Category Theory 2017 - Exercise sheet 8

1. Let $\mathbf{C}$ be a small category. Suppose we are given $P, Q \in \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$ and natural transformations $\phi, \psi\colon P \to Q$. Show that if for every representable yC and every natural transformation $\theta\colon yC \to P$ we have $\phi \circ \theta = \psi \circ \theta$ then $\phi = \psi$. (We say that the representables *generate* $\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$.)

2. Let $\mathbf{C}$ be a small category. Show that $\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$ is cartesian closed and that the Yoneda embedding $y\colon \mathbf{C} \to \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$ preserves any exponentials that exist in $\mathbf{C}$.

3. Let $\mathbf{C}$ be a small category. Show that $\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$ has a natural numbers object.

4. Let $\mathbb{C}$ be complete and cocomplete, and let $\mathbb{I}$ be a small category. Show that the diagonal functor $\mathbb{C} \to \mathbb{C}^{\mathbb{I}}$ has both a left and a right adjoint.

5. Prove that the forgetful functor $U\colon \mathbf{Top} \to \mathbf{Sets}$ has both a left and a right adjoint.