4th Homework sheet Model Theory

- Deadline: 11 April 2016.
- Submit your solutions by handing them to the lecturer at the *beginning of the lecture at 14:00*.
- Good luck!

Exercise 1 *(50 points)* Let M be an L-structure and A be a subset of M. We say that b is *algebraic over* A if there is an L-formula $\varphi(x, \overline{y})$ and a tuple $\overline{a}$ from A such that

$$M \models \varphi(b, \overline{a})$$

and the set

$$\{x \in M : M \models \varphi(x, \overline{a})\}$$

is finite. We write $\mathrm{acl}(A)$ for the set of elements in M that are algebraic over A.

(a) Show that $A \subseteq \mathrm{acl}(A)$.

(b) Show that $\mathrm{acl}(\mathrm{acl}(A)) = \mathrm{acl}(A)$.

(c) Write $T = \mathrm{Th}_{L_A}((M, a)_{a \in A})$, the set of all L_A-sentences true in M. Show that if b is algebraic over A, then $\mathrm{tp}_M^{L_A}(b)$ is isolated in T.

(d) Suppose that T is a nice ω-categorical theory. Show that there is a function $f : \mathbb{N} \to \mathbb{N}$ such that for any model M of T and any subset $A \subseteq M$ with $|A| \leq n$, we have $|\mathrm{acl}(A)| \leq f(n)$.

Exercise 2 *(50 points)* In this exercise n is a fixed natural number and κ is a fixed infinite cardinal. Suppose that T is a theory in a language L for which the type space $S_n(T)$ has at most κ many points. Prove that there are, up to logical equivalence over T, at most κ many L-formulas with free variables among $x_1, \ldots, x_n$.

Hint: Choose for each pair $p, q \in S_n(T)$ with $p \neq q$ a formula $\varphi_{p,q}$ such that $\varphi_{p,q} \in p$ and $\varphi_{p,q} \notin q$. Prove that each formula with free variables among $x_1, \ldots, x_n$ is equivalent over T to a boolean combination of the $\varphi_{p,q}$ (where ψ is a boolean combination of some formulas in Γ if ψ can be obtained from Γ using the propositional operations $\wedge, \vee, \to$ and $\neg$).