2nd Homework sheet Model Theory

- Deadline: 20 February, 13:00 sharp.
- Submit your solutions by handing them to the lecturer or the teaching assistant at the *beginning of the lecture*.
- Good luck!

Exercise 1 *(100 points)* We will call a formula φ *positive*, if it does not contain any negations $\neg$ or implications $\rightarrow$; in other words, if it can be obtained from atomic formulas using only $\wedge, \vee, \exists$ and $\forall$. In addition, we will call a homomorphism $f\colon M \to N$ of $\mathcal{L}$-structures *positive*, if

$$M \models \varphi(m_1, \ldots, m_n) \Rightarrow N \models \varphi(f(m_1), \ldots, f(m_n))$$

holds for all positive formulas $\varphi(x_1, \ldots, x_n)$ and all $m_1, \ldots, m_n \in M$.

(a) *(40 points)* Let T be a consistent $\mathcal{L}$-theory and write

$$T_0 = \{\psi\colon \psi \text{ is a positive sentence and } T \models \psi\}.$$

Prove that for any model A of T_0 there is a diagram of $\mathcal{L}$-structures

$$\begin{array}{ccc} & & B \\ & & \big\downarrow l \\ A & \xrightarrow[k]{} & C \end{array}$$

such that: (1) B is a model of T, (2) k is an elementary embedding, (3) l is a positive homomorphism, and (4) the image of k is contained in the image of l.

(b) *(20 points)* Let $f\colon D \to A$ be a positive homomorphism of $\mathcal{L}$-structures. Prove that there exists a commuting square of $\mathcal{L}$-structures

$$\begin{array}{ccc} D & \xrightarrow{k} & B \\ f\big\downarrow & & \big\downarrow g \\ A & \xrightarrow[l]{} & C \end{array}$$

in which the horizontal maps k and l are elementary embeddings, the vertical maps f and g are positive homomorphisms and the image of l is contained in the image of g.

(c) *(40 points)* Let T be a consistent $\mathcal{L}$-theory whose models are closed under surjective images: so if $f\colon M \to N$ is a surjective homomorphism and M is a model of T, then so is N. Use parts (a) and (b) to prove that T can be axiomatised using positive sentences.