2nd Homework sheet Model Theory

- Deadline: 7 March 2016.
- Submit your solutions by handing them to the lecturer at the *beginning of the lecture at 14:00*.
- Good luck!

Exercise 1 Consider a first-order language in a signature consisting of a finite set

$$\pi = \{P_1, \ldots, P_n\}$$

of unary predicate symbols. The aim of this exercise is to use game-theoretic means to prove that every sentence in this language can be rewritten to a certain normal form.

Subsets of π will be called *colours*, and for such a colour $\sigma \subseteq \pi$ and a variable x we define

$$\odot\sigma(x) := \bigwedge_{P \in \sigma} Px \wedge \bigwedge_{P \in \pi \setminus \sigma} \neg Px$$

and for a sequence $\bar{x}$ of variables (with $\bar{x} = x_1 \cdots x_k$), we set

$$\mathit{diff}(\bar{x}) := \bigwedge_{1 \leq i < j \leq k} x_i \neq x_j.$$

Given two sequence $\bar{\sigma} = \sigma_1, \ldots, \sigma_k$ and $\bar{\tau} = \tau_1, \ldots, \tau_m$ of colours, we define the formula

$$\chi_{\bar{\sigma},\bar{\tau}} := \exists x_1 \cdots x_k \Big(\mathit{diff}(\bar{x}) \wedge \bigwedge_i \odot\sigma_i(x_i) \wedge \forall z \big(\mathit{diff}(\bar{x}, z) \to \bigvee_j \odot\tau_j(z) \big) \Big)$$

We call a first-order sentence *special* if it is of the form $\chi_{\bar{\sigma},\bar{\tau}}$ for some $\bar{\sigma}, \bar{\tau}$.

(a) For each $k \in \mathbb{N}$ define an equivalence relation between structures by putting $A \sim_k B$ if for each colour σ either $|\sigma^A| = |\sigma^B| < k$ or $|\sigma^A|, |\sigma^B| \geq k$. Here $|\sigma^A|$ denotes the size of σ in A, i.e., the number of elements a in A such that $A \models \odot\sigma(a)$.

Prove that $A \sim_k B$ implies that $A \equiv_k B$.

(b) Show that every first-order sentence in this signature is equivalent to a disjunction of special formulas.