3rd Exercise sheet Model Theory
14 Feb 2017

Exercise 1 Assume T is an L-theory, $\varphi(x)$ is an L-formula and c is a constant not occurring in L.

(a) Prove: $T \models \varphi(c)$ iff $T \models \forall x\, \varphi(x)$.

(b) Prove: T is consistent with $\varphi(c)$ iff T is consistent with $\exists x\, \varphi(x)$.

Exercise 2 For any L-structure A we write $\mathrm{AtDiag}(A)$ for the collection of atomic L_A-sentences true in A. A formula is called *positive primitive* if it is of the form

$$\exists \overline{x} \bigwedge \psi_i(\overline{y}, \overline{x}),$$

where each $\psi_i(\overline{y}, \overline{x})$ is an atomic formula.

(a) Show that there is a one-to-one correspondence between L_A-structures B which model $\mathrm{AtDiag}(A)$ and L-structures C with a homomorphism $f\colon A \to C$.

(b) Show that the following are equivalent for an L-theory T:

(i) If $B \models T$ and $f\colon A \to B$ is a homomorphism, then $A \models T$.

(ii) There is a set of positive primitive sentences S such that $\{\neg\sigma : \sigma \in S\}$ and T have the same models.

Exercise 3 A class $\mathcal{K}$ of L-structures is a PC_Δ-*class*, if there is an extension L' of L and an L'-theory T' such that $\mathcal{K}$ consists of all reducts to L of models of T'.

Show that a PC_Δ-class of L-structures is L-elementary if and only if it is closed under L-elementary substructures.

Exercise 4 (Challenging!) An *existential sentence* is a sentence which consists of a string of existential quantifiers followed by a quantifier-free formula.

Show that a theory T can be axiomatised using existential sentences if and only if its models are closed under extensions.