4th Exercise sheet Model Theory
17 Feb 2017

Exercise 1 Prove Theorem 4.3 and Lemma 4.4 in the syllabus.

Exercise 2 The aim of this exercise is to prove the Chang-Łoś-Suszko Theorem. To state it we need a few definitions.

A $\forall\exists$-*sentence* is a sentence which consists first of a sequence of universal quantifiers, then a sequence of existential quantifiers and then a quantifier-free formula. A theory T can be axiomatised by $\forall\exists$-sentences if there is a set T' of $\forall\exists$-sentences such that T and T' have the same models.

In addition, we will say that a theory T is *preserved by directed unions* if for any directed system consisting of models of T and embeddings between them, the colimit is a model T as well. And T is *preserved by unions of chains* if for any chain of models of T and embeddings between them, the colimit is a model of T as well.

Show that the following statements are equivalent:

(1) T is preserved by directed unions.

(2) T is preserved by unions of chains.

(3) T can be axiomatised by $\forall\exists$-sentences.

Hint: To show (2) $\Rightarrow$ (3), suppose T is preserved by unions of chains and let

$$T_{\forall\exists} = \{\varphi\colon \varphi \text{ is a } \forall\exists\text{-sentence and } T \models \varphi\}.$$

Then prove that starting from any model B of $T_{\forall\exists}$ one can construct a chain of embeddings

$$B = B_0 \to A_0 \to B_1 \to A_1 \to B_2 \to A_2 \dots$$

such that:

1. Each A_n is a model of T.
2. The composed embeddings $B_n \to B_{n+1}$ are elementary.
3. Every universal sentence in the language L_{B_n} true in B_n is also true in A_n (when regarding A_n is an L_{B_n}-structure via the embedding $B_n \to A_n$).