12th Exercise sheet Model Theory
17 Mar 2017

Exercise 1 Throughout this exercise I will be some non-empty set. A collection $\mathcal{F}$ of subsets of I is called a *filter* (on I) if:

1. $I \in \mathcal{F}, \emptyset \notin \mathcal{F}$;
2. whenever $A, B \in \mathcal{F}$, then also $A \cap B \in \mathcal{F}$;
3. whenever $A \in \mathcal{F}$ and $A \subseteq B$, then also $B \in \mathcal{F}$.

A filter which is maximal in the inclusion ordering is called an *ultrafilter*. By Zorn's Lemma every filter can be extended to an ultrafilter. An ultrafilter on a set I is called *principal* if there is an element $i \in I$ such that $X \in \mathcal{U}$ iff $i \in X$.

In addition, we will assume that we have some collection $\{M_i : i \in I\}$ of L-structures and a filter $\mathcal{F}$ on I. This allows us to construct a new L-structure M, as follows. Its universe is

$$\prod_{i \in I} M_i = \{f : I \to \bigcup_i M_i : (\forall i \in I)\, f(i) \in M_i\},$$

quotiented by the following equivalence relation:

$$f \sim g \quad :\Leftrightarrow \quad \{i \in I : f(i) = g(i)\} \in \mathcal{F}.$$

In addition, if g is an n-ary function symbol belonging to L and $[f_1], \ldots, [f_n] \in M$, then

$$g^M([f_1], \ldots, [f_n]) = [i \mapsto g^{M_i}(f_1(i), \ldots, f_n(i))],$$

and if R is an n-ary relation symbol belonging to L and $[f_1], \ldots, [f_n] \in M$, then

$$([f_1], \ldots, [f_n]) \in R^M \quad :\Leftrightarrow \quad \{\, i \in I : (f_1(i), \ldots, f_n(i)) \in R^{M_i}\,\} \in \mathcal{F},$$

where one should check that everything is well-defined. The resulting structure M is often denoted by $\prod M_i/\mathcal{F}$. We will be most interested in the special case where $\mathcal{F}$ is an ultrafilter, in which case $\prod M_i/\mathcal{F}$ is called an *ultraproduct*.

(a) Show that a filter $\mathcal{U}$ is an ultrafilter iff for any $X \subseteq I$ either $X \in \mathcal{U}$ or $I \setminus X \in \mathcal{U}$.

(b) Show that every ultrafilter on a finite set I is principal, while there are non-principal ultrafilters on every infinite set I.

(c) (Łoś's Theorem) Show that if $\mathcal{U}$ is an ultrafilter on I, then for any formula $\varphi(x_1, \dots, x_n)$ and $[f_1], \dots, [f_n] \in \prod M_i/\mathcal{U}$ we have that

$$\prod M_i/\mathcal{U} \models \varphi([f_1], \dots, [f_n]) \quad \Leftrightarrow \quad \{i \in I \colon M_i \models \varphi(f_1(i), \dots, f_n(i))\} \in \mathcal{U}.$$

Exercise 2 The idea of this exercise is to use ultraproducts to give an alternative proof of the compactness theorem. So let T be a theory and assume that every finite subset Δ of T has a model M_Δ. Since our task is to prove that T has a model, we may just as well assume that T is infinite. We will write $I = \{\Delta \subseteq T \colon \Delta \text{ finite}\}$.

(a) For any $\Delta \in I$, write $\uparrow \Delta = \{\Delta' \in I \colon \Delta \subseteq \Delta'\}$. Let

$$\mathcal{F} = \{\, Y \subseteq I \colon (\exists \Delta \in I) \uparrow \Delta \subseteq Y \,\}.$$

Show that $\mathcal{F}$ is a filter on I.

(b) Let $\mathcal{U}$ be an ultrafilter on I with $\mathcal{U} \supseteq \mathcal{F}$. Show that $\prod M_\Delta/\mathcal{U} \models T$.

Exercise 3 Suppose $\mathcal{U}$ is an non-principal ultrafilter on $\mathbb{N}$. Let $(M_i)_{i\in\mathbb{N}}$ be a sequence of L-structures, and write ${}^*M = \prod M_i/\mathcal{U}$.

Let $A \subseteq {}^*M$ be arbitrary, and choose for each $a \in A$ an $f_a \in \prod M_i$ such that $a = [f_a]$. Let $p(x) = \{\varphi_i(x) \colon i < \omega\}$ be a set of L_A-formulas such that $p(x)$ is finitely satisfiable in ${}^*M$. By taking conjunctions, we may, withour loss of generality, assume that $\varphi_{i+1}(x) \to \varphi_i(x)$ for $i < \omega$. Let $\varphi_i(x)$ be $\theta_i(x, a_{i,1}, \dots, a_{i,m_i})$, where θ_i is an L-formula.

(a) Let

$$D_i = \{\, n \in \mathbb{N} \colon M_n \models \exists x\, \theta_i(x, f_{a_{i,1}}(n), \dots, f_{a_{i,m_i}}(n)) \,\}.$$

Show that $D_i \in \mathcal{U}$.

(b) Find $g \in \prod M_i$ such that if $i \leq n$ and $n \in D_i$, then

$$M_n \models \theta_i(g(n), f_{a_{i,1}}(n), \dots, f_{a_{i,m_i}}(n)).$$

(c) Show that $[g]$ realizes $p(x)$. Where do you use the fact that $\mathcal{U}$ is non-principal?

(d) Assume that L is countable. Conclude that ${}^*M$ is $\aleph_1$-saturated.

(e) Show that if the Continuum Hypothesis holds then every nice theory has a saturated model with size $\aleph_1$.