Category Theory 2017 - Exercise sheet 10

1. Let $\mathbb{C}$ be a small category and let X be a presheaf over $\mathbb{C}$. Verify that X is the colimit of the diagram defined as the composition $\int_{\mathbb{C}} X \stackrel{\pi_0}{\to} \mathbb{C} \stackrel{y}{\to} \hat{\mathbb{C}}$.
2. Let $\mathbb{C}$ be a small category and X a presheaf over $\mathbb{C}$. Show that $\mathrm{PSh}(\mathbb{C})/X \simeq \mathrm{PSh}(\int_{\mathbb{C}} X)$.
3. Show that the forgetful functor $U\colon M\mathbf{Sets} \to \mathbf{Sets}$ has both adjoints.
4. Let $\mathbb{C}$ be a category with all finite limits. We say $\mathbb{C}$ is *locally cartesian closed* if for every morphism $f\colon Y \to X$, the pullback functor $f^*\colon \mathbb{C}/X \to \mathbb{C}/Y$ has a right adjoint.
   (a) Show that $\mathbb{C}$ is locally cartesian closed if and only if for every object Z, the slice category $\mathbb{C}/Z$ is cartesian closed.
   (b) Deduce that if $\mathbb{D}$ is a small category then $\mathrm{PSh}(\mathbb{D})$ is locally cartesian closed.