Category Theory 2017 - Exercise sheet 9

1. Let X and Y be sets, and let $f\colon Y \to X$ be a function. Let $\mathcal{P}(X)$ be the poset of subsets of X ordered by inclusion and similarly define the poset $\mathcal{P}(Y)$ for Y. We define $f^*\colon \mathcal{P}(X) \to \mathcal{P}(Y)$ to be the preimage map (for $U \subseteq X$, $f^*(U) := \{y \in Y \mid f(y) \in U\}$). Show that f^* has both a left adjoint (which we denote $\exists_f$) and a right adjoint (which we denote $\forall_f$).

2. Let $\mathbb{C}$ be a category and X an object of $\mathbb{C}$. Consider the forgetful functor $\mathbb{C}/X \to \mathbb{C}$.

   (a) When does the functor have a right adjoint?

   (b) Suppose that it has a right adjoint $\mathbb{C} \to \mathbb{C}/X$. When does the right adjoint itself have its own right adjoint?

3. Suppose we have two functors F and G as below:

$$\mathbb{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathbb{D}$$

   (a) Suppose that F and G form an adjunction $F \dashv G$. Show how to construct natural transformations $\eta\colon 1_{\mathbb{C}} \Rightarrow GF$ and $\epsilon\colon FG \Rightarrow 1_{\mathbb{D}}$ making the following two diagrams commute for all objects X of $\mathbb{C}$ and all objects Y of $\mathbb{D}$:

$$\begin{array}{ccc} F(X) & \xrightarrow{F\eta_X} & FGF(X) \\ & {\scriptstyle 1_{F(X)}} \searrow & \downarrow {\scriptstyle \epsilon_{FX}} \\ & & F(X) \end{array} \qquad \begin{array}{ccc} G(Y) & \xrightarrow{\eta_{G(Y)}} & GFG(Y) \\ & {\scriptstyle 1_{G(Y)}} \searrow & \downarrow {\scriptstyle G\epsilon_Y} \\ & & G(Y) \end{array}$$

   (We refer to η and ϵ as the *unit* and *counit* of the adjunction respectively, and we refer to the commutative diagrams above as the *triangular identities*).

   (b) Conversely, suppose that we are given natural transformations $\eta\colon 1_{\mathbb{C}} \Rightarrow GF$ and $\epsilon\colon FG \Rightarrow 1_{\mathbb{D}}$ satisfying the triangular identities. Show that $F \dashv G$.

4. Suppose we are given $F\colon \mathbb{C} \to \mathbb{D}$. We say F is an *adjoint equivalence* if there exists $G\colon \mathbb{D} \to \mathbb{C}$ together with natural isomorphisms $\eta\colon 1_{\mathbb{C}} \Rightarrow GF$ and $\epsilon\colon FG \Rightarrow 1_{\mathbb{D}}$ that satisfy the triangular identities from question 3. Show that if F is full, faithful and essentially surjective then F is an adjoint equivalence.