Type theory and weak factorisation systems

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We will discuss two results. The first result is of Awodey and Warren, the second result is from Gambino and Garner.

1. When $C$ is a finitely complete category with a weak factorization system, then $C$ is a model of a form of Martin-Löf type theory with identity types [1].
2. When $\mathbb{T}$ is a dependent type theory with the axioms for identity types, then its syntactic category $Syn(\mathbb{T})$ admits a non-trivial weak factorisation system [2].

# 1 Preliminaries

Before we get into this, we will define a weak factorisation system. Before that we need the following definition:

**Definition 1.1** (Left lifting property (LLP)). Let $\mathbb{C}$ be a category. Given two maps $f : A \to B$ and $g : C \to D$ we say that $f$ has the *left lifting property* with respect to $g$, and $g$ has the right lifting property w.r.t. $f$, denoted by or $f \pitchfork g$, when for any commutative square as below:

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ {\scriptstyle f}\downarrow & \overset{l}{\nearrow} & \downarrow{\scriptstyle g} \\ B & \xrightarrow[k]{} & D \end{array}$$

there exists a map $l : B \to C$, the *diagonal filler*, such that $g \circ l = k$ and $l \circ f = h$

Let $\mathbb{C}$ be a category. For a collection of maps $\mathcal{M}$, we define ${}^{\pitchfork}\mathcal{M}$ to be the collection of maps in $\mathbb{C}$ having the LLP with respect to all maps in $\mathcal{M}$. The collection $\mathcal{M}^{\pitchfork}$ is defined similarly.

**Definition 1.2** (Weak factorization system). Let $\mathbb{C}$ be a category. A *weak factorisation system* on $\mathbb{C}$ consists of a pair of collections of maps $(\mathcal{L}, \mathcal{R})$, such that the following holds:

1. Every map $f$ in $\mathbb{C}$ admits a factorization $f = p \circ i$ where $i \in \mathcal{L}$ and $p \in \mathcal{R}$
2. $\mathcal{R} = \mathcal{L}^{\pitchfork}$ and $\mathcal{L} =^{\pitchfork} \mathcal{R}$

We remind you of the following definition:

**Definition 1.3** (Display map). A *display map* is a morphism between contexts, defined by "projecting away" a variable: $[\Gamma, x : A] \to \Gamma$, where $\Gamma$ is a context and $A$ is a type relative to $\Gamma$.

# 2 The result of Awodey and Warren

A model in Martin-Löf type theory is extensional if the following reflection rule is satisfied:

$$\frac{\vdash p : Id_A(a,b)}{\vdash a = b : A}$$

Type checking is decidable in the intensional theory, but not in extensional. That is the main reason why we should prefer intensional theories.

**Lemma 2.1** ([1]). *In the standard interpretation of type theory every locally cartesian closed category $\mathbb{C}$ is extensional.*

**Definition 2.2** (Model category). A model category is a bicomplete category $\mathbb{C}$ equipped with subcategories $\mathfrak{F}$ (fibrations), $\mathfrak{C}$ (cofibrations) and $\mathfrak{W}$ (weak equivalences) satisfying the following conditions:

1. (”Three-for-two”) given a commutative triangle

$$\begin{array}{ccc} A & \xrightarrow{\quad i \quad} & C \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle p} & \\ & B & \end{array} \qquad (*)$$

   if any two of $f$, $g$, $h$ are weak equivalences, then so is the third.

2. both $(\mathfrak{C}, \mathfrak{F} \cap \mathfrak{W})$ and $(\mathfrak{C} \cap \mathfrak{W}, \mathfrak{F})$ are weak factorization systems.

A map $f$ is an acyclic cofibration if it is in $\mathfrak{C} \cap \mathfrak{W}$, i.e. both cofibration and a weak equivalence. Similarly, an acyclic fibration is a map in $\mathfrak{F} \cap \mathfrak{W}$, i.e. which is simultaneously a fibration and a weak equivalence. An object $A$ is said to be fibrant if the canonical map $A \to 1$ is a fibration. Similarly, $A$ is cofibrant if $0 \to A$ is a cofibration.

In a model category $\mathbb{C}$ a path object $A^I$ for an object $A$ consists of a factorization

$$\begin{array}{ccc} A & \xrightarrow{\quad r \quad} & A^I \\ & {\scriptstyle \Delta}\searrow \quad \swarrow{\scriptstyle p} & \\ & A \times A & \end{array}$$

of the diagonal map $\Delta : A \to A \times A$ as an acyclic cofibration $r$ followed by a fibration $p$.

**Theorem 2.3.** *Let $\mathbb{C}$ be a finitely complete category with a weak factorization system and a functorial choice $(-)^I$ of path objects in $\mathbb{C}$, and all of its slices, which is stable under substitution, i.e. given any fibration $B \to A$ and any arrow $\sigma : A' \to A$, the evident comparison map is an isomorphism*

$$\sigma^*(B^I) \cong (\sigma^* B)^I .$$

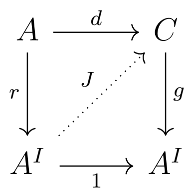
*Proof.* We may work in the empty context since the relevant structure is stable under slicing. Given a functorial choice of path objects ($*$), we interpret, given a fibrant object $A$, the judgement $x, y \vdash Id_A(x, y)$ as the path object fibration $p : A^I \to A \times A$. Because $p$ is a fibration, the formation is satisfied. Similarly, the introduction rule is valid because $r : A \to A^I$ is a section of $p$.
For the elimination and conversion rules, assume that the following premises are given

$$x : A, y : A, z : Id_A(x, y) \vdash D(x, y, z) \text{ type},$$
$$x : A \vdash d(x) : D(x, x, r_A(x)) .$$

We have, therefore, a fibration $g : D \to A^I$ together with a map $d : A \to D$ such that $g \circ d = r$. This data yields the following commutative square:

$$\begin{array}{ccc} A & \xrightarrow{d} & C \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle g} \\ A^I & \xrightarrow[1]{} & A^I \end{array}$$

Because $g$ is a fibration and $r$ is, by definition an acyclic cofibration, there exists a diagonal filler




Choose such a filler $J$ as the interpretation of the term:

$$x, y : A, z : Id_A(x, y) \vdash J_{A,D}(d, x, y, z) : D(x, y, z) .$$

Then commutativity of the bottom triangle on the diagram above is precisely the conclusion of the elimination rule nad commutativity of the top triangle is the computation rule. $\square$

# 3 The result of Gambino and Garner

Before we can prove the main theorem, we need to introduce a couple of definitions and lemma's.

We remind you of the following definition:

**Definition 3.1** (Syntactic category)**.** We have a category $Syn(\mathbb{T})$. Objects are the contexts of $\mathbb{T}$ and the morphisms are tuples of terms (context morphisms).

Let us consider a fixed context $\Gamma$.

**Definition 3.2** (Dependent context)**.** Let $\Phi = [x_0 : A_0, x_1 : A_1(x_0), \ldots, x_n : A_n(x_0, \ldots, x_{n-1})]$. We say that $\Phi$ is a *dependent context* relative to $\Gamma$ when we can derive $\Gamma \vdash \Phi : Cxt$, where we mean the following sequence of judgements:

$$\begin{aligned}
&\Gamma \vdash A_0 : Type \\
&\Gamma, x_0 : A_0 \vdash A_1(x_0) : Type \\
&\vdots \\
&\Gamma, x_0 : A_0, \ldots, x_{n-1} : A_{n-1}(x_0, \ldots, x_{n-1}) \vdash A_n(x_0, \ldots, x_{n-1}) : Type
\end{aligned}$$

Let $a = (a_0, a_1, \ldots, a_n)$. With $\Gamma \vdash a : \Phi$ we mean:

$$\begin{aligned}
&\Gamma \vdash a_0 : A_0 \\
&\Gamma \vdash a_1 : A_1(a_0) \\
&\vdots \\
&\Gamma \vdash a_n : A_n(a_0, \ldots, a_{n-1})
\end{aligned}$$

We say that $a$ is a *dependent element* of $\Phi$ with respect to $\Gamma$.

When we have a dependent context $\Phi$, relative to $\Gamma$, we obtain a new context $[\Gamma, \Phi]$. We also obtain the following morphisms:

**Definition 3.3** (Dependent projections)**.** A *dependent projection* is a map $[\Gamma, \Phi] \to \Gamma$, "projecting away" the variables in $\Phi$.

It is possible introduce expressions $\Gamma \vdash \Phi = \Psi : Cxt$ and $\Gamma \vdash a = b : \Phi$, such that these equalities satisfy reflexivity, symmetry and transitivity.

In addition to identity types we will introduce *identity contexts*:

**Definition 3.4.** For a context $\Phi$ and $a, b : \Phi$, we have an *identity context* $Id_\Phi(a, b)$.

We have the following deduction rules for identity contexts, where we leave implicit a context $\Gamma$, to which all notions are assumed to be relative:

$$\text{Formation:} \quad \frac{\vdash \Phi : Cxt}{a : \Phi, b : \Phi \vdash Id_\Phi(a, b) : Cxt}$$

$$\text{Introduction:} \quad \frac{\vdash \Phi : Cxt}{a : \Phi \vdash refl(a) : Id_\Phi(a, a)}$$

$$\text{Elimination:} \quad \frac{\begin{array}{c} a : \Phi, b : \Phi, u : Id_\Phi(a, b), \Delta(a, b, u) \vdash C(a, b, u) : Cxt \\ a : \Phi, \Delta(a, a, refl(a)) \vdash d(a) : C(a, a, refl(a)) \end{array}}{a : \Phi, b : \Phi, u : Id_\Phi(a, b), \Delta(a, b, u) \vdash J(d, a, b, u) : C(a, b, u)}$$

$$\text{Computation:} \quad \frac{a : \Phi, \Delta(a, a, refl(a)) \vdash d(a) : C(a, a, refl(a))}{a : \Phi, \Delta(a, a, refl(a)) \vdash J(d, a, a, refl(a)) = d(a) : C(a, a, refl(a))}$$

Here $\Delta(a, b, u)$ is a dependent context.

We will need the following lemma's:

**Lemma 3.5** ([2])**.** *For every context $\Phi$, we can derive a rule of the form*

$$\frac{a : \Phi \vdash \Phi(a) : Cxt}{a : \Phi, b : \Phi, u : Id_\Phi(a, b), e : \Phi(a) \vdash u_*(e) : \Phi(b)}$$

*such that*

$$\frac{a : \Phi, e : \Phi(a)}{(refl(a))_*(e) = e : \Phi(a)}$$

*holds*

**Lemma 3.6** ([2])**.** *We can derive rules of the form*

$$\frac{u : Id_\Phi(a, b), v : Id_\Phi(b, c)}{v \circ u : Id_\Phi(a, c)}$$

$$\frac{a : \Phi}{\mathbb{1}_a : Id_\Phi(a, a)}$$

*such that*

$$\frac{u : Id_\Phi(a, b)}{\mathbb{1}_b \circ u = u : Id_\Phi(a, b)}$$

*holds*

**Lemma 3.7** ([2]). *We can derive a rule*

$$\frac{u : Id_\Phi(a,b)}{\psi_u : Id_\Phi(u \circ \mathbb{1}_a, u)}$$

*such that*

$$\frac{a : \Phi}{\psi_{\mathbb{1}_a} = \mathbb{1}_{\mathbb{1}_a} : Id_\Phi(\mathbb{1}_a, \mathbb{1}_a)}$$

*holds*

**Lemma 3.8** (Retract argument, [3]). *Suppose $f = p \circ i$ and $f$ has the RLP with respect to $i$. Then $f$ is a retract of $p$.*

We are now ready to prove the main theorem.

**Theorem 3.9.** *Let $\mathbb{T}$ be a dependent type theory with axioms for identity types. Let $\mathcal{D}$ be the set of display maps in $Syn(\mathbb{T})$. The pair $(\mathcal{L}, \mathcal{R})$, where $\mathcal{L} :=^{\pitchfork} \mathcal{D}$ and $\mathcal{R} := \mathcal{L}^{\pitchfork}$, forms a weak factorisation system on $Syn(\mathbb{T})$.*

We will show the theorem by proving the following two lemma's:

**Lemma 3.10.** *Every map $f$ admits a factorisation $f = p \circ i$, where $i \in \mathcal{L}$ and $p$ is a dependent projection.*

**Lemma 3.11.** $\mathcal{L} =^{\pitchfork} \mathcal{R}$

*Proof of Theorem 3.9.* Note that a display map is a dependent projection. Also note that $\mathcal{D} \subseteq \mathcal{R}$. We have that $\mathcal{R}$ is closed under composition, and we can create all dependent projections from compositions of display maps, so $\mathcal{R}$ contains all dependent projections. Then Lemma 3.10 gives us axiom 1 in Definition 1.2. Then by definition of $(\mathcal{L}, \mathcal{R})$ and Lemma 3.11 we get axiom 2 in Definition 1.2, which proves the theorem. □

We will now continue to prove the lemma's that we used.

*Proof of Lemma 3.10.* Let $f : \Phi \to \Psi$ be a context morphism. Define $Id(f) := [x : \Phi, y : \Psi, u : Id_\Psi(f(x), y)]$. We will now show that $f = p_f \circ i_f$, where $p_f := [y]$ and $i_f := [x, f(x), 1_{f(x)}]$. The factorization is displayed in the following picture:

$$\Phi \xrightarrow{i_f} Id(f) \xrightarrow{p_f} \Psi$$

It is clear that $p_f$ is a dependent projection. So we only need to show that $i_f \in \mathcal{A}$, which means that $i_f$ has the LLP with respect to all display maps.

$$\begin{array}{ccc} \Phi & \xrightarrow{g} & [v : \Delta, z : D(v)] \\ {\scriptstyle i_f}\big\downarrow & \overset{df_1}{\nearrow} & \big\downarrow{\scriptstyle d} \\ Id(f) & \xrightarrow[h]{} & [v \in \Delta] \end{array}$$

We thus want to show that the commuting diagram above, where $d$ is some display map, has a diagonal filler, $df_1$. Display maps are closed under pullbacks (we proved this in one of the lectures).

This means that we also have a commuting diagram as below:

$$\begin{array}{ccc} X & \xrightarrow{j} & [v : \Delta, w : D(v)] \\ \big\downarrow{\scriptstyle \bar{d}} & & \big\downarrow{\scriptstyle d} \\ Id(f) & \xrightarrow{h} & [v : \Delta] \end{array}$$

And a unique morphism $e : \Phi \to X$, such that $\bar{d} \circ e = i_f$ and $j \circ e = g$. Moreover, $\bar{d}$ is also a pullback and so $X$ can be written as $[Id(f), z : C(x, y, u)]$ where $C(x, y, u)$ is a dependent type relative to $Id(f)$.

So if we can find a diagonal filler $df_2$ for this diagram:

$$\begin{array}{ccc} \Psi & \xrightarrow{e} & [Id(f), z : C(x, y, u)] \\ {\scriptstyle i_f}\big\downarrow & \overset{df_2}{\nearrow} & \big\downarrow{\scriptstyle \bar{d}} \\ Id(f) & \xrightarrow[\mathbb{1}_{Id(f)}]{} & Id(f) \end{array}$$

Then by concatenation of $df_2$ with $j$, we get a diagonal filler for the first diagram. The rest of the proof will be dedicated to finding $df_2$

We can derive

$$x : \Phi, y_0 : \Psi, y_1 : \Psi, v : Id_\Psi(y_0, y_1), u : Id_\Psi(f(x), y), z : C(x, y_0, u) \vdash C(x, y_1, v \circ u) : Type \tag{3.1}$$

since we can form $v \circ u : Id_\Psi(f(x), y_1)$ with Lemma 3.6 and thus a context $Id(f) = [x : \Phi, y_1 : \Psi, v \circ u : Id_\Psi(f(x), y_1)]$, so we can obtain the type $C(x, y_1, v \circ u)$ from the display map $\bar{d}$.

We can also derive

$$x : \Phi, y : \Psi, u : Id_\Psi(f(x), y), z : C(x, y, u) \vdash z : C(x, y, \mathbb{1}_y \circ u) \tag{3.2}$$

by the morphism $e$ and again using Lemma 3.6.

Then, by the elimination rule for identity contexts, we obtain from 3.1 and 3.2

$$x : \Phi, y_0 : \Psi, y_1 : \Psi, v : Id_\Psi(y_0, y_1), u : Id_\Psi(f(x), y), z : C(x, y_0, u) \vdash J(z, y_0, y_1, v) : C(x, y_1, v \circ u) \tag{3.3}$$

From 3.3 we can then obtain

$$x : \Phi, y : \Psi, u : Id_\Psi(f(x), y), z : C(x, f(x), \mathbb{1}_{f(x)}) \vdash J(z, f(x), y, u) : C(x, y, u \circ \mathbb{1}_{f(x)}) \tag{3.4}$$

Since here $z$ only depends on $x$, we can substitute it for $d(x)$ to get

$$x : \Phi, y : \Psi, u : Id_\Psi(f(x), y) \vdash J(d(x), f(x), y, u) : C(x, y, u \circ \mathbb{1}_{f(x)}) \tag{3.5}$$

Since we have $u : Id_\Psi(f(x), y)$, by Lemma 3.7 we also have $\psi_u : Id(u \circ \mathbb{1}_{f(x)}, u)$.
By this and by Lemma 3.5 we obtain

$$x : \Phi, y : \Psi, u : Id_\Psi(f(x), y) \vdash (\psi_u)_*(J(d(x), f(x), y, u) : C(x, y, u) \tag{3.6}$$

We now claim that the required filler, $df_2$ can be defined as $[x, y, u, (\psi_u)_*(J(d(x), f(x), y, u))]$.
That the bottom triangle commutes is obvious. The commutativity of the top triangle follows from the following equalities:
$(\psi_{\mathbb{1}_{f(x)}})_*(J(d(x), f(x), f(x), \mathbb{1}_{f(x)}) = J(d(x), f(x), f(x), \mathbb{1}_{f(x)}) = d(x)$

□

*Proof of Lemma 3.11.* Since $\mathcal{L} =^{\pitchfork} \mathcal{D}$ and $\mathcal{R} = \mathcal{L}^{\pitchfork}$, we have that $\mathcal{D} \subseteq \mathcal{R}$. This implies that ${}^{\pitchfork}\mathcal{R} \subseteq^{\pitchfork} \mathcal{D} = \mathcal{L}$. We still need to show that $\mathcal{L} \subseteq^{\pitchfork} \mathcal{R}$, that every map in $\mathcal{L}$ has the LLP with respect to every map in $\mathcal{R}$. We have that $\mathcal{L} =^{\pitchfork} \mathcal{D}$, so every map in $\mathcal{L}$ has the LLP with respect to every display map. But dependent projections are composites of display maps, so also every map in $\mathcal{L}$ has the LLP with respect to every dependent projection.

Lemma 3.8 and Lemma 3.10 tell us that every map in $\mathcal{R}$ is a retract of a dependent projection. From this we can conclude that $\mathcal{L} \subseteq^{\pitchfork} \mathcal{R}$. □

# 4 Exercises

In the following exercises, consider the category of sets **Set**

1. What class of functions is equal to $\{\emptyset \to \{*\}\}^{\pitchfork}$?

2. What class of functions is equal to ${}^{\pitchfork}\{\{a, b\} \to \{*\}\}$?

   We have that a function $f : X \to Y$ *has a section* when there is a function $g : Y \to X$ such that $f \circ g = \mathbb{1}_Y$

3. Let $\mathcal{L}$ be all monomorphisms and $\mathcal{R}$ be all epimorphisms. Show that $(\mathcal{L}, \mathcal{R})$ is a weak factorisation system for **Set** iff the Axiom of Choice holds (*Hint:* AC is equivalent to some function having a section).

# Bibliography

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[2] Nicola Gambino and Richard Garner. "The identity type weak factorisation system". In: *Theoretical Computer Science* 409.1 (2008), pp. 94–109.

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