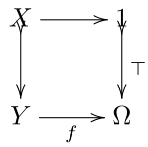
Category Theory 2017 - Exercise sheet 11

1. In the lecture we defined a subobject classifier to be a subobject $A \rightarrowtail \Omega$ such that for any subobject $X \rightarrowtail Y$ there exists a unique map $f\colon Y \to \Omega$ such that $f^*A = X$. Deduce from this that $A = 1$ and hence that this definition coincides with Awodey's.

2. Let $\mathbb{C}$ be a small category. Recall that we define the presheaf Ω by taking $\Omega(a)$ to be the set of sieves on a.

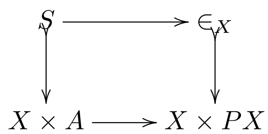
   (a) Show how to define Ω on morphisms and check that this is a presheaf.

   (b) Recall that we define the natural transformation $\top\colon 1 \to \Omega$ by taking $\top_a(*)$ to be the maximal sieve on a (the set of all morphisms into a). Let Y be a presheaf, and let X be a subpresheaf of Y. Define $f\colon Y \to \Omega$ by taking $f_c(y)$ to be $\{\alpha\colon d \to c \mid d \in \mathrm{Ob}(\mathbb{C}),\ y \cdot_B \alpha \in X(d)\}$. Show that the square below is a pullback:



   (c) Show that f is the unique morphism with this property and deduce that Ω is a subobject classifier.

3. Let X be an object of a category $\mathbb{C}$. A *power object on* X is an object PX, together with a subobject $\in_X \rightarrowtail X \times PX$ such that for every object A and every subobject $S \rightarrowtail X \times A$ there exists a unique map $A \to PX$ making the diagram below a pullback:



   (a) Show that in **Sets** every object has a power object.

   (b) Show that in any *topos* (a category with finite limits, exponentials and a subobject classifier) every object has a power object. (Hint: Take PX to be Ω^X.)

4. A topos is *2-valued* if a coproduct inclusion $1 \to 1 + 1$ is a subobject classifier. Show that M**Sets** is 2-valued if and only if M is a group. (Hint: An *ideal* in a monoid M is a subset $I \subseteq M$ such that if $g \in I$ and $h \in M$ then $gh \in I$. Show that a monoid is a group if and only if the only ideals are $\emptyset$ and M.)