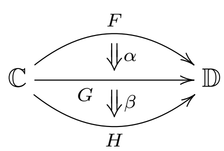
Category Theory 2017 - Exercise sheet 6

1. Show how to construct the following isomorphisms.
   (a) $(A \times B)^C \cong A^C \times B^C$
   (b) $(A^B)^C \cong A^{B \times C}$
   (c) $1^A \cong 1$

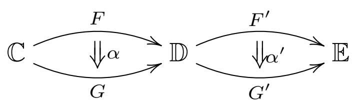
2. (a) Suppose we are given two natural transformations α from F to G and β from G to H as in the following diagram:



   Show how to define a natural transformation $\beta \circ \alpha$ from F to H. Deduce that there is a category $[\mathbb{C}, \mathbb{D}]$ where the objects are functors from $\mathbb{C}$ to $\mathbb{D}$ and the morphisms are natural transformations.

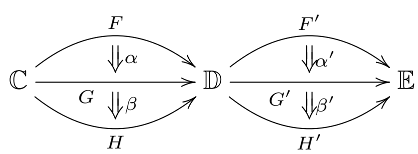
   (b) Show that $[\mathbb{C}, \mathbb{D}]$ is the exponential $\mathbb{D}^{\mathbb{C}}$ in **Cat**.

   (c) Suppose we are given two natural transformations α and α' like in the following diagram:



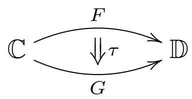
   Show how to define a natural transformation $\alpha' \bullet \alpha$ from $F' \circ F$ to $G' \circ G$.

   (d) Suppose we are given four natural transformations like in the following diagram:



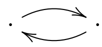
   Show that $(\beta' \bullet \beta) \circ (\alpha' \bullet \alpha) = (\beta' \circ \alpha') \bullet (\beta \circ \alpha)$.

3. (a) Let τ be a natural transformation as in the following diagram:



Show that τ is an isomorphism as a morphism in $[\mathbb{C}, \mathbb{D}]$ if and only if for all objects C of $\mathbb{C}$, τ_C is an isomorphism as a morphism in $\mathbb{D}$. (We refer to such maps as *natural isomorphisms*)

(b) (If time). Let $\mathbb{J}$ be the category with 2 objects, and two non-identity morphisms, as illustrated below:



Show that there is a correspondence between natural isomorphisms from $\mathbb{C}$ to $\mathbb{D}$ and maps $\mathbb{J} \times \mathbb{C} \to \mathbb{D}$.

4. (For students familiar with computability theory) We define the category **PER** of *partial equivalence relations* as follows. An object of **PER** is a pair (A, R) where $A \subseteq \mathbb{N}$ and R is an equivalence relation on A. A morphism from (A, R) to (B, S) is an equivalence class of partial computable functions ϕ such that for all $a \in A$, $\phi(a) \downarrow$ and for all $a, a' \in A$ aRa' implies $\phi(a) S \phi(a')$, with the equivalence relation $\phi \sim \phi'$ when for all $a \in A$, $\phi(a) S \phi'(a)$.

Show that **PER** is a cartesian closed category with a natural numbers object and all finite limits and finite colimits.