We will discuss two results. The first result is of Awodey and Warren, the second result is from Gambino and Garner.

1. When $C$ is a finitely complete category with a weak factorization system, then $C$ is a model of a form of Martin-Löf type theory with identity types [1].

2. When $\mathbb{T}$ is a dependent type theory with the axioms for identity types, then its syntactic category $Syn(\mathbb{T})$ admits a non-trivial weak factorisation system [2].
1 Preliminaries

Before we get into this, we will define a weak factorisation system. Before that we need the following definition:

**Definition 1.1** (Left lifting property (LLP)). Let $C$ be a category. Given two maps $f : A \to B$ and $g : C \to D$ we say that $f$ has the left lifting property with respect to $g$, and $g$ has the right lifting property w.r.t. $f$, denoted by $f \Leftarrow g$, when for any commutative square as below:

\[
\begin{array}{ccc}
A & \xrightarrow{h} & C \\
\downarrow{f} & & \downarrow{g} \\
B & \xrightarrow{k} & D
\end{array}
\]

there exists a map $l : B \to C$, the diagonal filler, such that $g \circ l = k$ and $l \circ f = h$

Let $C$ be a category. For a collection of maps $\mathcal{M}$, we define $\Leftarrow \mathcal{M}$ to be the collection of maps in $C$ having the LLP with respect to all maps in $\mathcal{M}$. The collection $\mathcal{M}^\Leftarrow$ is defined similarly.

**Definition 1.2** (Weak factorization system). Let $C$ be a category. A weak factorisation system on $C$ consists of a pair of collections of maps $(\mathcal{L}, \mathcal{R})$, such that the following holds:

1. Every map $f$ in $C$ admits a factorization $f = p \circ i$ where $i \in \mathcal{L}$ and $p \in \mathcal{R}$
2. $\mathcal{R} = \mathcal{L}^\Leftarrow$ and $\mathcal{L} = \mathcal{R}^\Leftarrow$

We remind you of the following definition:

**Definition 1.3** (Display map). A display map is a morphism between contexts, defined by "projecting away" a variable: $[\Gamma, x : A] \to \Gamma$, where $\Gamma$ is a context and $A$ is a type relative to $\Gamma$.
The result of Awodey and Warren

A model in Martin-Löf type theory is extensional if the following reflection rule is satisfied:

\[ \vdash p : \text{Id}_A(a, b) \quad \vdash a = b : A \]

Type checking is decidable in the intensional theory, but not in extensional. That is the main reason why we should prefer intensional theories.

Lemma 2.1 ([1]). In the standard interpretation of type theory every locally cartesian closed category \( \mathcal{C} \) is extensional.

Definition 2.2 (Model category). A model category is a bicomplete category \( \mathcal{C} \) equipped with subcategories \( \mathcal{F} \) (fibrations), \( \mathcal{C} \) (cofibrations) and \( \mathcal{W} \) (weak equivalences) satisfying the following conditions:

1. ("Three-for-two") given a commutative triangle

\[
\begin{array}{ccc}
A & \xrightarrow{i} & C \\
\downarrow f & & \downarrow p \\
B & \xleftarrow{g} & \end{array}
\tag{\ast}
\]

if any two of \( f, g, h \) are weak equivalences, then so is the third.

2. both \( (\mathcal{C}, \mathcal{F} \cap \mathcal{W}) \) and \( (\mathcal{C} \cap \mathcal{W}, \mathcal{F}) \) are weak factorization systems.

A map \( f \) is an acyclic cofibration if it is in \( \mathcal{C} \cap \mathcal{W} \), i.e. both cofibration and a weak equivalence. Similarly, an acyclic fibration is a map in \( \mathcal{F} \cap \mathcal{W} \), i.e. which is simultaneously a fibration and a weak equivalence. An object \( A \) is said to be fibrant if the canonical map \( A \to 1 \) is a fibration. Similarly, \( A \) is cofibrant if \( 0 \to A \) is a cofibration.

In a model category \( \mathcal{C} \) a path object \( A^I \) for an object \( A \) consists of a factorization

\[
\begin{array}{ccc}
A & \xrightarrow{r} & A^I \\
\downarrow \Delta & & \downarrow p \\
A \times A & \xleftarrow{p} & \end{array}
\]

of the diagonal map \( \Delta : A \to A \times A \) as an acyclic cofibration \( r \) followed by a fibration \( p \).
Theorem 2.3. Let $C$ be a finitely complete category with a weak factorization system and a functorial choice $(−)^I$ of path objects in $C$, and all of its slices, which is stable under substitution, i.e. given any fibration $B \to A$ and any arrow $σ : A' \to A$, the evident comparison map is an isomorphism

$$σ^*(B^I) \cong (σ^*B)^I.$$ 

Proof. We may work in the empty context since the relevant structure is stable under slicing. Given a functorial choice of path objects $(*)$, we interpret, given a fibrant object $A$, the judgement $x, y : Id_A(x, y)$ as the path object fibration $p : A^I \to A \times A$. Because $p$ is a fibration, the formation is satisfied. Similarly, the introduction rule is valid because $r : A \to A^I$ is a section of $p$.

For the elimination and conversion rules, assume that the following premises are given

$$x : A, y : A, z : Id_A(x, y) \vdash D(x, y, z) \text{ type},$$

$$x : A \vdash d(x) : D(x, x, r_A(x)).$$

We have, therefore, a fibration $g : D \to A^I$ together with a map $d : A \to D$ such that $g \circ d = r$. This data yields the following commutative square:

$$\begin{array}{ccc}
A & \xrightarrow{d} & C \\
\downarrow{r} & & \downarrow{g} \\
A^I & \xrightarrow{1} & A^I
\end{array}$$

Because $g$ is a fibration and $r$ is, by definition an acyclic cofibration, there exists a diagonal filler

$$\begin{array}{ccc}
A & \xrightarrow{d} & C \\
\downarrow{r} & \xrightarrow{J} & \downarrow{g} \\
A^I & \xrightarrow{1} & A^I
\end{array}$$

Choose such a filler $J$ as the interpretation of the term:

$$x, y : A, z : Id_A(x, y) \vdash J_{A, D}(d, x, y, z) : D(x, y, z).$$

Then commutativity of the bottom triangle on the diagram above is precisely the conclusion of the elimination rule nad commutativity of the top triangle is the computation rule. □
3 The result of Gambino and Garner

Before we can prove the main theorem, we need to introduce a couple of definitions and lemma’s.

We remind you of the following definition:

**Definition 3.1** (Syntactic category). We have a category $\text{Syn}(\mathbb{T})$. Objects are the contexts of $\mathbb{T}$ and the morphisms are tuples of terms (context morphisms).

Let us consider a fixed context $\Gamma$.

**Definition 3.2** (Dependent context). Let $\Phi = [x_0 : A_0, x_1 : A_1(x_0), \ldots, x_n : A_n(x_0, \ldots, x_{n-1})]$. We say that $\Phi$ is a dependent context relative to $\Gamma$ when we can derive $\Gamma \vdash \Phi : Cxt$, where we mean the following sequence of judgements:

\[
\begin{align*}
\Gamma &\vdash A_0 : Type \\
\Gamma, x_0 : A_0 &\vdash A_1(x_0) : Type \\
&\vdots \\
\Gamma, x_0 : A_0, \ldots, x_{n-1} : A_{n-1}(x_0, \ldots, x_{n-1}) &\vdash A_n(x_0, \ldots, x_{n-1}) : Type
\end{align*}
\]

Let $a = (a_0, a_1, \ldots, a_n)$. With $\Gamma \vdash a : \Phi$ we mean:

\[
\begin{align*}
\Gamma &\vdash a_0 : A_0 \\
\Gamma &\vdash a_1 : A_1(a_0) \\
&\vdots \\
\Gamma &\vdash a_n : A_n(a_0, \ldots, a_{n-1})
\end{align*}
\]

We say that $a$ is a dependent element of $\Phi$ with respect to $\Gamma$.

When we have a dependent context $\Phi$, relative to $\Gamma$, we obtain a new context $[\Gamma, \Phi]$. We also obtain the following morphisms:

**Definition 3.3** (Dependent projections). A dependent projection is a map $[\Gamma, \Phi] \rightarrow \Gamma$, "projecting away" the variables in $\Phi$.

It is possible introduce expressions $\Gamma \vdash \Phi = \Psi : Cxt$ and $\Gamma \vdash a = b : \Phi$, such that these equalities satisfy reflexivity, symmetry and transitivity.

In addition to identity types we will introduce identity contexts:
Definition 3.4. For a context $\Phi$ and $a, b : \Phi$, we have an identity context $Id_\Phi(a, b)$.

We have the following deduction rules for identity contexts, where we leave implicit a context $\Gamma$, to which all notions are assumed to be relative:

- **Formation:**
  \[
  \frac{\vdash \Phi : Cxt}{a : \Phi, b : \Phi \vdash Id_\Phi(a, b) : Cxt}
  \]

- **Introduction:**
  \[
  \frac{\vdash \Phi : Cxt}{a : \Phi \vdash \text{refl}(a) : Id_\Phi(a, a)}
  \]

- **Elimination:**
  \[
  \frac{a : \Phi, b : \Phi, u : Id_\Phi(a, b), \Delta(a, b, u) \vdash C(a, b, u) : Cxt}{a : \Phi, \Delta(a, a, \text{refl}(a)) \vdash d(a) : C(a, a, \text{refl}(a))}
  \]
  \[
  \frac{a : \Phi, b : \Phi, u : Id_\Phi(a, b), \Delta(a, b, u) \vdash J(d, a, b, u) : C(a, b, u)}{a : \Phi, b : \Phi, u : Id_\Phi(a, b), \Delta(a, a, \text{refl}(a)) \vdash J(d, a, a, \text{refl}(a)) : d(a) : C(a, a, \text{refl}(a))}
  \]

Here $\Delta(a, b, u)$ is a dependent context.

We will need the following lemma's:

**Lemma 3.5 ([2]).** For every context $\Phi$, we can derive a rule of the form

\[
\frac{a : \Phi \vdash \Phi(a) : Cxt}{a : \Phi, b : \Phi, u : Id_\Phi(a, b), e : \Phi(a) \vdash u_* (e) : \Phi(b)}
\]

such that

\[
\frac{a : \Phi, e : \Phi(a)}{(\text{refl}(a))_* (e) = e : \Phi(a)}
\]

holds

**Lemma 3.6 ([2]).** We can derive rules of the form

\[
\frac{u : Id_\Phi(a, b), v : Id_\Phi(b, c)}{v \circ u : Id_\Phi(a, c)}
\]

such that

\[
\frac{a : \Phi}{\mathbb{1}_a : Id_\Phi(a, a)}
\]

such that

\[
\frac{u : Id_\Phi(a, b)}{\mathbb{1}_b \circ u = u : Id_\Phi(a, b)}
\]

holds
Lemma 3.7 ([2]). We can derive a rule
\[
\frac{u : \text{Id}_\Phi(a,b)}{\psi_u : \text{Id}_\Phi(u \circ \mathbb{1}_a, u)}
\]
such that
\[
\frac{a : \Phi}{\psi_{\mathbb{1}_a} = \mathbb{1} : \text{Id}_\Phi(\mathbb{1}_a, \mathbb{1}_a)}
\]
holds

Lemma 3.8 (Retract argument, [3]). Suppose \( f = p \circ i \) and \( f \) has the RLP with respect to \( i \). Then \( f \) is a retract of \( p \).

We are now ready to prove the main theorem.

Theorem 3.9. Let \( \mathbb{T} \) be a dependent type theory with axioms for identity types. Let \( D \) be the set of display maps in \( \text{Syn}(\mathbb{T}) \). The pair \((L, R)\), where \( L := \mathbb{r} D \) and \( R := L^\mathbb{r} \), forms a weak factorisation system on \( \text{Syn}(\mathbb{T}) \).

We will show the theorem by proving the following two lemma’s:

Lemma 3.10. Every map \( f \) admits a factorisation \( f = p \circ i \), where \( i \in L \) and \( p \) is a dependent projection.

Lemma 3.11. \( L = \mathbb{r} R \)

Proof of Theorem 3.9. Note that a display map is a dependent projection. Also note that \( D \subseteq R \). We have that \( R \) is closed under composition, and we can create all dependent projections from compositions of display maps, so \( R \) contains all dependent projections. Then Lemma 3.10 gives us axiom 1 in Definition 1.2. Then by definition of \((L, R)\) and Lemma 3.11 we get axiom 2 in Definition 1.2, which proves the theorem.

We will now continue to prove the lemma’s that we used.

Proof of Lemma 3.10. Let \( f : \Phi \to \Psi \) be a context morphism. Define \( \text{Id}(f) := [x : \Phi, y : \Psi, u : \text{Id}_\Phi(f(x), y)] \). We will now show that \( f = p_f \circ i_f \), where \( p_f := [y] \) and \( i_f := [x, f(x), 1_f(x)] \). The factorization is displayed in the following picture:

\[
\Phi \xrightarrow{i_f} \text{Id}(f) \xrightarrow{p_f} \Psi
\]

It is clear that \( p_f \) is a dependent projection. So we only need to show that \( i_f \in A \), which means that \( i_f \) has the LLP with respect to all display maps.
We thus want to show that the commuting diagram above, where $d$ is some display map, has a diagonal filler, $df_1$. Display maps are closed under pullbacks (we proved this in one of the lectures).

This means that we also have a commuting diagram as below:

$$
\begin{array}{c}
\Phi \xrightarrow{g} [v : \Delta, z : D(v)] \\
\downarrow_{ij} \quad \downarrow_{d} \\
Id(f) \xrightarrow{h} [v \in \Delta]
\end{array}
$$

And a unique morphism $e : \Phi \to X$, such that $d \circ e = i_f$ and $j \circ e = g$. Moreover, $\bar{d}$ is also a pullback and so $X$ can be written as $[Id(f), z : C(x, y, u)]$ where $C(x, y, u)$ is a dependent type relative to $Id(f)$.

So if we can find a diagonal filler $df_2$ for this diagram:

$$
\begin{array}{c}
\Psi \xrightarrow{e} [Id(f), z : C(x, y, u)] \\
\downarrow_{ij} \quad \downarrow_{d} \\
Id(f) \xrightarrow{1_{Id(f)}} \xrightarrow{id} Id(f)
\end{array}
$$

Then by concatenation of $df_2$ with $j$, we get a diagonal filler for the first diagram.

The rest of the proof will be dedicated to finding $df_2$.

We can derive

$$
x : \Phi, y_0 : \Psi, y_1 : \Psi, v : Id_{\Psi}(y_0, y_1), u : Id_{\Psi}(f(x), y), z : C(x, y_0, u) \vdash C(x, y_1, v \circ u) : Type
$$

(3.1)

since we can form $v \circ u : Id_{\Psi}(f(x), y_1)$ with Lemma 3.6 and thus a context $Id(f) = [x : \Phi, y_1 : \Psi, v \circ u : Id_{\Psi}(f(x), y_1)]$, so we can obtain the type $C(x, y_1, v \circ u)$ from the display map $\bar{d}$.

We can also derive

$$
x : \Phi, y : \Psi, u : Id_{\Psi}(f(x), y), z : C(x, y, u) \vdash z : C(x, y, 1_y \circ y)
$$

(3.2)

by the morphism $e$ and again using Lemma 3.6.
Then, by the elimination rule for identity contexts, we obtain from 3.1 and 3.2

\[ x : \Phi, y_0 : \Psi, y_1 : \Psi, v : Id_{\Psi}(y_0, y_1), u : Id_{\Psi}(f(x), y), z : C(x, y_0, u) \vdash J(z, y_0, y_1, v) : C(x, y_1, v \circ u) \]  

(3.3)

From 3.3 we can then obtain

\[ x : \Phi, y : \Psi, u : Id_{\Psi}(f(x), y), z : C(x, f(x), u) \vdash J(z, f(x), y, u) : C(x, y, u \circ u_{f(x)}) \]  

(3.4)

Since here \( z \) only depends on \( x \), we can substitute it for \( d(x) \) to get

\[ x : \Phi, y : \Psi, u : Id_{\Psi}(f(x), y) \vdash J(d(x), f(x), y, u) : C(x, y, u \circ u_{f(x)}) \]  

(3.5)

Since we have \( u : Id_{\Psi}(f(x), y) \), by Lemma 3.7 we also have \( \psi_u : Id(u \circ u_{f(x)}, u) \).

By this and by Lemma 3.5 we obtain

\[ x : \Phi, y : \Psi, u : Id_{\Psi}(f(x), y) \vdash (\psi_u)_*(J(d(x), f(x), y, u) : C(x, y, u)) \]  

(3.6)

We now claim that the required filler, \( df_2 \) can be defined as \( [x, y, u, (\psi_u)_*(J(f(x), y, u, d(x)))] \).

That the bottom triangle commutes is obvious. The commutativity of the top triangle follows from the following equalities:

\[ (\psi_{u_{f(x)}})_*(J(d(x), f(x), y, u)) = J(d(x), f(x), f(x), u_{f(x)}) = d(x) \]

\[ \square \]

Proof of Lemma 3.11. Since \( L =^h D \) and \( R = L^h \), we have that \( D \subseteq R \). This implies that \( ^h B \subseteq^h D = L \). We still need to show that \( L \subseteq^h R \), that every map in \( L \) has the LLP with respect to every map in \( R \). We have that \( L =^h D \), so every map in \( L \) has the LLP with respect to every display map. But dependent projections are composites of display maps, so also every map in \( L \) has the LLP with respect to every dependent projection.

Lemma 3.8 and Lemma 3.10 tell us that every map in \( R \) is a retract of a dependent projection. From this we can conclude that \( L \subseteq^h R \).  

\[ \square \]
4 Exercises

In the following exercises, consider the category of sets $\text{Set}$

1. What class of functions is equal to $\{\emptyset \to \{\ast\}\}$?

2. What class of functions is equal to $\Rightarrow\{\{a,b\} \to \{\ast\}\}$?

   We have that a function $f : X \to Y$ has a section when there is a function $g : Y \to X$ such that $f \circ g = 1_X$.

3. Let $\mathcal{L}$ be all monomorphisms and $\mathcal{R}$ be all epimorphisms. Show that $(\mathcal{L}, \mathcal{R})$ is a weak factorisation system for $\text{Set}$ iff the Axiom of Choice holds ($\text{Hint: AC is equivalent to some function having a section}$).
Bibliography

