Category Theory 2017 - Exercise sheet 4

1. Let $f\colon A \to X$ and $g\colon B \to X$. Show that the product of f and g as objects in the slice category $\mathbb{C}/X$ is the same as their pullback in $\mathbb{C}$ below:

$$\begin{array}{ccc} A \times_X B & \xrightarrow{p_2} & B \\ {\scriptstyle p_1}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & Y \end{array}$$

2. Consider the commutative diagram below.

$$\begin{array}{ccccc} F & \xrightarrow{f'} & E & \xrightarrow{g'} & D \\ {\scriptstyle h''}\downarrow & & \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle h} \\ A & \xrightarrow[f]{} & B & \xrightarrow[g]{} & C \end{array}$$

   (a) Show that if both squares are pullbacks, then so is the whole rectangle.

   (b) Show that if the right hand square and whole rectangle are pullbacks, then the left hand square is also a pullback.

3. Let $\mathbb{C}$ be a category with pullbacks. Let $f\colon Y \to X$ be a morphism in $\mathbb{C}$. Show how to define a functor $f^*\colon \mathbb{C}/X \to \mathbb{C}/Y$ where the action on objects is defined using pullback, as follows. Given an object of $\mathbb{C}/X$ of the form $h\colon A \to X$, define $f^*(h)$ to be given by the pullback diagram below:

$$\begin{array}{ccc} f^*(A) & \longrightarrow & A \\ {\scriptstyle f^*(h)}\downarrow & \lrcorner & \downarrow{\scriptstyle h} \\ Y & \xrightarrow[f]{} & X \end{array}$$

4. Let $\mathbb{C}$ be a category with pullbacks.

   (a) Show that an arrow $m\colon M \to X$ is monic if and only if the diagram below is a pullback.

$$\begin{array}{ccc} M & \xrightarrow{1_M} & M \\ {\scriptstyle 1_M}\downarrow & & \downarrow{\scriptstyle m} \\ M & \xrightarrow[m]{} & X \end{array}$$

   Deduce that m is monic iff as an object in $\mathbb{C}/X$, $m \times m \cong m$.

(b) Show that the pullback along an arrow $f\colon Y \to X$ of a pullback square over X,

$$\begin{array}{ccc} A \times_X B & \longrightarrow & B \\ \downarrow & \lrcorner & \downarrow \\ A & \longrightarrow & X \end{array}$$

is again a pullback square over Y. (Hint: draw a cube and use exercise 2.) Conclude that the pullback functor $f^*\colon \mathbb{C}/X \to \mathbb{C}/Y$ preserves products.

(c) Conclude that in a pullback square as below,

$$\begin{array}{ccc} M' & \longrightarrow & M \\ {\scriptstyle m'}\downarrow & \lrcorner & \downarrow{\scriptstyle m} \\ A' & \xrightarrow[f]{} & A \end{array}$$

if m is monic, then so is m'.

5. Show directly that in any category, in a pullback square as below,

$$\begin{array}{ccc} M' & \longrightarrow & M \\ {\scriptstyle m'}\downarrow & \lrcorner & \downarrow{\scriptstyle m} \\ A' & \xrightarrow[f]{} & A \end{array}$$

if m is monic, then so is m'.