Homotopy Type Theory presentation 2: Equivalences

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2017-10-6

Today we will discuss equivalences of types. We have seen some aspects of the rich theory of equality of terms. As it turns out, there is a similar richness in equalities of types, which we call *equivalences*. These equivalences will be the isomorphisms of types.

1 Defining quasi-inverses

In set theory, we declare that the isomorphisms between A and B are exactly the functions $f : A \to B$ such that f is a *bijection*, so f has an inverse $g : B \to A$ with $g(f(x)) = x$ and $f(g(y)) = y$ for all $x \in A$, $y \in B$. In type theory, we can make a similar definition of equivalence of types, but first we need to define an equivalence of functions.

Definition 1. Given types A, B and functions $f, g : A \to B$, we say f and g are *homotopic* (written $f \sim g$) if they are equal at each point: for each $x : A$ the type $\mathrm{Id}(f(x), g(x))$ is inhabited.

Formally, we define a type family $(\sim)$ as follows:

$$(\sim) : \prod_{A\,B:\mathcal{U}} \prod_{f\,g:A\to B} \mathcal{U}$$
$$f \sim g = \prod_{x:A} \mathrm{Id}(f(x), g(x)).$$

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The name is no accident: if we use the topological interpretation of the identity type, then functions that are extensionally equal are in fact homotopic according to the topological definition.

Definition 2. Given types A, B and a function $f : A \to B$, we say f has a *quasi-inverse* if there exists a $g : B \to A$ such that the compositions $g \circ f$ and $f \circ g$ are homotopic to the identity functions $\mathbb{1}_A : A \to A$ and $\mathbb{1}_B : B \to B$ respectively.

Formally, we define a type family qinv as follows:

$$\mathrm{qinv} : \prod_{A:\mathcal{U}} \prod_{B:\mathcal{U}} \prod_{f:A\to B} \mathcal{U}$$
$$\mathrm{qinv}(f) = \sum_{g:B\to A} \left((g \circ f \sim \mathbb{1}_A) \times (f \circ g \sim \mathbb{1}_B) \right).$$

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Topologists will recognise this as stating that spaces A and B are homotopy equivalent, the most popular notion of isomorphism used in algebraic topology.

For each type X we have a very useful function with quasi-inverse: the identity function $\mathbb{1}$, with quasi-inverse $\mathbb{1}$ itself. Homotopies $\mathbb{1} \circ \mathbb{1} \sim \mathbb{1}$ are given by taking the path refl_x for each $x \in X$.

2 Multiple quasi-inverses

The main issue with this definition of quasi-inverses, and the reason they are not called true inverses, consists of the fact that $\mathrm{qinv}(f)$ is not necessarily a proposition. The argument relies on the following lemma:

Lemma 3. *Let A be a type. The type of quasi-inverses of $\mathbb{1}_A$ is equivalent to $\prod_{x:A} \mathrm{Id}(x,x)$.*

Proof. The type $\mathrm{qinv}(\mathbb{1}_A)$ is defined as:

$$\mathrm{qinv}(\mathbb{1}_A) = \sum_{g:A\to A} \left((g \circ \mathbb{1}_A \sim \mathbb{1}_A) \times (\mathbb{1}_A \circ g \sim \mathbb{1}_A)\right).$$

Since the identity function is the identity for composition, we get:

$$\begin{aligned}\mathrm{qinv}(\mathbb{1}_A) &= \sum_{g:A\to A} \left((g \sim \mathbb{1}_A) \times (g \sim \mathbb{1}_A)\right) \\ &= \sum_{g:A\to A} \left(\prod_{x:A} \mathrm{Id}(g(x), \mathbb{1}_A(x)) \times \prod_{x:A} \mathrm{Id}(g(x), \mathbb{1}_A(x))\right).\end{aligned}$$

Extensionality of functions gives that $\mathrm{Id}(g(x), \mathbb{1}_A(x))$ is equivalent to $\mathrm{Id}(g, \mathbb{1}_A)$:

$$\mathrm{qinv}(\mathbb{1}_A) \simeq \sum_{g:A\to A} \left(\mathrm{Id}(g, \mathbb{1}_A) \times \prod_{x:A} \mathrm{Id}(g(x), \mathbb{1}_A(x))\right).$$

By the characterization of paths in a sum type, we get that the type of paths between $(g,p) : \sum_{g:A\to A} \mathrm{Id}(g, \mathbb{1}_A)$ and $(\mathbb{1}_A, \mathrm{refl})$ is equivalent to $\sum_{q:\mathrm{Id}(\mathbb{1},g)} \mathrm{Id}(q_*(\mathrm{refl}), p)$. Taking q to be equal to p, we get for each (g,p) a path of type $\mathrm{Id}((\mathbb{1}, \mathrm{refl}_{\mathbb{1}}), (g,p))$, so $\sum_{g:A\to A} \mathrm{Id}(g, \mathbb{1}_A)$ is contractible. Performing this contraction, we get $\mathrm{qinv}(\mathbb{1}_A) \simeq \prod_{x:A} \mathrm{Id}(\mathbb{1}_A(x), \mathbb{1}_A(x)) = \prod_{x:A} \mathrm{Id}(x,x)$. □

Lemma 4.1.1 of [1] is an extended version of this lemma that also works for arbitrary $f : A \to B$ with a quasi-inverse. For the argument, we only need the case we just proved.

The next lemma is quite technical but we will just use it once to construct a function that we need.

Lemma 4. *Let A be a type, $a : A$ and $q : \mathrm{Id}(a,a)$. If A is contractible with center a, and $\mathrm{Id}(a,a)$ is a set such that for all $p : \mathrm{Id}(a,a)$ we have $\mathrm{Id}(p \bullet q, q \bullet p)$, then there exists a function $f : \prod_{x:A} \mathrm{Id}(x,x)$ such that $f(a) = q$.*

Proof. See [1, Lemma 4.1.2]. □

Lemma 5. *There is a type X such that $\mathrm{qinv}(\mathbb{1}_X)$ is not a proposition.*

Proof. Let $\mathbf{2}$ be the type with two unique elements and $X = \sum_{A:\mathcal{U}} \|\mathrm{Id}(A, \mathbf{2})\|$. Note that for all types X we have that $\mathrm{qinv}(\mathbb{1}_X)$ is inhabited by $(\mathbb{1}_X, \mathrm{refl}, \mathrm{refl})$, so Lemma 3 gives that it suffices to find two distinct inhabitants of $\prod_{x:X} \mathrm{Id}(x,x)$. Certainly refl is an inhabitant, so we will construct an $f : \prod_{x:X} \mathrm{Id}(x,x)$ with $f \neq \mathrm{refl}$.

Let $a : X$ be given by $a = |(\mathbf{2}, \mathrm{refl}_{\mathbf{2}})|$. Since the second component, $\|\mathrm{Id}(A, \mathbf{2})\|$, is a proposition by definition, elements of $\sum_{A:\mathcal{U}} \|\mathrm{Id}(A, \mathbf{2})\|$ are equal if and only if the projection onto the first component is equal, so in other words $\mathrm{Id}(a,a) \simeq \mathrm{Id}(\mathbf{2}, \mathbf{2})$. By univalence, we get $\mathrm{Id}(a,a) \simeq (\mathbf{2} \simeq \mathbf{2})$. Note that $\mathbf{2}$ has two distinct quasi-invertible functions, $\mathbb{1}_{\mathbf{2}}$ and not, where $\mathsf{not}\ 0 = 1$ and $\mathsf{not}\ 1 = 0$. Let $q : \mathrm{Id}(a,a)$ be the path equivalent to not, so we have $q \neq \mathrm{refl}_a$.

Now we are ready to apply Lemma 4, which states that it suffices to find a path $q \neq \mathrm{refl} : \mathrm{Id}(a,a)$ if $\mathrm{Id}(a,a)$ is a set, for all $x : X$ we have $\|\mathrm{Id}(x,a)\|$ and all paths $p : \mathrm{Id}(a,a)$ commute with q: $p \cdot q = q \cdot p$. As we noted before, $\mathrm{Id}(a,a) \simeq (\mathbf{2} \simeq \mathbf{2})$, and $\mathbf{2} \simeq \mathbf{2}$ is a discrete set consisting of $\mathbb{1}_{\mathbf{2}}$ and not, so the first condition is satisfied. The second condition is satisfied since an element $x = (A,p) : X$ can be projected to $p : \mathrm{Id}(A, \mathbf{2})$, which is exactly the path between the first components of x and a that we need. The third condition is satisfied as the only path $p : \mathrm{Id}(a,a)$ distinct from q is refl_a, and $\mathrm{refl}_a \cdot q = q = q \cdot \mathrm{refl}_a$. □

3 Fibres and half adjoint equivalences

To get a good definition of equivalence, we will build a type that is logically equivalent to qinv, but each instance is a proposition. The easiest way to make the quasi-inverse type better behaved is to add an extra homotopy to the type of equivalences. Adding this homotopy will allow us to create all the paths in the type that we need to make it into a proposition.

Definition 6. Given types A, B and a function $f : A \to B$, we say f is a *half-adjoint equivalence* if there exists a $g : B \to A$ such that there are homotopies $\eta : g \circ f \sim \mathbb{1}_A$ and $\epsilon : f \circ g \sim \mathbb{1}_B$ and a homotopy of homotopies $\tau : f \circ \eta \sim \epsilon \circ f$.

Formally, we define a type family ishae as follows:

$$\begin{aligned} \text{ishae} &: \prod_{A:\mathcal{U}} \prod_{B:\mathcal{U}} \prod_{f:A\to B} \mathcal{U} \\ \text{ishae}(f) &= \sum_{g:B\to A} \sum_{\eta:g\circ f\sim\mathbb{1}_A} \sum_{\epsilon:f\circ g\sim\mathbb{1}_B} (f\circ\eta \sim \epsilon\circ f). \end{aligned}$$

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It turns out that this condition on f is logically equivalent to the analogous condition on g, that $g\circ\epsilon \sim \eta\circ g$, as is proved in [1, Lemma 4.2.2].

Theorem 7. *For all types A, B and $f : A \to B$, the types* $\text{qinv}(f)$ *and* $\text{ishae}(f)$ *are logically equivalent.*

Proof. Going from half adjoint equivalences to quasi-inverses is simple: simply drop out the homotopy τ. The converse, building a half adjoint equivalence from a quasi-inverse, is somewhat more complicated.

Suppose $g' : B \to A$ is a quasi-inverse to f with homotopies $\eta' : g \circ f \sim \mathbb{1}_A$ and $\epsilon' : f \circ g \sim \mathbb{1}_B$. To construct an element $(g, \eta, \epsilon, \tau) : \text{ishae}(f)$, we can simply copy $g = g'$ and $\eta = \eta'$. If we let $\epsilon = \epsilon'$ we get exactly the problem we started out with, so we need a better choice. It turns out defining $\epsilon(b) = \epsilon'(f(g(b)))^{-1} \cdot f(\eta(g(b))) \cdot \epsilon'(b)$ works. Finally, we need to find a τ that proves that $f(\eta'(a)) = (\epsilon'(f(g(f(a)))))^{-1} \cdot f(\eta'(g(f(a)))) \cdot \epsilon'(f(a))$. Since applying homotopies commutes with applying functions, we get $f(\eta'(g(f(a)))) = f(g(f(\eta'(a))))$. Now ϵ' is a homotopy from $f \circ g$ to $\mathbb{1}_B$, and $f(\eta(a))$ is a path from $f(g(f(a)))$ to $f(a)$, which gives us $f(g(f(\eta'(a)))) \cdot \epsilon'(f(a)) = \epsilon'(f(g(f(\eta'(a))))) \cdot f(\eta'(a))$. Combining this, we get $(\epsilon'(f(g(f(a)))))^{-1} \cdot f(\eta'(g(f(a)))) \cdot \epsilon'(f(a)) = (\epsilon'(f(g(f(a)))))^{-1} \cdot \epsilon'(f(g(f(\eta'(a))))) \cdot f(\eta'(a)) = f(\eta'(a))$, which proves τ. □

The proof that $\text{ishae}(f)$ is a proposition requires some more work. We first introduce the type-theoretical definition of fibres of a function, which should remind you of the inverse image of a function in set-theoretical situations.

Definition 8. Let A, B be types, $f : A \to B$ and $y : B$. The *fibre of f over y* consists of all points in x that f maps to y:

$$\text{fib}_f(y) = \sum_{x:A} \text{Id}(f(x), y).$$

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In set-theoretical situations, having at least one inhabitant per fibre means f is surjective, as each $y \in B$ has an $x \in A$ with $f(x) = y$. Having at most one inhabitant per fibre means f is injective, as $f(x) = f(x')$ implies x and x' are in the same fibre, therefore $x = x'$. Therefore, we will show that half-adjoint equivalences have exactly one inhabitant per fibre, up to homotopy.

Lemma 9. *For all half-adjoint equivalences $f : A \to B$ and $y : B$, the type* $\text{fib}_f(y)$ *is contractible.*

Proof. Let $(g, \epsilon, \eta, \tau)$: ishae(f). Since $\mathrm{fib}_f(y)$ is defined as $\sum_{x:A} \mathrm{Id}(f(x), y)$, let $x : A$ and $p : \mathrm{Id}(f(x), y)$. We will show a path (γ, δ) exists from (x, p) to $(g(y), \epsilon(y))$. Note that $g \circ p^{-1} : \mathrm{Id}(g(y), g(f(x)))$ and $\eta(x) : \mathrm{Id}(g(f(x)), x)$, so we can define for the first component $\gamma : \mathrm{Id}(g(y), x)$ as $(g \circ p^{-1}) \cdot \eta(x)$.

Filling this in for the second component, we need to show $\delta : \mathrm{Id}((f \circ \gamma) \cdot p, \epsilon(y))$. Substituting our definition of γ we get that the type is equal to $\mathrm{Id}((f \circ ((g \circ p^{-1}) \cdot \eta(x))) \cdot p, \epsilon(y)) = \mathrm{Id}((f \circ g \circ p^{-1}) \cdot (f \circ \eta(x)) \cdot p, \epsilon(y))$. Since $\tau : f \circ \eta \sim \epsilon \circ f$, we get a homotopy of type $\mathrm{Id}((f \circ g \circ p^{-1}) \cdot (f \circ \eta(x)) \cdot p, (f \circ g \circ p^{-1}) \cdot (\epsilon(f(x))) \cdot p)$. Naturality of the homotopy ϵ gives another homotopy of type $\mathrm{Id}((f \circ g \circ p^{-1}) \cdot \epsilon(f(x))) \cdot p, \epsilon(y) \cdot p^{-1} \cdot p) = \mathrm{Id}((f \circ g \circ p^{-1}) \cdot \epsilon(f(x)) \cdot p, \epsilon(y))$. Composing these homotopies gives $(\gamma, \delta) : \mathrm{Id}((x, p), (g(y), \epsilon(y)))$. Since $(g(y), \epsilon(y))$ is independent of (x, p), the whole type $\mathrm{fib}_f(y)$ is contractible. □

Theorem 10. *For all types A, B and $f : A \to B$, the type* ishae(f) *is a proposition.*

Proof. It suffices to show this in the case that ishae(f) is inhabited, so let $(g, \epsilon, \eta, \tau)$: ishae(f). We defined

$$\mathrm{ishae}(f) = \sum_{h:B \to A} \sum_{\alpha: h \circ f \sim \mathbb{1}_A} \sum_{\beta: f \circ h \sim \mathbb{1}_B} (f \circ \alpha \sim \beta \circ f),$$

where we can move the last sum into the first:

$$\mathrm{ishae}(f) = \sum_{(h,\beta):\sum_{h:B \to A} f \circ h \sim \mathbb{1}_A} \sum_{\alpha: h \circ f \sim \mathbb{1}_B} (f \circ \alpha \sim \beta \circ f).$$

Splitting this up into two components, it suffices to give a contraction for the second component of type $\sum_{\alpha: h \circ f \sim \mathbb{1}_B} (f \circ \alpha \sim \beta \circ f)$, keeping h and α fixed, and then contracting the first component of type $\sum_{h:B \to A} f \circ h \sim \mathbb{1}_A$.

By definition, the type of the second component is equal to $\sum_{\alpha: h \circ f \sim \mathbb{1}_B} \prod_{x:A} \mathrm{Id}(f(\alpha(x)), \beta(f(x)))$. Since the product is independent of the sum, it suffices to show for each $x : A$ that $\sum_{\alpha: h \circ f \sim \mathbb{1}_B} \mathrm{Id}(f(\alpha(x)), \beta(f(x)))$ is contractible. Since each αx is simply a path from $h(f(x))$ to x and composition of paths with $\alpha^{-1} x$ is an equivalence, this type is equivalent to $\mathrm{Id}((h(f(x)), \beta(f(x))), (x, \mathrm{refl}_{f(x)}))$, where $(h(f(x)), \beta f(x)) : \sum_{x:A} \mathrm{Id}(f(x), f(x))$. Lemma 9 gives that $\sum_{x:A} \mathrm{Id}(f(x), f(x))$ is contractible, since it is the fibre of f over $f(x)$. All paths in a contractible type are themselves contractible, so $\mathrm{Id}((h(f(x)), \beta f(x)), (x, \mathrm{refl}_{f(x)}))$ is contractible. This is equivalent to $\sum_{\alpha: h \circ f \sim \mathbb{1}_B} \prod_{x:A} \mathrm{Id}(f(\alpha(x)), \beta(f(x)))$ which is therefore contractible too.

Finally, note that $\sum_{h:B \to A} f \circ h \sim \mathbb{1}_A$ is the fibre of $(f \circ)$ over $\mathbb{1}_A$, where $(f \circ)$ denotes precomposition with f. Since $\eta : g \circ f \sim \mathbb{1}_A$ implies $\eta'(h) : g \circ f \circ h \sim \mathbb{1}_A \circ h$, by function extensionality we have $\eta' : \prod_{h:C \to A} \mathrm{Id}(g \circ f \circ h, h)$ and similarly $\epsilon' : \prod_{h:C \to B} \mathrm{Id}(f \circ g \circ h, h)$. Therefore, $(g \circ, \eta', \epsilon') : \mathrm{qinv}(f \circ)$, and by Theorem 7 this is equivalent to ishae$(f \circ)$. Fibres of half-adjoint equivalences are contractible, so $\sum_{h:B \to A} f \circ h \sim \mathbb{1}_A$ is contractible.

Since ishae(f) is equivalent to the sum of a contractible type over a contractible type, it is itself contractible. Since it is contractible if it is inhabited, it is a proposition. □

4 Exercise

Prove for all types A and B and functions $f : A \to B$, if f has a quasi-inverse, then the type $\mathrm{qinv}\, f$ is equivalent to $\prod_{x:A} \mathrm{Id}(x, x)$. (If you know the univalence axiom, do not use it.)

References

[1] Univalent Foundations Program, *Homotopy Type Theory*, version 1075-g3c53219.