6th Exercise sheet Model Theory
24 Feb 2017

Exercise 1 Give a direct proof of Proposition 6.2, that is, without using Theorem 6.3.

Hint: Simply use induction on n.

Exercise 2 Let $I = \{0, 1\}$ be the linear order in which $0 < 1$ and put $M := (\mathbb{Z}, <)$ and $N := (I, <) \times (\mathbb{Z}, <)$, where N is ordered lexicographically. Show that for any $i \in I$ the embedding $h_i : M \to N$ obtained by mapping $z \in \mathbb{Z}$ to (i, z) is elementary.

Hint: Let $A \subseteq \mathbb{Z}$ be finite and let $f : A \to N$ be an order and distance preserving function. Show that $(M, a)_{a \in A} \equiv (N, f(a))_{a \in A}$ (that is, show that if we consider the two L_A-structures obtained by interpreting any $a \in A$ as a itself in M and as $f(a)$ in N, then these two structures are L_A-elementarily equivalent).

Exercise 3 The circle of length $N \in \mathbb{N}$ is the structure $\mathcal{C}_N := (C_N, R)$, where $C_N = \{0, \dots, N-1\}$ and $R = \{(i, j) \in C_N \times C_N : j = i + 1 \mod N\}$.

(a) Give a function $f : \mathbb{N} \to \mathbb{N}$ such that $\mathcal{C}_N \equiv_n \mathcal{C}_{N'}$ whenever $N, N' \geq f(n)$.

(b) Is there a first-order formula φ such that $\mathcal{C}_N \models \varphi$ if and only if N is even?

Exercise 4 Show that there is no formula of first-order logic which expresses "(a, b) is in the transitive closure of R", even on finite structures. (For infinite structures it is easy to show there is no such formula.)