10th Exercise sheet Model Theory
10 Mar 2017

Exercise 1 Suppose T is some nice L-theory. For any L-structure A the group $\mathrm{Aut}(A)$ of automorphisms of A acts on the set A^n as follows:

$$f \cdot (a_1, \ldots, a_n) := (f(a_1), \ldots, f(a_n)),$$

where $f \in \mathrm{Aut}(A)$ and $(a_1, \ldots, a_n) \in A^n$. The *orbit* of $(a_1, \ldots, a_n)$ is the set

$$\{ (f(a_1), \ldots, f(a_n)) : f \in \mathrm{Aut}(A) \}.$$

Show that the following are equivalent for T:

(i) T is ω-categorical.

(ii) If A is a countable model of T and $n \in \mathbb{N}$, the collection of orbits under the action of $\mathrm{Aut}(A)$ on A^n is finite.

Exercise 2 A theory T has *quantifier elimination* if for any formula $\varphi(\overline{x})$ there is a quantifier-free formula $\psi(\overline{x})$ such that

$$T \models \varphi(\overline{x}) \leftrightarrow \psi(\overline{x}).$$

(a) Show that T has quantifier elimination if and only if every type over T is implied by its quantifier-free part.

(b) Suppose L is a finite language with no function symbols and T is a nice L-theory with quantifier elimination. Prove that T is ω-categorical.

(c) Suppose T is a some theory and each $p \in S_n(T)$ contains a complete formula which is also quantifier-free. Deduce that T has quantifier elimination.

(d) Use (c) to show that $T = DLO$ and $T = RG$ have quantifier elimination.

Exercise 3 (Hard!) Give an example of a complete theory T in an uncountable language which has exactly one countable model but for which not all $S_n(T)$ are finite.