8th Exercise sheet Model Theory
3 Mar 2017

Exercise 1 Consider $M = (\mathbb{N}, \cdot)$ and $T = \text{Th}(M)$.

(a) Show that there are first-order formulas expressing that "x is prime" and "x is the product of two distinct primes".

(b) Show that both formulas in (a) are complete over T.

  Hint: Use the existence and uniqueness of prime factorisations to show that if $P \subseteq \mathbb{N}$ is the set of all primes, then any permutation of P can be extended to an automorphism of M.

(c) Show that M only realises isolated types and conclude that a type consistent with T is isolated if and only if it is realised in M.

(d) Show that there are non-isolated 1-types consistent with T.

Exercise 2 Consider $M = (\mathbb{R}, <, q)_{q \in \mathbb{Q}}$, that is, the reals with the usual ordering and with a name for every rational, and let $T = \text{Th}(M)$.

(a) Show that each $r \in \mathbb{R}$ realizes a unique 1-type over T. Which of these 1-types are isolated and which are not?

  Hint: Show that $(\mathbb{Q}, <, q)_{q \in \mathbb{Q}} \equiv M$.

(b) Are there 1-types over T that are not realised in M? If so, how many are there? Justify your answers!

Exercise 3 In this exercise we look at the theory $VS_{\mathbb{Q}}$ of vector spaces over $\mathbb{Q}$ of positive dimension. The language of this theory contains symbols $+$ and 0, for vector addition and the null vector, as well as unary operations m_q, one for every $q \in \mathbb{Q}$, for scalar multiplication with q. The theory $VS_{\mathbb{Q}}$ has axioms expressing that $(+, 0)$ is an infinite Abelian group on which $\mathbb{Q}$ acts as a set of scalars.

(a) For which infinite κ is $VS_{\mathbb{Q}}$ κ-categorical?

(b) Show that $VS_{\mathbb{Q}}$ is complete.

(c) Describe $S_1(T)$ and $S_2(T)$ for $T = VS_\mathbb{Q}$. Which elements in these type spaces are isolated?

Exercise 4 Show that the theory of $(\mathbb{R}, 0, +)$ has exactly two 1-types and $\aleph_0$ many 2-types. *Hint:* Think of the previous exercise.