1st Exercise sheet Model Theory
7 Feb 2017

Exercise 1 A theory T is *consistent* if it has a model and *complete* if it is consistent and for any formula φ we have

$$T \models \varphi \quad \text{or} \quad T \models \neg\varphi.$$

Show that the following are equivalent for a consistent theory T:

(1) T is complete.

(2) All models of T are elementarily equivalent.

(3) There is a structure M such that T and $\mathrm{Th}(M)$ have the same models.

Exercise 2 Let M_1, M_2 be L-structures. Define an L-structure $M_1 \times M_2$ whose underlying set is the cartesian product of the underlying sets of M_1 and M_2 and such that the projections $\pi_i \colon M_1 \times M_2 \to M_i$ for $i = 1, 2$ are homomorphisms satisfying the following universal property: given an L-structure N and homomorphisms $\varphi_i \colon N \to M_i$ for $i = 1, 2$, there is a unique homomorphism $\psi \colon N \to M_1 \times M_2$ such that $\pi_i \circ \psi = \varphi_i$ for $i = 1, 2$.

Exercise 3 An *isomorphism* is a bijective homomorphism $h \colon M \to N$ of L-structures whose inverse h^{-1} is a homomorphism as well.

Show that if $h \colon M \to N$ is an isomorphism and $\varphi(x_1, \ldots, x_k)$ is any L-formula, then

$$M \models \varphi(m_1, \ldots, m_k) \Leftrightarrow N \models \varphi(h(m_1), \ldots, h(m_k))$$

for any $m_1, \ldots, m_k \in M$.

Exercise 4 An element a in an L-structure M is *definable* if there is an L-formula $\varphi(x)$ such that for any $m \in M$

$$M \models \varphi(m) \Leftrightarrow a = m.$$

(a) What are the definable elements in $(\mathbb{N}, +)$? And in $(\mathbb{Z}, +)$?

(b) (Challenging!) Define an equivalence relation on the integers by putting $a \sim b$ if for any formula $\varphi(x)$ with one free variable x in the language of $(\mathbb{Z}, \cdot)$, we have

$$(\mathbb{Z}, \cdot) \models \varphi(a) \Leftrightarrow (\mathbb{Z}, \cdot) \models \varphi(b).$$

Describe an algorithm for determining whether two integers are equivalent in this sense.