Category Theory 2017 - Exercise sheet 2

1. Show that for any category $\mathbb{C}$, there is a functor $H\colon \mathbb{C}^{\text{op}} \times \mathbb{C} \to \mathbf{Sets}$ such that for objects C and D of $\mathbb{C}$, $H(C, D) = \text{Hom}(C, D)$.

2. Let $\mathbb{C}$ be a category and let $f\colon A \to B$ be an isomorphism in $\mathbb{C}$.
   (a) Show that the inverse of f is unique (i.e. if g and g' are both inverses of f then $g = g'$) and that the inverse is also an isomorphism.
   (b) Show that for any functor $F\colon \mathbb{C} \to \mathbb{D}$, $F(f)$ is also an isomorphism and its inverse is $F(f^{-1})$.

3. Let f, g, h be morphisms such that $h = g \circ f$.
   (a) Show that if g and f are both monomorphisms, then so is h.
   (b) Show that if g and f are both epimorphisms, then so is h.
   (c) Show that if h is monic, then so is f.
   (d) Show that if h is epic, then so is g.
   (e) Show that if any two of h, f and g are isomorphisms, then so is the third.

4. (a) Show that a morphism $f\colon A \to B$ in **Sets** is monic if and only if it is an injective function.
   (b) Show that a morphism $f\colon A \to B$ in **Sets** is epic if and only if it is a surjective function.
   (c) Show that a morphism $f\colon A \to B$ in **Mon** is monic if and only if it is an injective function.
   (d) Show that the inclusion $i\colon \mathbb{N} \to \mathbb{Z}$ is an epimorphism in **Mon**. Deduce that there is a morphism in **Mon** which is both monic and epic, but not an isomorphism.

5. A morphism $f\colon A \to B$ in a category $\mathbb{C}$ is an *absolute epimorphism* if for all functors $F\colon \mathbb{C} \to \mathbb{D}$, $F(f)$ is an epimorphism; f is a *split epimorphism* if for some $g : B \to A$, $f \circ g = 1_B$. Show that f is an absolute epimorphism if and only if it is a split epimorphism. (Hint for ($\Rightarrow$): Consider the functor $\text{Hom}(B, -)\colon \mathbb{C} \to \mathbf{Sets}$, and use question 4.(b))