Category Theory 2017 - Exercise sheet 1

1. The objects of **Rel** are sets and an arrow $A \to B$ is a subset $R \subseteq A \times B$. The identity arrow on a set A is $\{\langle a, a\rangle \in A \times A \;\; a \in A\}$. For $R\colon A \to B$ and $S\colon B \to C$, the composition $S \circ R$ is given by

$$S \circ R := \{\langle a, c\rangle \in A \times C \mid \exists b \in B\, (\langle a, b\rangle \in R \;\wedge\; \langle b, c\rangle \in S)\}$$

   (a) Show that **Rel** is a category.

   (b) Show that there is a functor **Sets** $\to$ **Rel** taking objects to themselves and a function $f\colon A \to B$ to its graph,

$$G(f) := \{\langle a, f(a)\rangle \mid a \in A\}$$

2. Show that there is an isomorphism $F\colon \mathbf{Rel} \to \mathbf{Rel}^{\mathrm{op}}$. (An *isomorphism* from a category $\mathbb{C}$ to a category $\mathbb{D}$ is a functor $F\colon \mathbb{C} \to \mathbb{D}$ such that there exists a functor $G\colon \mathbb{D} \to \mathbb{C}$ with $F \circ G = 1_{\mathbb{D}}$ and $G \circ F = 1_{\mathbb{C}}$.)

3. Show how to define a functor $F\colon \mathbf{Cat} \to \mathbf{Cat}$ such that for each category $\mathbb{C}$, $F(\mathbb{C}) = \mathbb{C}^{\mathrm{op}}$.

4. Let $\mathbb{C}$ be a category. We will define a new category $\mathbb{C}^{\to}$, called the *arrow category on* $\mathbb{C}$. Each object of the arrow category is an arrow $f\colon A \to B$ of $\mathbb{C}$. An arrow from $f\colon A \to B$ to $g\colon C \to D$, is a pair of arrows $h\colon A \to C$ and $k\colon B \to D$ such that $k \circ f = g \circ h$.

   (a) Check that $\mathbb{C}^{\to}$ is a category.

   (b) Show how to define a functor $F\colon \mathbf{Cat} \to \mathbf{Cat}$ such that $F(\mathbb{C}) = \mathbb{C}^{\to}$.