbers of the red and green particles respectively.
\( \tilde{\sigma} \): permutes these wave numbers, but does not mix red and green ones.

The eigenvalue must be

\[ \Lambda = \left( \prod_j t u_j \right) \left( \prod_k t v_k^{-1} \right) \]

and the amplitudes \( A \) satisfy:

\[ A_{..., j, k, ...}^{r, g, ...} = t^{-1} \left( u_j^2 + v_k^{-2} \right) A_{..., k, j, ...}^{g, r, ...} \]

\[ A_{..., j, k, ...}^{r, r, ...} = -A_{..., k, j, ...}^{r, r, ...} \]

\[ A_{..., j, k, ...}^{g, g, ...} = -A_{..., k, j, ...}^{g, g, ...} \]
Assume periodic boundary conditions:

\[ \psi (\cdots, c, 2L) = \psi (c, \cdots) \]

this is satisfied if:

\[ A^{\cdots, c}_k (w^c_k)^{2L} = A^c_{k, \cdots} \]

\( L \): horizontal system size in lattice edges.

The left- and right-hand side of this equation can also be related by successive interchanges of \( \frac{c}{k} \) with its predecessor.

\( \Rightarrow \) the Bethe Ansatz equations (BAE)

\[ u^{2L}_j = (-)^{n_r + 1} t^{-n_g} \prod_{k=1}^{n_g} u^2_j + v^{-2}_k \]

\[ v^{-2L}_k = (-)^{n_g + 1} t^{-n_r} \prod_{j=1}^{n_r} u^2_j + v^{-2}_k \]

\[ \Lambda = \left( \prod_{j=1}^{n_r} t u_j \right) \left( \prod_{k=1}^{n_g} t v^{-1}_k \right) \]

Convenient to introduce \( \xi = u^2 \) and \( \eta = -v^{-2} \).
In these new variables the BAE read

\[
\xi^L_j = (-)^{ng+1} t^{-ng} \prod_{k=1}^{ng} \xi_j - \eta_k
\]

\[
\eta^L_k = (-)^{L+ng+nr+1} t^{-nr} \prod_{j=1}^{nr} \eta_k - \xi_j
\]

And the eigenvalue:

\[
\lambda = \left( \prod_{j=1}^{nr} t \xi_j^{1/2} \right) \left( \prod_{k=1}^{ng} t (-\eta_k)^{1/2} \right)
\]

Introduce the functions:

\[
F_r(z) = \log z - \sum_{k=1}^{ng} \frac{1}{L} \log(z - \eta_k) + \frac{ng}{L} \log t
\]

\[
F_g(z) = \log z - \sum_{j=1}^{nr} \frac{1}{L} \log(z - \xi_j) + \frac{nr}{L} \log t
\]

Then the BAE read:

\[
LF_r(\xi_j) = (0 \text{ or } \pi i) \mod 2\pi i
\]

\[
LF_g(\eta_j) = (0 \text{ or } \pi i) \mod 2\pi i
\]
Expect solutions along curves in the complex plane, given by

\[
\text{Re } F_r(\xi_j) = 0 \quad \text{Re } F_g(\eta_k) = 0
\]
At successive roots:

$$F_r(\xi_{j+1}) - F_r(\xi_j) = \frac{2\pi i}{L}$$

Therefore in the thermodynamic limit

$$f_r(z) \equiv \frac{\partial F_r(z)}{\partial z} = \frac{2\pi i}{L} \text{ (root density)}$$

With this knowledge we can transform the sums into integrals:

$$f_r(z) = \frac{1}{z} - \frac{1}{L} \sum_{k=1}^{n_g} \frac{1}{z - \eta_k}$$

$$= \frac{1}{z} - \frac{1}{2\pi i} \int_{\eta_{ng}}^{\eta_1} \frac{f_g(\eta)}{\eta - z} \, d\eta$$

(double sign change: \(z \leftrightarrow \eta\) and \(\eta_1 \leftrightarrow \eta_{ng}\))

likewise

$$f_g(z) = \frac{1}{z} + \frac{1}{2\pi i} \int_{\xi_1}^{\xi_{nr}} \frac{f_r(\xi)}{\xi - z} \, d\xi$$

So, for the analyticity of \(f_r\) and \(f_g\):
What is the change of these functions at the cuts?

\[
fr(\eta + \varepsilon) - fr(\eta - \varepsilon)
\]

\[
\uparrow \quad \uparrow \quad \Uparrow
\]

\[
\bullet - \bullet = \odot = - \odot
\]

\[
fr(z) = \frac{1}{z} - \frac{1}{2\pi i} \int_{\eta = \mu} \frac{fg(\eta)}{\eta - z} \, d\eta
\]

\[
\Rightarrow
\]

\[
fr(z + \varepsilon) - fr(z - \varepsilon) = \frac{1}{2\pi i} \int_{|\eta - z| = \varepsilon} \frac{fg(\eta)}{\eta - z} \, d\eta
\]

\[
= fg(z)
\]
Solution of the integral equations (P. Kalugin)

So $f_r$ jumps by $f_g$ across the ($\eta$) cut.

$\Rightarrow$ analytic continuation through the cut:

$$f_r \rightarrow f_r - f_g$$

Likewise $f_g$ taken through the $\xi$-cut changes:

$$f_g \rightarrow f_r + f_g$$

Now inspect the function

$$f(z) = a_r f_r(z) + a_g f_g(z).$$

Through the $\eta$-cut $f_g$ is analytic, and $f_r$ changes by $-f_g$.

This effectively multiplies the vector $\begin{pmatrix} a_r \\ a_g \end{pmatrix}$ by the matrix $\Gamma_g = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$.

At the $\xi$-cut the corresponding monodromy transformation is $\Gamma_r = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. 
Together $\Gamma_g$ and $\Gamma_r$ generate the entire group $\text{SL}_2(\mathbb{Z})$. (2×2 integer matrices with determinant 1).

However, when the triangles cover half the area (the point of 12-fold symmetry):
And also when $\tau > \sigma$ (hexagonal symmetry)

The cuts appear to have a common end point in the thermodynamic limit.
When that is the case there is no way to run through one cut and return to the point of origin without going through the other cut.

Effectively there is only one monodromy operator:

\[
\Gamma_g \Gamma_r = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}
\]

Note that:

\[
(\Gamma_g \Gamma_r)^2 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}
\]

therefore

\[
(\Gamma_g \Gamma_r)^3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}
\]

and

\[
(\Gamma_g \Gamma_r)^6 = 1.
\]

But can we believe that the cuts close on the basis of numerical evidence?
We have three parameters to play with: \( n_r, n_g, \) and \( t \).
The restriction that the cuts close is one complex equation;
So it is reasonable to assume that it can be satisfied.

Let, therefore, in the thermodynamic limit

\[
\begin{align*}
\eta_1 & \rightarrow b \\
\eta_{ng} & \rightarrow b^* \\
\eta_{nr} & \rightarrow b \\
\xi_1 & \rightarrow b^*
\end{align*}
\]

Then, because \((\Gamma g \Gamma r)^6 = 1\), \( f_r(z) \) and \( f_g(z) \)
must be single-valued functions of \( w \)

\[
\begin{align*}
w^6 &= \frac{z/b - 1}{1 - z/b^*} \\
z &= \frac{w^6 + 1}{w^6/b^* + 1/b}
\end{align*}
\]

Let

\[
f(w)dw = f_r(z)dz
\]

In this way, poles of \( f_r(z) \) translate into poles of \( f(w) \) with the same residues.
The BAE:

\[ f_r(z) = \frac{1}{z} - \frac{1}{L} \sum_{k=1}^{n_g} \frac{1}{z - \eta_k} \]

\[ f_g(z) = \frac{1}{z} - \frac{1}{L} \sum_{j=1}^{n_r} \frac{1}{z - \xi_j} \]

Therefore \( f_r(z)dz \) and \( f_g(z)dz \) have poles in the origin, (residu 1) and at \( \infty \) (residu \( n_g/L - 1 \) and \( n_r/L - 1 \) respectively).

What does that imply for \( f(w)dw \)?

First define:

\[ b = i|b|e^{-i\gamma} \]
\[ \omega = e^{\pi i/6} \]
\[ \theta = e^{\gamma i/3} \]
Then the poles of $f(w)dw$ are:

<table>
<thead>
<tr>
<th>$m$</th>
<th>$z$</th>
<th>$w_m$</th>
<th>$f(w)dw/dz$</th>
<th>residue $r_m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>$\omega$</td>
<td>$f_r(z)$</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>$\infty$</td>
<td>$\omega^2 \theta$</td>
<td>$f_r(z)$</td>
<td>$n_g/L - 1$</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>$i$</td>
<td>$f_r(z) - f_g(z)$</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>$\infty$</td>
<td>$\omega^4 \theta$</td>
<td>$-f_g(z)$</td>
<td>$1 - n_r/L$</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>$-\omega^{-1}$</td>
<td>$-f_g(z)$</td>
<td>$-1$</td>
</tr>
<tr>
<td>6</td>
<td>$\infty$</td>
<td>$-\theta$</td>
<td>$-f_r(z) - f_g(z)$</td>
<td>$2 - (n_r + n_g)/L$</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>$-\omega$</td>
<td>$-f_r(z)$</td>
<td>$-1$</td>
</tr>
<tr>
<td>8</td>
<td>$\infty$</td>
<td>$-\omega^2 \theta$</td>
<td>$-f_r(z)$</td>
<td>$1 - n_g/L$</td>
</tr>
<tr>
<td>9</td>
<td>0</td>
<td>$-i$</td>
<td>$-f_r(z) + f_g(z)$</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>$\infty$</td>
<td>$-\omega^4 \theta$</td>
<td>$f_g(z)$</td>
<td>$n_r/L - 1$</td>
</tr>
<tr>
<td>11</td>
<td>0</td>
<td>$\omega^{-1}$</td>
<td>$f_g(z)$</td>
<td>1</td>
</tr>
<tr>
<td>12</td>
<td>$\infty$</td>
<td>$\theta$</td>
<td>$f_r(z) + f_g(z)$</td>
<td>$(n_r + n_g)/L - 2$</td>
</tr>
</tbody>
</table>
\( f(w) \) is determined completely by its poles:

\[
f(w)dw = \sum_{m=1}^{12} \frac{r_m \, dw}{w - w_m}.
\]

This is the solution of \( f(w) \) in terms of \( b \), \( n_r/L \) and \( n_g/L \).

The locus of the roots was determined by

\[
\text{Re} \, F_r(\xi_j) = 0
\]

\[
\text{Re} \, F_g(\eta_k) = 0
\]

Therefore also

\[
0 = \text{Re} \, dF_r(z) = \text{Re} \, f_r(z) \, dz
\]

\[
0 = \text{Re} \, dF_g(z) = \text{Re} \, f_g(z) \, dz
\]

Since in \( z = b \) two cuts join from different directions, \( f_r(z = b) = f_g(z = b) = 0 \).

Likewise \( f(w = 0) = 0 \), i.e.

\[
\sum_{m=1}^{12} \frac{r_m}{w_m} = 0
\]
When the values of $r_m$ and $w_m$ are substituted, this (complex) equation reads

$$(1 - n_g/L)\omega^{-1}\theta^{-1} + (1 - n_r/L)\omega\theta^{-1} = 1$$

The solution:

$$n_r/L = 1 - \frac{2}{\sqrt{3}}\cos\frac{\pi - \gamma}{3}$$

$$n_g/L = 1 - \frac{2}{\sqrt{3}}\cos\frac{\pi + \gamma}{3}$$

is consistent with hexagonal symmetry. (from the geometrical considerations)

with $\gamma$ related to the triangle density:

$$\sin\frac{\gamma}{3} = \frac{\sqrt{2\tau - 1}}{(1 - \tau)\sqrt{3} + 2\tau}$$

i.e. \( \tau = \frac{2\sqrt{3} + 4 - \cos(\gamma/3)(2\sqrt{3} + 3)}{\cos(\gamma/3) + 1} \)

For $\gamma = 0$, $\tau = \sigma = 1/2$, consistent with the maximal, i.e. twelve-fold, rotational symmetry.
We now know \( f_r(z), f_g(z) \), but we still have not calculated the eigenvalue.

\[
\Lambda = \prod_{j=1}^{n_r} t \xi_j^{1/2} \prod_{k=1}^{ng} t (-\eta_k)^{1/2}
\]

So we need to know \( t, \prod \xi \) and \( \prod \eta \).

Remember:

\[
F_r(z) = \log z - \sum_{k=1}^{ng} \frac{1}{L} \log \frac{z - \eta_k}{t}
\]

\[
F_g(z) = \log z - \sum_{j=1}^{nr} \frac{1}{L} \log \frac{z - \xi_j}{t}
\]

We know that \( \text{Re} \ F_r(b) = \text{Re} \ F_g(b) = 0 \).

\( \text{Re} \ F_r(z \to 0) \) is much like \( \sum_k \log(t/\eta_k)/L \), except for the logarithmic singularity.

Likewise in the limit \( z \to \infty \), \( F_r \) has a finite contribution \( ng/L \log t \), modulo the logarithmic divergence.
Therefore we define

\[ F_1(z) = F_r(z) - \log z \]

\[ F_2(z) = F_r(z) - \left(1 - \frac{ng}{L}\right) \log z \]

Note:
(1) Since \( b \) is on the locus of the roots, \( F_r(z = b) \) is imaginary.
(2) The limits \( F_1(z \to 0) \) and \( F_2(z \to \infty) \) are regularized, so that

\[ \text{Re}(F_1(0) - F_1(b)) = \log |b| + \frac{1}{L} \sum_{k=1}^{ng} \log \frac{t}{\eta_k} \]

\[ \text{Re}(F_2(\infty) - F_2(b)) = \log |b| + \frac{ng}{L} \log \frac{t}{|b|} \]

These can also be calculated directly from integrals of the known functions \( f_r \) and \( f_g \) (most conveniently in the \( w \)-plane).

\[ F_1(0) - F_1(b) = \int_b^0 dz \left( f_r(z) - \frac{1}{z} \right) \]

\[ = \int_0^{w_1} dw \left( f(w) - \frac{1}{z} \right) \]
This is a simple integral, since

\[ f(w) = \sum_{m=1}^{12} \frac{r_m}{w - w_m} \]

and likewise

\[ \frac{1}{z} \frac{dz}{dw} = \sum_{m=1}^{12} \frac{(-1)^{m+1}}{w - w_m} \]

\(F_2\) is subjected to the same treatment:

\[ F_2(\infty) - F_2(b) = \int_b^\infty dz \left( f_r(z) - \frac{1 - n_g}{z} \right) \]

\[ = \int_0^{w_2} dw \left( f(w) - \frac{(1 - n_g) dz}{z} \right) \]

Analogously, \(F_3\) and \(F_4\) are defined from \(F_g\):

\[ F_3(z) = F_g(z) - \log z \]

\[ F_4(z) = F_g(z) - \left(1 - \frac{n_r}{L}\right) \log z \]
Altogether we have four linear equations in the four (yet) unknowns: \( \log(t) \), \( \log |b| \), \( \sum_j \log |\xi_j| \) and \( \sum_k \log |\eta_k| \).

From these the eigenvalue \( \Lambda \) can be found:

\[
\Lambda = \prod_{j=1}^{n_r} t |\xi_j|^{1/2} \prod_{k=1}^{n_g} t |\eta_k|^{1/2}
\]

which is the partition sum per row.

Let \( T_{row} \) be the number of triangles per row:

\[
\frac{T_{row}}{L} = 8 - 4\sqrt{3}\cos \frac{\gamma}{3}
\]

(from geometric relations with \( n_g \) and \( n_r \).)

Then the entropy per vertex is equal to

\[
S_v = \frac{1}{L} \log \Lambda - \frac{T_{row}}{2L} \log t
\]

resulting in:
\[ S_V = \log \frac{108}{\cos^2 \gamma} + \]
\[ + 2 \cos \left( \frac{\pi}{6} + \frac{\gamma}{3} \right) \log \left[ \tan \left( \frac{\pi}{12} + \frac{\gamma}{6} \right) \tan \left( \frac{\pi}{4} + \frac{\gamma}{6} \right) \right] \]
\[ + 2 \cos \left( \frac{\pi}{6} - \frac{\gamma}{3} \right) \log \left[ \tan \left( \frac{\pi}{12} - \frac{\gamma}{6} \right) \tan \left( \frac{\pi}{4} - \frac{\gamma}{6} \right) \right] \]

For \( \tau < \frac{1}{2} \) solution only numeric.
The finite size correction of the entropy gives the **central charge**:

\[ S_a = S_\infty + \frac{c \pi}{6L^2} \]

When we calculate this as function of the **triangle density** \( \tau \):

Does \( c \) really depend continuously on \( \tau \)? Can it be a conformal theory?
The spectrum of the transfer matrix scales as a scale-invariant theory:

$$\log \frac{\Lambda_j}{\Lambda_0} = \frac{2\pi \Delta_j}{L}$$

all correlation lengths are thus proportional to the circumference of the cylinder.

In a conformal theory there are many gaps \((\Lambda_j - \Lambda_k)\) that scale with an integer amplitude \(\Delta_{jk}\).

How does that come out in this system?

Gaps \(\Delta_{jk}\) that are integer for \(\tau > .5\) turn out complex for \(\tau < .5\), but continue to satisfy

$$|\Delta_j| \in \mathbb{Z}$$

What does that mean?
The resolution of this question turns out to be the following:

We propose that the two families of particles even though they interact with each other, effectively behave as free fermions.

The only effect of the interaction is that they stretch each other's space in an anisotropic way.

Though this effect is anisotropic, the combination of the effects of both families has again square symmetry.

Each of the families has central charge 1, but the contribution to the effective finite size correction is modified by a geometric factor as a result of the deformation.
Solvable random tilings:

\[ b = 2 \sin \frac{\beta}{2} \]

Tilings of rectangles and isosceles triangles have been solved, i.e. the free energy has been calculated in the thermodynamic limit.

Special values of \( \beta \) support special symmetries, in particular if \( \beta = \frac{2\pi}{n} \):
\[ \beta = \frac{2\pi}{4} : \text{ octogonal} \]

de Gier & BN:

\[ \beta = \frac{2\pi}{5} : \text{ decagonal} \]

de Gier & BN:

\[ \beta = \frac{2\pi}{6} : \text{ duodecagonal} \]

Kalugin:
de Gier & BN:
Sketch of the solution of the octogonal rectangle-triangle tiling.

The first step is to deform the tiles to fit them to a square lattice.

The short edges are rotated to the nearest axial direction and the long edges to the nearest diagonal direction.

The rotated edges are colored to code for the original orientation.

The resulting lattice model can be attacked by standard techniques such as the transfer matrix.
The perfect tiling condition (i.e. tiles fit together without holes or overlap) translates into continuity of the red and blue lines.

Every horizontal cuts an equal number of red (blue) lines. ⇒ Conserved quantities under the action of a transfer matrix.
The eigenstates of the transfer matrix satisfy a nested Bethe Ansatz.
i.e. piecewise plane wave solution, with amplitudes changing at "collisions".
As a result the eigenvalues $\Lambda$ follow from Bethe Ansatz equations.

\[
(u_j^\pm)^{-L} \prod_{k=1}^m (w_k - u_j^\pm) = 1
\]

\[
\prod_{i=1}^{n^+} (w_k - u_i^+) \prod_{j=1}^{n^-} (w_k - u_j^-) = 1
\]

\[
\Lambda = \left( \prod_{k=1}^m w_k \right) \left( \prod_{i=1}^{n^+} \frac{1}{u_i^+} \right) \left( \prod_{j=1}^{n^-} \frac{1}{u_j^-} \right)
\]
The Bethe Ansatz equations

\[
(u_j^\pm)^{-L} \prod_{k=1}^{m} (w_k - u_j^\pm) = 1
\]

\[
\prod_{i=1}^{n^+} (w_k - u_i^+) \prod_{j=1}^{n^-} (w_k - u_j^-) = 1
\]

can be solved as follows:

- introduce the functions
  \[ F(z) \equiv -L \log(z) + \sum_k \log(w_k - z) \]
  \[ G(z) \equiv \sum_i \log(z - u_i^+) \sum_j \log(z - u_j^-) \]

- Then each \( u_j^\pm \) is a solution of
  \[ F\left(u_j^\pm\right) = 0 \mod 2\pi i \]
  and each \( w_k \) of \( G\left(w_k\right) = 0 \mod 2\pi i \)

- Hypothesis: The equations for consecutive roots differ by \( 2\pi i \).

- Then the functions \( f(z) = dF(z)/dz \) and \( g(z) = dG(z)/dz \) measure the density of roots.
• In the thermodynamic limit the sums can be turned into integrals, so that the Bethe Ansatz equations are now integral equations with integration contours running over the locus of the roots:

\[
f(z) = \frac{L}{z} - \frac{m}{2\pi i} \int \frac{g(w)}{z-w} \, dw
\]

\[
g(z) = \frac{n}{\pi i} \int \frac{f(w)}{z-w} \, dw
\]

where we have chosen \( n^+ = n^- = n \).

• The loci of the roots are thus branch cuts in the functions \( f(z) \) and \( g(z) \) with a linear monodromy group acting on the analytic continuation \( af(z) + bg(z) \).
The entropy per unit area is given by:

\[
S_a = \frac{2 + \sqrt{2}}{4 \cos^2 \gamma/4} \left[ \log \frac{4}{\cos \gamma} + \cos \left( \frac{\pi}{4} + \frac{\gamma}{2} \right) \log \tan \left( \frac{\pi}{8} + \frac{\gamma}{4} \right) + \cos \left( \frac{\pi}{4} - \frac{\gamma}{2} \right) \log \tan \left( \frac{\pi}{8} - \frac{\gamma}{4} \right) \right]
\]

An analytic solution is available only for triangle density \( \tau > 1/2 \):

\[
\tau = (\sqrt{2} + 1) \frac{\sqrt{2} - \cos \gamma/2}{1 + \cos \gamma/2}
\]
Leaves many questions:

1. Correlation functions
2. Diffraction patterns
3. Nature of the phases: critical?
4. Scale- vs Conformal invariance.
5. What makes it solvable?
6. Other solvable tilings.
   In particular: rect.tri. $\frac{2\pi}{n}$
7. Are the cases $n \in \{4, 5, 6\}$ exceptional like $E_{6,7,8}$
8. What properties are universal?
   (i.e. dependent on symmetry alone)
9. Convexity of the entropy:
   Related to solvability?
10. 3D random binary/ternary packings
references

For the Bethe Ansatz:


M. Gaudin *La fonction d’onde de Bethe*, Masson (1983)

For some background on Quasicrystals see e.g. *Quasi-crystals the state of the art*, ed. P.J. Steinhardt and D.P. Di Vincenzo. (World Scientific, 1991) Chapter 15, by C.L. Henley is devoted to random tiling models.

More specific on a one of the models discussed: *Random square-triangle tilings, a model of 12-fold quasicrystals*, M. Oxborrow, C.L. Henley, Phys.Rev. B 48, 6966 (1993)
Recent results discussed in the lectures:


Most of the material of these lectures is covered in: