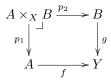
Category Theory 2017 - Exercise sheet 4

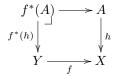
1. Let $f: A \to X$ and $g: B \to X$. Show that the product of f and g as objects in the slice category \mathbb{C}/X is the same as their pullback in \mathbb{C} below:



2. Consider the commutative diagram below.

$$\begin{array}{c|c} F \xrightarrow{f'} E \xrightarrow{g'} D \\ h'' & & \downarrow h' & \downarrow h \\ A \xrightarrow{f'} B \xrightarrow{g'} C \end{array}$$

- (a) Show that if both squares are pullbacks, then so is the whole rectangle.
- (b) Show that if the right hand square and whole rectangle are pullbacks, then the left hand square is also a pullback.
- 3. Let \mathbb{C} be a category with pullbacks. Let $f: Y \to X$ be a morphism in \mathbb{C} . Show how to define a functor $f^*: \mathbb{C}/X \to \mathbb{C}/Y$ where the action on objects is defined using pullback, as follows. Given an object of \mathbb{C}/X of the form $h: A \to X$, define $f^*(h)$ to be given by the pullback diagram below:



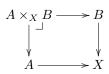
- 4. Let \mathbb{C} be a category with pullbacks.
 - (a) Show that an arrow $m\colon M\to X$ is monic if and only if the diagram below is a pullback.



Deduce that m is monic iff as an object in \mathbb{C}/X , $m \times m \cong m$.

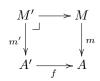
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(b) Show that the pullback along an arrow $f: Y \to X$ of a pullback square over X,



is again a pullback square over Y. (Hint: draw a cube and use exercise 2.) Conclude that the pullback functor $f^* \colon \mathbb{C}/X \to \mathbb{C}/Y$ preserves products.

(c) Conclude that in a pullback square as below,



if m is monic, then so is m'.

5. Show directly that in any category, in a pullback square as below,

$$\begin{array}{ccc} M' \longrightarrow M \\ & & \\ m' & \downarrow & & \\ m' & & & \\ M' & & & \\ A' & \xrightarrow{f} & A \end{array}$$

if m is monic, then so is m'.