CHAPTER 1

Basic definitions

1. On language and interpretation

DEFINITION 1.1. A *language* or *signature* L consists of:

(1) a set of constants.
(2) a set of function symbols, each with an arity $n \in \mathbb{N}$.
(3) a set of relation symbols, each with an arity $n \in \mathbb{N}$.

Once and for all, we fix a countably infinite set of variables.

DEFINITION 1.2. The *terms* in a signature L are the smallest set of expressions such that:

(1) all constants are terms.
(2) all variables are terms.
(3) if $t_1, \ldots, t_n$ are terms and f is an n-ary function symbol, then also $f(t_1, \ldots, t_n)$ is a term.

Terms which do not contain any variables are called *closed*.

DEFINITION 1.3. The *atomic formulas* is an expression of the form

(1) $s = t$, where s and t are terms, or
(2) $P(t_1, \ldots, t_n)$, where $t_1, \ldots, t_n$ are terms and P is a n-ary relation symbol.

DEFINITION 1.4. The set of *formulas* is the smallest set of expressions which:

(1) contains the atomic formulas.
(2) contains $\varphi \wedge \psi, \varphi \vee \psi, \varphi \to \psi, \neg\varphi$ whenever φ and ψ are formulas.
(3) contains $\exists x\, \varphi$ and $\forall x\, \varphi$, if φ is a formula.

A formula which does not contain any quantifiers, so can be obtained by applying rules (1) and (2) only, is called *quantifier-free*. A *sentence* is a formula which does not contain any free variables. A set of sentences is called a *theory*.

We will often write $\varphi(x_1, \ldots, x_n)$ instead of φ. The notation $\varphi(x_1, \ldots, x_n)$ is meant to indicate that φ is a formula whose free variables are contained in $\{x_1, \ldots, x_n\}$.

DEFINITION 1.5. A *structure* or *model* M in a language L consists of:

(1) a non-empty set M (the *domain* or the *universe*).
(2) interpretations $c^M \in M$ of all the constants in L,
(3) interpretations $f^M\colon M^n \to M$ of all n-ary function symbols in L,
(4) interpretations $R^M \subseteq M^n$ of all n-ary relation symbols in L.

If $A \subseteq M$, then we will write L_A for the language obtained by adding to L fresh constants $\{c_a : a \in A\}$. In this case M could also be considered an L_A-structure in which c_a is interpreted as a. We will often just write a instead of c_a (!!).

If M is a model then the interpretation in M of constants in the language L_M can be extended to all closed terms in the language L_M by putting:

$$f(t_1, \ldots, t_n)^M = f^M(t_1^M, \ldots, f_n^M).$$

DEFINITION 1.6. If M is a model in in the language L and φ is a sentence in the language L_M, then we will write:

- $M \models s = t$ if $s^M = t^M$;
- $M \models P(t_1, \ldots, t_n)$ if $(t_1, \ldots, t_n) \in P^M$;
- $M \models \varphi \wedge \psi$ if $M \models \varphi$ and $M \models \psi$;
- $M \models \varphi \vee \psi$ if $M \models \varphi$ or $M \models \psi$;
- $M \models \varphi \to \psi$ if $M \models \varphi$ implies $M \models \psi$;
- $M \models \neg\varphi$ if not $M \models \varphi$;
- $M \models \exists x\, \varphi(x)$ if there is an $m \in M$ such that $M \models \varphi(m)$;
- $M \models \forall x\, \varphi(x)$ if for all $m \in M$ we have $M \models \varphi(m)$.

If $M \models \varphi$ we say that φ *holds in* M or *is true in* M.

DEFINITION 1.7. If M is a model in a language L, then $\mathrm{Th}(M)$ is the collection of all L-sentences true in M. If N is another model in the language L, then we write $M \equiv N$ and call M and N *elementarily equivalent*, whenever $\mathrm{Th}(M) = \mathrm{Th}(N)$.

DEFINITION 1.8. Let Γ and Δ be theories. If $M \models \varphi$ for all $\varphi \in \Gamma$, then M is called a *model* of Γ. We will write $\Gamma \models \Delta$ if every model of Γ is a model of Δ as well. We write $\Gamma \models \varphi$ for $\Gamma \models \{\varphi\}$ and $\varphi \models \psi$ for $\{\varphi\} \models \{\psi\}$.

DEFINITION 1.9. If $L \subseteq L'$ and M is an L'-structure, then we can obtain an L-structure N by taking the universe of M and forgetting the interpretations of the symbols which do not occur in L. In that case, M is an *expansion* of N and N is the *L-reduct* of M.

LEMMA 1.10. *If $L \subseteq L'$ and M is an L'-structure and N is its L-reduct, then we have $N \models \varphi(m_1, \ldots, m_n)$ iff $M \models \varphi(m_1, \ldots, m_n)$ for all formulas $\varphi(x_1, \ldots, x_n)$ in the language L and all elements $m_1, \ldots, m_n$ from M.*

2. Morphisms

Any structure in mathematics comes with a notion of homomorphism: a mapping preserving that structure.

DEFINITION 1.11. Let M and N be two L-structures. A *homomorphism* $h\colon M \to N$ is a function $h\colon M \to N$ such that:

(1) $h(c^M) = c^N$ for all constants c in L;
(2) $h(f^M(m_1, \ldots, m_n)) = f^N(h(m_1), \ldots, h(m_n))$ for all function symbols f in L and elements $m_1, \ldots, m_n \in M$;
(3) $(m_1, \ldots, m_n) \in R^M$ implies $(h(m_1), \ldots, h(m_n)) \in R^N$.

A homomorphism which is bijective and whose inverse f^{-1} is also a homomorphism is called an *isomorphism*. If there exists an isomorphism between structures M and N, then M and N are called *isomorphic*. An isomorphism from a structure to itself is called an *automorphism*.

Actually, in model theory the general notion of homomorphism turns out to of limited usefulness. More important are the embeddings.

DEFINITION 1.12. A homomorphism $h: M \to N$ is an *embedding* if

(1) h is injective;
(2) $(h(m_1), \ldots, h(m_n)) \in R^N$ implies $(m_1, \ldots, m_n) \in R^M$.

LEMMA 1.13. *The following are equivalent for a homomorphism $h: M \to N$:*

(i) *h is an embedding.*
(ii) *$M \models \varphi(m_1, \ldots, m_n) \Leftrightarrow N \models \varphi(h(m_1), \ldots, h(m_n))$ for all $m_1, \ldots, m_n \in M$ and atomic formulas $\varphi(x_1, \ldots, x_n)$.*
(iii) *$M \models \varphi(m_1, \ldots, m_n) \Leftrightarrow N \models \varphi(h(m_1), \ldots, h(m_n))$ for all $m_1, \ldots, m_n \in M$ and quantifier-free formulas $\varphi(x_1, \ldots, x_n)$.*

DEFINITION 1.14. If M and N are two models and the inclusion $M \subseteq N$ is an embedding, then M is a *substructure* of N and N is an *extension* of M.

But the most important notion of morphism in model theory is that of an elementary embedding.

DEFINITION 1.15. An embedding is called *elementary*, if

$$M \models \varphi(m_1, \ldots, m_n) \Leftrightarrow N \models \varphi(h(m_1), \ldots, h(m_n))$$

for all $m_1, \ldots, m_n \in M$ and *all* formulas $\varphi(x_1, \ldots, x_n)$.

REMARK 1.16. In the definition of an elementary embedding the equivalence

$$M \models \varphi(m_1, \ldots, m_n) \Leftrightarrow N \models \varphi(h(m_1), \ldots, h(m_n))$$

holds as soon as the implication from left to right or from right to left holds. (Why? *Hint:* Negation!) A similar remark applies to point (iii) of Lemma 1.13.

LEMMA 1.17. *Any isomorphism $h: M \to N$ is also an elementary embedding. If $h: M \to N$ is an elementary embedding, then $M \equiv N$.*

3. Exercises

EXERCISE 1. A theory T is *consistent* if it has a model and *complete* if it is consistent and for any formula φ we have

$$T \models \varphi \quad \text{or} \quad T \models \neg\varphi.$$

Show that the following are equivalent for a consistent theory T:

(1) T is complete.
(2) All models of T are elementarily equivalent.
(3) There is a structure M such that T and $\mathrm{Th}(M)$ have the same models.

EXERCISE 2. An element a in an L-structure M is *definable* if there is an L-formula $\varphi(x)$ such that for any $m \in M$

$$M \models \varphi(m) \Leftrightarrow a = m.$$

(a) What are the definable elements in $(\mathbb{N}, +)$? And in $(\mathbb{Z}, +)$? Justify your answers.
(b) Is the embedding $(\mathbb{N}, +) \subseteq (\mathbb{Z}, +)$ elementary? And the embedding $(\mathbb{N}, \cdot) \subseteq (\mathbb{Z}, \cdot)$? And the embedding $(\mathbb{Z}, \cdot) \subseteq (\mathbb{Q}, \cdot)$? And the embedding $(\mathbb{Q}, \cdot) \subseteq (\mathbb{R}, \cdot)$? And the embedding $(\mathbb{R}, \cdot) \subseteq (\mathbb{C}, \cdot)$?

EXERCISE 3. (For the algebraists.) Let $L_r = \{0, 1, +, -, \cdot\}$ be the language of (unital) rings with binary operations $+$ and $\cdot$, a unary operation $-$ and constants $0, 1$. Let CR be the theory of commutative rings, saying that both $+$ and $\cdot$ are associative and commutative with units 0 and 1, respectively, plus an axiom saying that $-x$ is an additive inverse for x and the distributive law $x \cdot (y + z) = x \cdot y + x \cdot z$. The theory ID of integral domains is the theory CR together with the axioms $0 \neq 1$ and $\forall x \forall y\, (\, x \cdot y = 0 \to x = 0 \vee y = 0)$, while the theory F of fields is the theory CR together with $0 \neq 1$ and $\forall x\, (\, x \neq 0 \to \exists y\, x \cdot y = 1\,)$.

(a) A *universal sentence* is one of the form $\forall x_1, \ldots, x_n \varphi(x_1, \ldots, x_n)$ where $\varphi(x_1, \ldots, x_n)$ is quantifier-free. A theory T can be *axiomatised using universal sentences* if there is a collection of universal sentences S such that S and T have the same models.

Show that CR and ID can be axiomatised using universal sentences, while this is impossible for F. *Hint:* Check that universal sentences are preserved by substructures.

(b) Write $T_\forall = \{\varphi \colon T \models \varphi \text{ and } \varphi \text{ is universal}\}$. Show that $F_\forall$ and ID have the same models. *Hint:* Use that any integral domain can be embedded into a field (its field of fractions) by mimicking the construction of $\mathbb{Q}$ out of $\mathbb{Z}$.

EXERCISE 4. Let L be signature and M and N be two L-structures. Show that if M is finite and M and N are elementarily equivalent, then M and N are isomorphic. *Hint:* You may find it helpful to first think about the special case where the language L is finite.

CHAPTER 2

Compactness theorem

The most important result in model theory is:

THEOREM 2.1. *Let T be a theory in language L. If every finite subset of T has a model, then T has a model.*

I suspect many of you have seen a proof of this already. In fact, it is often obtained as a direct corollary of the completeness theorem for first-order logic. But one can give a purely model-theoretic proof (without any proof calculus in sight) and such a proof will be sketched below.

1. A proof

For convenience let us temporarily call a theory T *finitely consistent* if any finite subset of T has a model. The goal is to show that finitely consistent theories are consistent (that is, have a model). The first step is to reduce the problem to showing that maximal finitely consistent theories have models.

DEFINITION 2.2. A theory T in a language L is maximal finitely consistent if there is no finitely consistent L-theory T' with $T \subset T'$ (in other words, adding any new sentence to T destroys its finite consistency).

The following is a direct consequence of Zorn's Lemma (see below).

LEMMA 2.3. *Any finitely consistent L-theory T can be extended to a maximal finitely consistent L-theory T'.*

PROOF. Consider the collection P of all finitely consistent L-theories which extend T and order P by inclusion. Since every linearly subset X of P has an upper bound (simply take the union of all theories in X), Zorn's Lemma tells us that P has a maximal element. Such a maximal element is a maximal finitely consistent theory T' extending T. □

LEMMA 2.4. *Let T be maximal finitely consistent L-theory.*

(1) *For any sentence φ the theory T contains either φ or $\neg\varphi$.*
(2) *If T_0 is a finite subset of T and $T_0 \models \varphi$, then $\varphi \in T$.*

PROOF. (i): Suppose T is a maximal finitely consistent L-theory and $\varphi \notin T$. Since T was maximal, $T \cup \{\varphi\}$ cannot be finitely consistent, so there is a finite subset $T_2 \subseteq T$ such that $T_2 \cup \{\varphi\}$ has no models.

We want to show that $\neg\varphi \in T$. For this it suffices to prove that $T \cup \{\neg\varphi\}$ is finitely consistent; indeed, this can only be compatible with the maximality of T if $T \cup \{\neg\varphi\} = T$, or, in other words, if $\neg\varphi \in T$.

To see that $T \cup \{\neg\varphi\}$ is finitely consistent, let $T_0 \subseteq T \cup \{\neg\varphi\}$ be finite. Then T_0 is a subset of a set of form $T_1 \cup \{\neg\varphi\}$ with T_1 a finite subset of T.

Consider $T_1 \cup T_2$. This is a finite subset of T and since T is finitely consistent, the set $T_1 \cup T_2$ has a model M. Because M is a model of T_2, it cannot be a model of φ. So $M \models T_1$ and $M \models \varphi$. Hence M is a model of T_0 and since T_0 was an arbitrary finite subset of $T \cup \{\neg\varphi\}$, we have shown that $T \cup \{\neg\varphi\}$ is finitely consistent, as desired.

(ii): Assume T_0 is a finite subset of a maximal finitely consistent L-theory T and $T_0 \models \varphi$. It follows that $\varphi \in T$. For if $\varphi \notin T$, then $\neg\varphi \in T$ by (i). But then $T_0 \cup \{\neg\varphi\}$ is a finite subset of T, so has a model M. But then M is a model of T_0 in which φ does not hold, contradiction. □

Proposition 2.5. *Suppose T is a finitely consistent theory in a language L and C is a set of constants in L. If for any formula $\psi(x)$ in the language L there is a constant $c \in C$ such that*

$$\exists x\, \psi(x) \to \psi(c) \in T,$$

then T has a model whose universe consists entirely of interpretations of constants in C.

Proof. In view of Lemma 2.3 it suffices to prove the statement for maximal finitely consistent T. In this case we construct a model M by taking the closed terms in L and identifying closed terms s and t whenever the expression $s = t$ belongs to T: it follows from part (ii) of the previous lemma that this is an equivalence relation.

We have to show how to interpret constants as well as function and relation symbols in M. If c is any constant in L, then we put $c^M = [c]$, whilst if any f is any n-ary function symbol and $t_1, \ldots, t_n$ are closed L-terms, then we set

$$f^M([t_1], \ldots, [t_n]) := [f(t_1, \ldots, t_n)].$$

Another appeal to part (ii) of the previous lemma is needed to show that this is well-defined.

Finally, if R is an n-ary relation symbol, then we will say that $([t_1], \ldots, [t_n]) \in R^M$ in case $R(t_1, \ldots, t_n) \in T$. Part (ii) of the previous lemma should again to be used to justify this definition.

Now one can easily show by induction on the structure of the term t that $t^M = [t]$ and the structure of the formula φ that $M \models \varphi$ if and only if $\varphi \in T$. In short, M is a model of T.

It remains to verify that any element in M is an interpretation of a constant $c \in C$. We know that any element in M is of the form $[t]$ for some closed term t. But because $\exists x\, (x = t)$ is a tautology and there exists an element $c \in C$ for which

$$T \models \exists x\, (x = t) \to c = t$$

by hypothesis, there is an element $c \in C$ with $c = t \in T$. So $M \models c = t$ and $c^M = t^M = [t]$. □

Lemma 2.6. *Suppose T is a finitely consistent L-theory. Then L can be extended to a language L' and T to a finitely consistent L'-theory T' such that for any L'-formula $\varphi(x)$ there is a constant c in L' such that*

$$T' \models \exists x\, \varphi(x) \to \varphi(c).$$

PROOF. We define by induction a sequence of languages L_n and L_n-theories T_n. We start by putting $L_0 = L$ and $T_0 = T$.

If L_n and T_n have been defined, we obtain L_{n+1} by adding to L_n a fresh constant c_φ for any L_n-formula $\varphi(x)$. Moreover, T_{n+1} is obtained by adding to T_n for any L_n-formula $\varphi(x)$ the sentence

$$\exists x\, \varphi(x) \to \varphi(c_\varphi).$$

One easily proves by induction on n that each T_n is finitely consistent.

Finally, we put $L' = \bigcup_{n\in\mathbb{N}} L_n$ and $T' = \bigcup_{n\in\mathbb{N}} T_n$. Then T' is finitely consistent (see exercise 5 below). Moreover, any L'-formula $\varphi(x)$ is already an L_n-formula for some n (see again exercise 5 below). So

$$\exists x\, \varphi(x) \to \varphi(c_\varphi) \in T_{n+1} \subseteq T,$$

as desired. □

THEOREM 2.7. (Compactness Theorem) *Let T be a theory in language L. If every finite subset of T has a model, then T has a model.*

PROOF. Let T be a finitely consistent L-theory. Combining the previous lemma with the previous proposition, one sees that L can be extended to a language L' and T to an L'-theory T' such that T' has a model M. So if N is the reduct of M to L, then N is a model of T by Lemma 1.10. □

2. Appendix: statement of Zorn's Lemma

DEFINITION 2.8. A *partial order* is a set P together with a binary relation $\leq$ which is

(i) reflexive, so $x \leq x$ for any $x \in P$.
(ii) anti-symmetric, so $x \leq y$ and $y \leq x$ imply $x = y$.
(iii) transitive, so $x \leq y$ and $y \leq z$ imply $x \leq z$.

A subset $X \subseteq P$ is called a *chain* if for any two elements $x, y \in X$ we have either $x \leq y$ or $y \leq x$. An *upper bound* for a set $X \subseteq P$ is an element $y \in P$ such that $x \leq y$ for all $x \in X$. An element $x \in P$ is *maximal* if $x \leq y$ implies $x = y$.

LEMMA 2.9. (Zorn's Lemma) *Let $(P, \leq)$ be a partial order and assume that any chain in P has an upper bound. Then P contains at least one maximal element.*

PROOF. A proof can be found in most textbooks on set theory (for example, on page 114 of Moschovakis, *Notes on Set Theory*, second edition, Springer-Verlag, 2006). □

3. Exercises

EXERCISE 5. (a) Let $A_0 \subseteq A_1 \subseteq A_2 \subseteq \ldots$ be an increasing sequence of sets, and write $A := \bigcup_{n\in\mathbb{N}} A_n$. Show that any finite subset of A is already a finite subset of some A_n.
(b) Suppose that $L_0 \subseteq L_1 \subseteq L_2 \subseteq \ldots$ is an increasing sequence of languages and $L = \bigcup_{n\in\mathbb{N}} L_n$. Show that any L-formula is also an L_n-formula for some n.
(c) Suppose that $T_0 \subseteq T_1 \subseteq T_2 \subseteq \ldots$ is an increasing sequence of finitely consistent theories. Prove that $\bigcup_{n\in\mathbb{N}} T_n$ is finitely consistent as well.

EXERCISE 6. A class of models $\mathcal{K}$ in some fixed signature is called an *elementary class* if there is a first-order theory such that $\mathcal{K}$ consists of precisely those L-structures that are models of T.

Show that if $\mathcal{K}$ is a class of L-structures and both $\mathcal{K}$ and its complement (in the class of all L-structures) are elementary, then there is a sentence φ such that M belongs to $\mathcal{K}$ if and only if $M \models \varphi$.

EXERCISE 7. We work over the empty language L (no constants, function or relations symbols). Show that the class of infinite L-structures is elementary, but the class of finite L-structures is not. Deduce that there is no sentence φ that is true in an L-structure if and only if the L-structure is infinite.