The syntax category

Definition. Let $\mathbb{T}$ be a type theory. The *syntax category of* $\mathbb{T}$, written $\mathrm{Syn}(\mathbb{T})$, is defined as follows:

- Objects: the contexts Γ of $\mathbb{T}$.
- Morphisms: tuples of terms

  $$\Gamma \xrightarrow{[t_1,\ldots,t_n]} \Delta$$
  $$= [x_1 : A_1, x_2 : A_2(x_1), \ldots, A_n(x_1, \ldots x_{n-1})]$$

  such that

  $$\begin{aligned} &\Gamma \vdash t_1 : A_1 \\ &\Gamma \vdash t_2 : A_2(x_1) \\ &\vdots \\ &\Gamma \vdash t_n : A_n(x_1, \ldots, x_{n-1}). \end{aligned}$$

  For any $\Delta \vdash A$ Type, we write $A\{f\}$ for $A[t_1/x_1, \ldots, t_n/x_n]$. Analogous for contexts and terms.
- Composition: if

  $$\begin{array}{ccc} \Gamma & \xrightarrow{f=[t_1,\ldots,t_n]} & \Delta \\ & & \big\downarrow g=[s_1,\ldots,s_m] \\ & & \Theta = [y_1 : B_1, \ldots, y_m : B_m(y_1, \ldots, y_{m-1})], \end{array}$$

  then $g \circ f = [s_1\{f\}, \ldots s_m\{f\}]$.
- Identity: $\mathrm{id}_\Gamma = [x_1, \ldots, x_n]$ for any

  $$\Gamma = [x_1 : A_1, \ldots, x_n : A_n(x_1, \ldots, x_{n-1})].$$

Proposition. *$\mathrm{Syn}(\mathbb{T})$ is a category.*

<u>Terms and types in $\mathrm{Syn}(\mathbb{T})$</u> For any context <u>$\Gamma = [x_1 : A_1, \ldots, x_n : A_n(x_1, \ldots, x_{n-1})]$</u> of $\mathbb{T}$, the types A of Γ correspond to *display maps* and the terms t to its sections, as depicted in the following picture:

$$\Gamma \underset{[x_1,\ldots,x_n,t]}{\overset{[x_1,\ldots,x_n]}{\leftrightarrows}} \Gamma.A$$

We write $\mathsf{p}(\Gamma.A)$ for the display map corresponding to the type A and $\bar{t}$ for the section corresponding to its term t.

<u>Substitution in $\mathrm{Syn}(\mathbb{T})$: a coherence problem</u> For $\Delta \vdash$ <u>A Type and $f : \Gamma \to \Delta$, the type $A\{f\}$ of Γ</u> is given by the pullback of f along $\mathsf{p}(\Gamma.A)$:

$$\begin{array}{ccc} \Gamma.A\{f\} & \xrightarrow{\mathsf{q}(f,A)} & \Delta.A \\ {\scriptstyle \mathsf{p}(\Gamma.A\{f\})}\big\downarrow & & \big\downarrow{\scriptstyle \mathsf{p}(\Delta.A)} \\ \Gamma & \xrightarrow[f]{} & \Delta. \end{array}$$

where $\mathsf{q}(f, A) := [t_1, \ldots, t_n, y]$, for $f = [t_1, \ldots, t_n]$ and $\Gamma.A\{f\} = \Gamma, y : A\{f\}$, is the *weakening* of f with A. Because pullbacks are only defined up to isomorphism, substitution in the syntax category is not strictly associative, as it is in the syntax itself.

Categories with families

Definition. A *category with families* is a structure $(\mathbb{C}, Ty, Tm, -\{-\}, \top, \langle\rangle_-, -.-, \mathsf{p}, \mathsf{v}_-, \langle -, -\rangle_-)$, where

- $\mathbb{C}$ is a category with terminal object $\top$ and arrows $\langle\rangle_\Gamma : \Gamma \to \top$.
- For every $\Gamma \in \mathbb{C}$ collections:
  - $\mathsf{Ty}(\Gamma)$;
  - $\mathsf{Tm}(\Gamma, A)$ for all $A \in \mathsf{Ty}(\Gamma)$.
- For each morphism $f : \Gamma \to \Delta$ functions:
  - $-\{f\} : \mathsf{Ty}(\Delta) \to \mathsf{Ty}(\Gamma)$;
  - $-\{f\} : \mathsf{Tm}(\Delta, A) \to \mathsf{Tm}(\Gamma, A\{f\})$.
- For every $\Delta \in \mathbb{C}$ and $A \in \mathsf{Ty}(\Gamma)$,
  - $\Delta.A \in \mathbb{C}$ with corresponding:
  - $\mathsf{p}(A) : \Delta.A \to \Delta$;
  - $\mathsf{v}_A \in \mathsf{Tm}(\Delta.A, A\{\mathsf{p}(A)\})$;
  - for every $f : \Gamma \to \Delta$ and $t \in \mathsf{Tm}(\Gamma, A\{f\})$, $\langle f, t\rangle_A : \Gamma \to \Delta.A$.

such that for each $\Gamma, \Delta, \Theta \in \mathbb{C}$, $f : \Gamma \to \Delta$, $g : \Delta \to \Theta$, $A \in Ty(\Theta)$, $t \in Tm(\Theta, A)$ and $s \in Tm(\Delta, A\{g\})$,

$$\begin{array}{rll} A\{\mathrm{id}_\Theta\} = A & & \in Ty(\Theta) \\ A\{g \circ f\} = A\{g\}\{f\} & & \in Ty(\Gamma) \\ t\{\mathrm{id}_\Theta\} = t & & \in Tm(\Theta, A) \\ t\{g \circ f\} = t\{g\}\{f\} & & \in Tm(\Gamma, A\{g \circ f\}) \\ \mathsf{p}(A) \circ \langle g, s\rangle_A = g & & : \Delta \to \Theta \\ \mathsf{v}_A\{\langle g, s\rangle_A\} = s & & \in Tm(\Delta, A\{g\}) \\ \langle g, s\rangle_A \circ f = \langle g \circ f, s\{f\}\rangle_A & & : \Gamma \to \Theta.A \\ \langle \mathsf{p}(A), \mathsf{v}_A\rangle_A = \mathrm{id}_{\Theta.A} & & : \Theta.A \to \Theta.A. \end{array}$$

Definition. For any $t \in \mathsf{Tm}(\Delta, A)$, we define

$$\bar{t} = \langle id_\delta, t\rangle_A : \Delta \to \Delta.A$$

Proposition. $\bar{\cdot}$ *is a bijective map from* $\mathsf{Tm}(\Delta, A)$ *to the collection of sections of* $\mathsf{p}(A)$*.*

Definition. For $f : \Gamma \to \Delta$, the *weakening* of f by A is given by

$$\mathsf{q}(f, A) = \langle f \circ \mathsf{p}(A)\{f\}, \mathsf{v}_{A\{f\}}\rangle_A : \Gamma.A\{f\} \to \Delta.A$$

Interpreting type formers

To avoid ambiguities and make clear which projection we mean, we may write $\mathsf{p}(\Gamma.A)$ for $\mathsf{p}(A) : \Gamma.A \to \Gamma$ (or similarly, $\mathsf{p}(\Gamma.A.B)$ for $\mathsf{p}(B) : \Gamma.A.B \to \Gamma.B$).

Definition. *A Category with Families supports* Π-*types* if for any context Γ and any two types $A \in \mathsf{Ty}(\Gamma)$ and $B \in \mathsf{Ty}(\Gamma.A)$ we have that

(1) there is a type $\Pi(A, B) \in \mathsf{Ty}(\Gamma)$,

(2) for any $t \in \mathsf{Tm}(\Gamma.A, B)$, there is a term $\lambda_{A,B}(t) \in \mathsf{Tm}(\Gamma, \Pi(A, B))$,

(3) there is a morphism

$$\mathsf{App}_{A,B} : \Gamma.A.\Pi(A, B)\{\mathsf{p}(A)\} \to \Gamma.A.B$$

such that

$$\mathsf{p}(\Gamma.A.B) \circ \mathsf{App}_{A,B} = \mathsf{p}(\Gamma.A.\Pi(A, B))$$

and,

$$\mathsf{App}_{A,B} \circ \overline{(\lambda_{A,B}(t))\{\mathsf{p}(\Gamma.A)\}} = \bar{t},$$

for any $t \in \mathsf{Tm}(\Gamma.A, B)$,

(4) all of these construct are stable under substitution, i.e., for $f : \Delta \to \Gamma$, we have

  (a) $\Pi(A, B)\{f\} = \Pi(A\{f\}, B\{\mathsf{q}(f, A)\})$,
  (b) $(\lambda_{A,B})(t)\{f\} = \lambda_{A\{f\},B\{\mathsf{q}(f,A)\}}(t\{\mathsf{q}(f, A)\})$,
  (c) $\mathsf{App}_{A,B} \circ \mathsf{q}(\mathsf{q}(f, A), \Pi(A, B)\{\mathsf{p}(A)\}) = \mathsf{q}(\mathsf{q}(f, A), B) \circ \mathsf{App}_{A\{f\},B\{\mathsf{q}(f,A)\}}$.

Definition. *A Category with Families supports identity types* if for any context Γ and any type $A \in \mathsf{Ty}(\Gamma)$ we have that

(1) there is a type $\mathsf{Id}_A \in \mathsf{Ty}(\Gamma.A.A\{\mathsf{p}(A)\})$,

(2) there is a morphism

$$\mathsf{Refl}_A : \Gamma.A \to \Gamma.A.A\{\mathsf{p}(A)\}.\mathsf{Id}_A$$

such that $\mathsf{p}(\mathsf{Id}_A) \circ \mathsf{Refl}_A = \bar{v}_A$,

(3) for every type $B \in \mathsf{Ty}(\Gamma.A.A\{\mathsf{p}(A)\}.\mathsf{Id}_A)$ and term $H \in \mathsf{Tm}(\Gamma.A, B\{\mathsf{Refl}_A\})$ there is a term $R^{\mathsf{Id}}(H) \in \mathsf{Tm}(\Gamma.A.A\{\mathsf{p}(A)\}, B)$ such that $R^{\mathsf{Id}}(H)\{\mathsf{Refl}\} = H$,

(4) all of these constructs are stable under substitution, i.e.,

  (a) $\mathsf{Id}_A\{\mathsf{q}(\mathsf{q}(f, A), A\{\mathsf{p}(A)\})\} = \mathsf{Id}_{A\{f\}}$,
  (b) $\mathsf{q}(\mathsf{q}(\mathsf{q}(f, A), A\{\mathsf{p}(A)\}), \mathsf{Id}_A) \circ \mathsf{Refl}_{A\{f\}} = \mathsf{Refl}_A \circ \mathsf{q}(f, A)$.

Soundness and Completeness of CwF

Theorem. *There is a sound and complete interpretation function of type theory in categories with families.*

Example: Heyting Algebras and Peano's Third Axiom

Reminder. A *Heyting algebra* is a lattice H which as a poset admits an operation of implication $\to: A \to B$ satisfying the condition (really a universal property) $(x \wedge a) \leq b$ if and only if $x \leq (a \to b)$. We denote with $\mathsf{1}$ and $\mathsf{0}$ the maximal and minimal elements of H, respectively.

Let H be a Heyting algebra and consider it as a category $\mathcal{C}_H$ in the usual way (i.e., the objects of $\mathcal{C}_H$ are the elements of H and there is a unique morphism from $a \in H$ to $b \in H$ if and only if $a \leq b$). This category can be equipped with the structure of a category with families:

- $\mathcal{C}_H$ has the terminal object $\mathsf{1}$,
- for any context $\Gamma \in \mathcal{C}_H$, we let $\mathsf{Ty}(\Gamma) = H$, and $\mathsf{Tm}(\Gamma, A) = \mathsf{Hom}_{\mathcal{C}_H}(\Gamma, A)$,
- for comprehension of $\Gamma \in \mathcal{C}_H$ and $A \in \mathsf{Ty}(\Gamma) = H$ we define $\Gamma.A = \Gamma \wedge A$.
- Both substitutions $-\{f\}$ are the identity.

We interpret type constructors as follows:

Type	**Interpretation**
$\Pi(A, B)$	$A \to B$
$\Sigma(A, B)$	$A \wedge B$
Id_A	$\mathsf{1}$
N	$\mathsf{1}$
0	$\mathsf{0}$

Theorem. *Every Heyting algebra H exhibits the structure of a category with families $\mathcal{C}_H$ that supports Π-types, Σ-types, identity types, natural numbers and the empty type.*

Recall Peano's third axiom:

$$x \in \mathbb{N} \to Sx \neq 0 \qquad \text{(P)}$$

Proposition. *Peano's third axiom (P) is provable in type theory with universes.*

Proposition. *Let H be a Heyting algebra. Then judgements of the form $p : \mathsf{Id}_A(a, b) \vdash t(p) : \mathsf{0}$ are not valid in $\mathcal{C}_H$.*

Corollary. *For any Heyting algebra H, (P) is not provable in $\mathcal{C}_H$.*

Corollary. *Peano's third axiom (P) is independent of type theory.*

Homework

Exercise. Let H be a Heyting algebra and $\mathcal{C}_H$ be the associated category with families. Show that $\mathcal{C}_H$ supports Π-types. (Hint: you are allowed to use all well-known facts about Heyting algebras and categories that arise from a partial order.)