Category Theory 2017 - Exercise sheet 5

1. Let X be an object in some category $\mathbb{C}$.

   (a) Show that on the collection of monomorphisms in $\mathbb{C}$ with codomain X we can define an equivalence relation by declaring two monomorphisms $m : A \to X$ and $n : B \to X$ to be equivalent if there is an isomorphism $f : A \to B$ such that $nf = m$.

   (b) An equivalence class of monomorphisms with codomain X with respect to the equivalence relation in (a) is called a *subobject* of X. Show that the collection of subobjects $\mathrm{Sub}(X)$ is partially ordered by saying that $[m] \leq [n]$ if there is a map f such that $nf = m$. Show that this is well-defined and that the morphism f is a monomorphism and unique whenever it exists.

   (c) Show that in the category **Sets** the poset $\mathrm{Sub}(X)$ is isomorphic to the power set $\mathcal{P}(X)$ of X, ordered by inclusion.

   (d) Show that if $\mathbb{C}$ has pullbacks, then for every object X, $\mathrm{Sub}(X)$ has all finite meets. Also show that these finite meets are preserved by the pullback functors $f^* : \mathrm{Sub}(X) \to \mathrm{Sub}(Y)$ for any $f : Y \to X$.

2. Let $\mathbb{C}$ be a complete category. Show that $\mathbb{C}^{\to}$ is also complete.

3. Define the following functor $\mathcal{P} \colon \mathbf{Sets}^{\mathrm{op}} \to \mathbf{Sets}$. For a set X, let $\mathcal{P}(X)$ be the power set of X. For a morphism $f \colon X \to Y$ (a function $Y \to X$), and $A \in \mathcal{P}(X)$, define $\mathcal{P}(f)(A)$ to be the preimage $f^{-1}(A)$.

   (a) Check that $\mathcal{P}$ is a well defined functor.

   (b) Show that $\mathcal{P}$ preserves all small limits.

4. Let $F \colon \mathbf{Sets} \to \mathbf{Mon}$ be the free monoid functor. Show that F preserves all small colimits. (Hint: Let $U \colon \mathbf{Mon} \to \mathbf{Sets}$ be the forgetful functor and use the universal mapping property of free monoids).